

Lyapunov design of event-based controllers for the rendez-vous of coupled systems

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Abstract—The objective is to present a new type of triggering conditions together with new proof concepts for the event-based coordination of multi-agents. As a first step, we focus on the rendez-vous of two identical systems modeled as double integrators with additional damping in the velocity dynamics. The reduced number of agents in the system allows us to illustrate the design of the new triggering rule while avoiding technical issues that arise in the full multi-agent system. We first propose an event-triggering policy which relies on a designed clock variable whose dynamics depend on the relative position of the agents. We then explain how to derive self-triggered and time-triggered rules. The overall system is modeled as a hybrid system in each case, and an invariance principle is used to conclude about the rendez-vous. Simulation results are provided to illustrate the efficiency of the proposed controllers.

I. INTRODUCTION

Event-based sampling paradigms have recently been developed to implement controllers in order to provide alternatives to the traditional periodic sampling. This line of research is motivated by the rise of networked control systems and embedded systems which are subject to communication and/or computation constraints and for which periodic sampling may not be suitable. Indeed, it seems more natural in this context to use the CPU and the communication channel only when it is needed, in other words at some time instants which depend on the system state, rather than periodically. Following this idea, event-triggered control has been introduced in [4], [5]. It consists in closing the feedback loop whenever a designed state-dependent criterion is satisfied. Many works have then proposed event-triggered controllers mostly to stabilize (nonlinear) systems, see [8], [11], [13], [21], [28], [31] and the references therein, where it is shown that event-triggered control may significantly reduce the usage of the network compared to periodic sampling. A potential drawback of event-triggered control is that it requires to continuously evaluate the triggering law, which may be difficult to realize in practice. To overcome this issue, self-triggered control strategies have been developed to decide the next time instant at which the loop must be closed based on the last received measurement, see *e.g.*

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[1], [2], [29], [30]. This approach usually leads to more transmissions than event-triggered control, but necessitates less computation resources. Another alternative to event-triggered control is periodic event-triggered control [10], [20], where the triggering condition is periodically evaluated; we will not investigate it in this paper.

These event-based sampling approaches are particularly relevant in the context of multi-agent systems, for which the communication and the computation capacities are often (strongly) limited. On the other hand, multi-agent systems exhibit the following features which are absent in the works cited above:

- (i) *Weak Lyapunov function.* The stability analysis usually relies on a weak Lyapunov function, in the sense that the derivative of the Lyapunov function along the system solution is non-positive, as opposed to strong Lyapunov functions for which it is strictly negative (outside the attractor). This is an important difference with the vast majority of stabilizing event-triggered control techniques, which require the knowledge of strong Lyapunov function; the only exception is [22] to the best of our knowledge. This point induces non-trivial technical difficulties, which makes existing event-triggering results not applicable for the event-based control of multi-agent systems.
- (ii) *Distributed control.* Multi-agent systems are generally distributed as each agent has only access to its own state and the state of its neighbours (and not to the state of the overall system). Hence, it is necessary to design distributed triggering conditions which only depend on the local variables.

Various results have been proposed to address these challenges, see *e.g.* [7], [15], [17], [16], [18], [27]. In this paper, we focus on the rendez-vous of agents modeled as double integrators with a damping term in the velocity dynamics. By rendez-vous, we mean that the agent positions must asymptotically converge towards each other. We choose to focus on the particular case where the system is composed of two agents in order to concentrate on (i). Indeed, the aim of this paper is to present a new type of triggering conditions together with new proof concepts to address (i). The presented approach is extended to systems with an arbitrary finite number of agents in our forthcoming work [19], and we think that it can be used to address other event-based coordination problems, as we will show in future work.

We first focus on event-triggered control and we present

a triggering rule which relies on a clock whose dynamics depends on the relative position of the agents. At each triggering instant, the clock is reset to a fixed constant $b > 0$, it then decreases according to some state-dependent dynamics until it reaches $a \in [0, b)$, where a and b are designed by the agent. This technique is easy to construct and facilitates the analysis of the existence of a (uniform) amount of time between two successive triggering instants. Indeed, it suffices to verify that a boundedness property holds (which directly follows from the Lyapunov analysis) to ensure it. The existence of such a time is crucial in practice in order to respect the hardware which prevents the occurrence of arbitrarily close-in-time events. We then explain how to derive self-triggered and time-triggered rules. The idea of adding a variable to define the triggering condition is borrowed from [21] and the proposed techniques are inspired by [6], [23]. The overall system is modeled as a hybrid system using the formalism of [9]. The analysis is based on a hybrid Lyapunov function which is shown to be weak. An invariance principle from [26] is then applied to conclude about the desired rendez-vous.

The main novelty with respect to the existing literature on event-based control of distributed coordinated systems is that the triggering rule is designed to compensate for sampling-induced perturbative terms that appear in the Lyapunov analysis of the system. In that regard, the design is more systematic than in previously proposed approaches. The other added value of our contribution lies in the use of a hybrid invariance principle that allows us to infer the convergence result in a straightforward way. We believe that these two features (Lyapunov-based event-triggering function design and hybrid-invariance-principle-based analysis) can lead to a systematic analysis and design of event-based coordination algorithms. We focus here on the simplest example of coordination system with two agents to better highlight the two features mentioned above and to avoid technical issues that would divert the attention from the main focus of the contribution. The extension to the case of a general number of agents and the related technical details are addressed in [19].

The paper is organized as follows. Preliminaries on hybrid systems are recalled in Section II and the problem is stated in Section III. Event-triggered control is addressed in Section IV and we then explain respectively in Sections V and VI how to derive self-triggered and time-triggered rules. In Section VII, simulation results are presented and Section VIII concludes the paper.

II. PRELIMINARIES

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_{\geq 0} = [0, \infty)$, $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$, and $\mathbb{Z}_{> 0} = \{1, 2, \dots\}$. For $(x, y) \in \mathbb{R}^{n+m}$, (x, y) stands for $[x^T, y^T]^T$. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $r \in \mathbb{R}$, we denote by $f^{-1}(r)$ the set $\{x \in \mathbb{R}^n : f(x) = r\}$. A function $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class \mathcal{K} if it is continuous, zero at zero and strictly increasing and it is of class \mathcal{K}_{∞} if in addition it is unbounded.

We will study hybrid systems of the form below, using the formalism of [9],

$$\dot{x} = F(x) \quad \text{for } x \in C, \quad x^+ = G(x) \quad \text{for } x \in D, \quad (1)$$

where $x \in \mathbb{R}^n$ is the state, F is the flow map, G is the jump map, C is the flow set and D is the jump set. We assume that C and D are closed subsets of \mathbb{R}^n and that F and G are continuous. These conditions ensure that system (1) is well-posed, see Chapter 6 in [9]. We recall some definitions related to [9]. A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a *hybrid time domain* if for all $(T, K) \in E$, $E \cap ([0, T] \times \{0, \dots, K\}) = \bigcup_{k \in \{0, 1, \dots, K-1\}} ([t_k, t_{k+1}], k)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq \dots \leq t_K$. A function $\phi : E \rightarrow \mathbb{R}^{n_x}$ is a *hybrid arc* if E is a hybrid time domain and if for each $k \in \mathbb{Z}_{\geq 0}$, $t \mapsto \phi(t, k)$ is locally absolutely continuous on $\mathcal{I}^k = \{t : (t, k) \in E\}$. The hybrid arc $\phi : \text{dom } \phi \rightarrow \mathbb{R}^{n_x}$ is a *solution* to (1) if: (i) $\phi(0, 0) \in C \cup D$; (ii) for any $k \in \mathbb{Z}_{\geq 0}$, $\phi(t, k) \in C$ and $\frac{d}{dt}\phi(t, k) = F(\phi(t, k))$ for almost all $t \in \mathcal{I}^k$, where $\mathcal{I}^k = \{t : (t, k) \in \text{dom } \phi\}$; (iii) for every $(t, k) \in \text{dom } \phi$ such that $(t, k+1) \in \text{dom } \phi$, $\phi(t, k) \in D$ and $\phi(t, k+1) = G(\phi(t, k))$. A solution ϕ to (1) is:

- *nontrivial* if its domain contains at least two points;
- *maximal* if it cannot be extended;
- *complete* if $\text{dom } \phi$ is unbounded;
- *precompact* if it is complete and the closure of its range is compact, where the range of ϕ is $\text{rge } \phi := \{y \in \mathbb{R}^n : \exists (t, k) \in \text{dom } \phi \text{ such that } y = \phi(t, k)\}$.

We will use the definition below.

Definition 1: The solutions to (1) have a *uniform semiglobal dwell-time* if for any $\Delta \geq 0$, there exists $\tau(\Delta) > 0$ such that for any solution ϕ to (1) with $|\phi(0, 0)| \leq \Delta$, $\sup \mathcal{I}^k - \inf \mathcal{I}^k \geq \tau(\Delta)$ for any $k \in \mathbb{Z}_{> 0}$, where $\mathcal{I}^k = \{t : (t, k) \in \text{dom } \phi\}$. We say that the solutions to (1) have a *uniform global dwell-time* if there exists $\tau > 0$ such that $\sup \mathcal{I}^k - \inf \mathcal{I}^k \geq \tau$ for any $k \in \mathbb{Z}_{> 0}$ and any solution. \square

We recall the following invariance definition, see Definition 6.19 in [9].

Definition 2: A set $S \subset \mathbb{R}^{n_x}$ is *weakly invariant* for system (1) if it is:

- *weakly forward invariant*, i.e. for any $\xi \in S$ there exists at least one complete solution ϕ with initial condition ξ such that $\text{rge } \phi \subset S$;
- *weakly backward invariant*, i.e. for any $\xi \in S$ and $\tau > 0$, there exists at least one solution ϕ such that for some $(t^*, k^*) \in \text{dom } \phi$, $t^* + k^* \geq \tau$, it is the case that $\phi(t^*, k^*) = \xi$ and $\phi(t, k) \in S$ for all $(t, k) \in \text{dom } \phi$ with $t + k \leq t^* + k^*$. \square

Finally, we say that a solution ϕ *approaches the set* $S \subset \mathbb{R}^{n_x}$ ([26]) if for any $\epsilon > 0$ there exists $(t^*, k^*) \in \text{dom } \phi$ such that for all $(t, k) \in \text{dom } \phi$ with $t + k \geq t^* + k^*$, $\phi(t, k) \in S + \epsilon \mathbb{B}$, where \mathbb{B} is the unit ball of \mathbb{R}^n .

III. PROBLEM STATEMENT

The objective is to construct distributed controllers to ensure the rendez-vous of two identical agents with limited communication and/or computation capacities. The dynamics of each agent is given by, for $i \in \{1, 2\}$,

$$\begin{aligned}\dot{p}_i &= v_i \\ \dot{v}_i &= -v_i + u_i\end{aligned}\quad (2)$$

where $p_i \in \mathbb{R}$ is the position, $v_i \in \mathbb{R}$ is the velocity, $u_i \in \mathbb{R}$ is the control input. This dynamics corresponds to agents modeled as double integrators with the addition of a damping term possibly due to friction. In continuous-time, the control input u_i is defined as, like in [3],

$$u_i = \psi(z_i) \quad (3)$$

where

$$z_i = p_j - p_i, \quad j \in \{1, 2\} \setminus \{i\} \quad (4)$$

is the relative position of agent i with respect to agent j and we similarly call $v_j - v_i$ the relative velocity. The function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is designed and it is required to be continuously differentiable, nondecreasing and odd, that implies that $(x-y)(\psi(x) - \psi(y)) \geq 0$ and $\psi(-x) = -\psi(x)$ for $x, y \in \mathbb{R}$. According to [3], the controllers (3) guarantee that the positions p_i , $i \in \{1, 2\}$, asymptotically converge towards each other, which means that the rendez-vous is achieved.

In this paper, we take into account the resources limitations of the agents in terms of communication and/or computation. Hence, we envision a setting where the agents only receive measurements from their neighbour and/or update their control input at some given time instants t_k , $k \in \mathbb{Z}_{\geq 0}$, to be determined. The control input u_i in (3) becomes, for $i \in \{1, 2\}$,

$$u_i = \psi(\hat{z}_i) \quad (5)$$

where \hat{z}_i is a sampled version of z_i , which is locally maintained by agent i . The variable z_i is held constant between two successive updates, *i.e.*

$$\dot{\hat{z}}_i = 0, \quad (6)$$

and is reset to the actual values z_i at the update time instant, which leads to the jump equation

$$\hat{z}_i^+ = z_i. \quad (7)$$

Our goal is to define the sequence t_k , $k \in \mathbb{Z}_{\geq 0}$, which we term as *events*, in order to save resources while still ensuring the rendez-vous. We present solutions for the three scenarios listed below:

- *Event-triggered control*: the agents know their relative position at any time instant and the control input is only updated whenever a certain triggering condition is satisfied (see *e.g.* [7], [24], [27]). This scenario is relevant to reduce the actuators wear as well as the actuators energy consumption by limiting the number of control input updates. It requires that the agents are equipped with local sensors which measure the relative

positions with their neighbour at a high frequency or that the agents communicate with their neighbour via a high-bandwidth communication channel. In that way, we can make the approximation that the agents continuously have access to their neighbour relative position.

- *Self-triggered control*: the agents know their relative position and velocity and update their control input *only* at t_k , $k \in \mathbb{Z}_{\geq 0}$ (see [7], [17], [18]). The time instant t_{k+1} , $k \in \mathbb{Z}_{\geq 0}$, is predetermined by the values of $p_2 - p_1$ and $v_2 - v_1$ at t_k . This scheme reduces both the usage of the CPU and of the agents sensors or of the communication channel. This paradigm neither requires the continuous measurement of the neighbours relative position nor the continuous evaluation of a triggering condition.
- *Time-triggered control*: the setup is the same as for the item above, except that the sequence t_k , $k \in \mathbb{Z}_{\geq 0}$, is periodic (and not state-dependent). For related works, see [32] for example.

The proposed strategies ensure the existence of a strictly positive amount of time between two successive event time instants. This property is crucial as it prevents arbitrarily close-in-time events which would exceed the hardware capacities and render the proposed hybrid controllers not realizable.

IV. EVENT-TRIGGERED CONTROL

A. Main result

To define the events, we introduce a variable $\phi \in \mathbb{R}$, which we call a *clock* variable. The idea is to reset ϕ to $b > 0$ after each event and to trigger the next one when ϕ becomes equal to $a \in [0, b)$. The constants a and b are designed parameters and we explain how to tune these below. Between two successive events, ϕ is given by the solution to the ordinary differential equation below

$$\dot{\phi} = -\frac{1}{\sigma} \left(1 + \phi^2 (\nabla\psi(z))^2 \right), \quad (8)$$

where σ is any constant in $(0, \frac{1}{2})$ and we recall that $z = p_2 - p_1$ (note that $(\nabla\psi(p_2 - p_1))^2 = (\nabla\psi(p_1 - p_2))^2$ as ψ is odd). We notice that ϕ strictly decreases between two successive events in view of (8). The clock ϕ can be locally implemented on any (or both) of the agents provided continuous measurements of z are available. This type of triggering condition is inspired by the work in [6] where the stabilization of nonlinear sampled-data systems is investigated, in which case the dynamics of ϕ only depends on ϕ and on some constants which leads to a periodic rule. This technique has then been extended to event-triggered control in [23] by allowing the ϕ -dynamics to depend on the system state. In this paper, we show that this approach can be particularly useful to address event-based coordination control problems. We see that the inter-event time depends

on the choice of the constants a and b . Taking a small and b large may help enlarging the inter-event time, at the price of a degraded speed of convergence as the evolution of the velocities depend on the sampled control input, see for an illustration the simulation results in Section VII.

We model the overall system using the hybrid formalism of [9] for which a jump describes a control update

$$\left. \begin{aligned} \dot{p}_1 &= v_1 \\ \dot{v}_1 &= -v_1 + \psi(\hat{z}) \\ \dot{p}_2 &= v_2 \\ \dot{v}_2 &= -v_2 + \psi(-\hat{z}) = -v_2 - \psi(\hat{z}) \\ \dot{\hat{z}} &= 0 \\ \dot{\phi} &= -\frac{1}{\sigma} \left(1 + \phi^2 (\nabla\psi(p_2 - p_1))^2\right) \end{aligned} \right\} \phi \in [a, b],$$

$$\left. \begin{aligned} p_1^+ &= p_1 \\ v_1^+ &= v_1 \\ p_2^+ &= p_2 \\ v_2^+ &= v_2 \\ \hat{z}^+ &= p_2 - p_1 \\ \phi^+ &= b \end{aligned} \right\} \phi = a. \quad (9)$$

We have used the fact that ψ is odd to obtain the right hand-side of \dot{v}_2 in (9). The proposed event-triggered controllers ensure the rendez-vous of system (9) as formalized by the theorem below whose proof is given in the next subsection.

Theorem 1: Consider system (9). The solutions have a uniform semiglobal dwell-time and the maximal solutions are precompact and approach the set $\{(p_1, v_1, p_2, v_2, \hat{z}, \phi) : p_1 = p_2, v_1 = v_2 = \hat{z} = 0 \text{ and } \phi \in [a, b]\}$. \square

B. Proof of Theorem 1

1) *Change of coordinates:* We are interested in the convergence of $p_2 - p_1$ towards the origin, we therefore use the variable $z = p_2 - p_1$. System (9) becomes in the coordinates $(p_1, z, \hat{z}, v_1, v_2, \phi)$

$$\left. \begin{aligned} \dot{p}_1 &= v_1 \\ \dot{z} &= v_2 - v_1 \\ \dot{\hat{z}} &= 0 \\ \dot{v}_1 &= -v_1 + \psi(\hat{z}) \\ \dot{v}_2 &= -v_2 - \psi(\hat{z}) \\ \dot{\phi} &= -\frac{1}{\sigma} \left(1 + \phi^2 (\nabla\psi(z))^2\right) \end{aligned} \right\} \phi \in [a, b]$$

$$\left. \begin{aligned} p_1^+ &= p_1 \\ z^+ &= z \\ \hat{z}^+ &= z \\ v_1^+ &= v_1 \\ v_2^+ &= v_2 \\ \phi^+ &= b \end{aligned} \right\} \phi = a. \quad (10)$$

For the sake of convenience, we write system (10) as

$$\dot{q} = f(q) \quad \text{for } q \in C, \quad q^+ = g(q) \quad \text{for } q \in D, \quad (11)$$

where $q = (p_1, z, \hat{z}, v_1, v_2, \phi)$ and

$$C = \{q : \phi \in [a, b]\}, \quad D = \{q : \phi = a\}. \quad (12)$$

It can be noted that f and g are continuous and that C and D are closed, which implies that system (11), (12) is well-posed according to Chapter 6 in [9].

2) *Maximal solutions are complete:* We first investigate the completeness of the maximal solutions to (11), (12). We define

$$U(q) := U_{\text{phys}}(q) + U_{\text{cyber}}(q) \quad \forall q \in C \cup D, \quad (13)$$

where $U_{\text{phys}}(q) := \frac{1}{2}(v_1^2 + v_2^2) + \int_0^z \psi(s) ds$ is an energy-like function which takes into account the physical component of the system and $U_{\text{cyber}}(q) := \frac{1}{2}\phi(\psi(\hat{z}) - \psi(z))^2$ is a term which takes into account the cyberphysical nature of the system. Let $q \in D$, in view of (10),

$$\begin{aligned} U(g(q)) &= \frac{1}{2}(v_1^2 + v_2^2) + \int_0^z \psi(s) ds + \frac{1}{2}b(\psi(z) - \psi(z)) \\ &\leq U(q). \end{aligned} \quad (14)$$

Let $q \in C$. The function U is differentiable since ψ is assumed to be continuously differentiable. Hence it holds that, according to (10) and after some direct calculations,

$$\begin{aligned} \langle \nabla U(q), f(q) \rangle &= -v_1^2 - v_2^2 + (v_1 - v_2)(\psi(\hat{z}) - \psi(z)) \\ &\quad - \frac{1}{2}\frac{1}{\sigma} \left(1 + \phi^2 (\nabla\psi(z))^2\right) (\psi(\hat{z}) - \psi(z))^2 \\ &\quad - \phi(\psi(\hat{z}) - \psi(z))\nabla\psi(z)(v_2 - v_1). \end{aligned} \quad (15)$$

We introduce $e := \psi(\hat{z}) - \psi(z)$ and rewrite the equality above as

$$\begin{aligned} \langle \nabla U(q), f(q) \rangle &= -v_1^2 - v_2^2 + (v_1 - v_2)e \\ &\quad - \frac{1}{2}\frac{1}{\sigma} \left(1 + \phi^2 (\nabla\psi(z))^2\right) e^2 \\ &\quad - \phi e \nabla\psi(z)(v_2 - v_1). \end{aligned} \quad (16)$$

We use the bounds below, which follow from the facts that $xy \leq \frac{1}{2\eta}x^2 + \frac{\eta}{2}y^2$ and $(x-y)^2 \leq 2(x^2+y^2)$ for any $x, y \in \mathbb{R}$ and $\eta > 0$,

$$\begin{aligned} (v_1 - v_2)e &\leq \frac{\sigma}{2}(v_1 - v_2)^2 + \frac{1}{2\sigma}e^2 \\ &\leq \sigma(v_1^2 + v_2^2) + \frac{1}{2\sigma}e^2 \\ -\phi e \nabla\psi(z)(v_2 - v_1) &\leq \frac{\sigma}{2}(v_2 - v_1)^2 + \frac{1}{2\sigma}\phi^2 e^2 (\nabla\psi(z))^2 \\ &\leq \sigma(v_1^2 + v_2^2) + \frac{1}{2\sigma}\phi^2 e^2 (\nabla\psi(z))^2, \end{aligned} \quad (17)$$

where σ comes from (10), to derive that

$$\begin{aligned} \langle \nabla U(q), f(q) \rangle &\leq -v_1^2 - v_2^2 + \sigma(v_1^2 + v_2^2) + \frac{1}{2\sigma}e^2 \\ &\quad - \frac{1}{2}\frac{1}{\sigma} \left(1 + \phi^2 (\nabla\psi(z))^2\right) e^2 \\ &\quad + \sigma(v_1^2 + v_2^2) + \frac{1}{2\sigma}\phi^2 e^2 (\nabla\psi(z))^2 \\ &= (-1 + 2\sigma)(v_1^2 + v_2^2). \end{aligned} \quad (18)$$

Let ξ be a maximal solution to (11), (12). We want to prove that ξ is complete. We first need to ensure that ξ is a nontrivial solution. We use Proposition 6.10 in [9] for that purpose, in particular we aim to prove that for any $q \in C \setminus D$, $f(q)$ belongs to the tangent cone¹ of C at q . Let $q \in C \setminus D$. Consider the case where $\phi \in (a, b)$, the tangent cone to C at q is \mathbb{R}^6 and the desired result holds. Let

¹The tangent cone to C at x is the set of all vectors $w \in \mathbb{R}^6$ for which there exist $x_i \in C$, $\tau_i > 0$ with $x_i \rightarrow x$, $\tau_i \rightarrow 0$ as $i \rightarrow \infty$ such that $w = \lim_{i \rightarrow \infty} (x_i - x)/\tau_i$; see Definition 5.12 in [9].

$\phi = b$, the aforementioned tangent cone is $\mathbb{R}^5 \times (-\infty, 0]$. In view of the dynamics of ϕ on flows in (10), we deduce that $f(q) \in \mathbb{R}^5 \times (-\infty, 0]$. Consequently, ξ is not a trivial solution to (11), (12) according to Proposition 6.10 in [9].

The solution ξ cannot leave the flow and the jump sets as $g(D) \subset C$ in view of (12). Hence, the only potential obstacle to the completeness of ξ is if it explodes in finite time, according to Proposition 6.10 in [9]. Consider the definition of U in (13). The facts that ψ is monotone and odd and that $\phi \in [a, b]$ ensure that there exist $\underline{\alpha}_U, \bar{\alpha}_U \in \mathcal{K}_\infty$ such that, for any $q \in C \cup D$,

$$\underline{\alpha}_U(|(z, v_1, v_2)|) \leq U(q) \leq \bar{\alpha}_U(|(z, v_1, v_2, \psi(z), \psi(\hat{z}))|). \quad (19)$$

On the other hand, we derive from (14) and (18) that

$$U(\xi(t, k)) \leq U(\xi(0, 0)) \quad \forall (t, k) \in \text{dom } \xi. \quad (20)$$

Therefore, for all $(t, k) \in \text{dom } \xi$,

$$\underline{\alpha}_U(|(z(t, k), v_1(t, k), v_2(t, k))|) \leq U(q(0, 0)). \quad (21)$$

We conclude that the (z, v_1, v_2) -component of ξ is bounded, from which we derive that \hat{z} is also bounded in view of (10). Furthermore, we deduce from (10) and (21) that p_1 cannot explode in finite time as its dynamics on flows is bounded by a constant. Finally, we note that ϕ remains in $[a, b]$ by definition of the sets C and D . Consequently, ξ cannot explode in finite time and it is therefore complete.

3) *Maximal solutions are precompact:* We would like to apply an invariance principle for hybrid systems from [26] to (11), (12) to conclude about the desired rendezvous property. For this purpose, we need to ensure that the maximal solutions to (11) are precompact, *i.e.* these solutions are complete (which is the case) and bounded. Let ξ be a maximal solution to (11), (12). We have already shown above that the components $z, \hat{z}, v_1, v_2, \phi$ of ξ are bounded. We focus on p_1 . Let $\text{ave}(p_1, p_2) = \frac{p_1 + p_2}{2}$ and $\text{ave}(v_1, v_2) = \frac{v_1 + v_2}{2}$ (where $p_2 = p_1 + z$). For almost all $t \in \mathcal{I}^k$, where $\mathcal{I}^k = \{t : (t, k) \in \text{dom } \xi\}$ with $k \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} \frac{d}{dt} \text{ave}(p_1(t, k), p_2(t, k)) &= \text{ave}(v_1(t, k), v_2(t, k)) \\ \frac{d}{dt} \text{ave}(v_1(t, k), v_2(t, k)) &= -\text{ave}(v_1(t, k), v_2(t, k)) \end{aligned} \quad (22)$$

and for any $(t, k) \in \text{dom } \xi$ such that $(t, k+1) \in \text{dom } \xi$,

$$\begin{aligned} \text{ave}(p_1(t, k+1), p_2(t, k+1)) &= \text{ave}(p_1(t, k), p_2(t, k)), \\ \text{ave}(v_1(t, k+1), v_2(t, k+1)) &= \text{ave}(v_1(t, k), v_2(t, k)). \end{aligned} \quad (23)$$

As a result, we see that, for any $(t, k) \in \text{dom } \xi$,

$$\begin{aligned} \text{ave}(p_1(t, k), p_2(t, k)) &= \text{ave}(p_1(0, 0), p_2(0, 0)) \\ &\quad + (1 - \exp(-t)) \text{ave}(v_1(0, 0), v_2(0, 0)), \end{aligned} \quad (24)$$

thus showing that $\text{ave}(p_1, p_2)$ is bounded. On the other hand,

$$p_1 = \text{ave}(p_1, p_2) + \frac{p_1 - p_2}{2} = \text{ave}(p_1, p_2) - \frac{z}{2}. \quad (25)$$

Consequently, the boundedness of z implies the boundedness of the absolute position p_1 . Hence ξ is precompact.

4) *Solutions have a uniform semiglobal dwell-time:* We now verify that the conditions of Corollary 4.4 in [26] are verified by system (11), (12). We note that (2) in [26] holds with $u_c(q) = (-1 + 2\sigma)(v_1^2 + v_2^2)$ when $q \in C$ and $u_c(q) = -\infty$ when $q \in \mathbb{R}^6 \setminus C$ and $u_d(q) = 0$ when $q \in D$ and $u_d(q) = -\infty$ when $q \in \mathbb{R}^6 \setminus D$ in view of (14) and (18) and that these functions are non-positive (notice that $-1 + 2\sigma < 0$ since $\sigma \in (0, \frac{1}{2})$) as required by Theorem 4.3 in [26]. The conclusions of Corollary 4.4 in [26] only apply to precompact solutions. We have proved above that any maximal solution to (11), (12) is indeed precompact. The last point to verify before applying Corollary 4.4 in [26] is the existence of a dwell-time for any solution to (11), (12), *i.e.* for any solution ξ , there exist $K, \gamma > 0$ such that for all $k \geq K$, $\sup \mathcal{I}^k - \inf \mathcal{I}^k \geq \gamma$ for any $k \in \mathbb{Z}_{>0}$, where $\mathcal{I}^k = \{t : (t, k) \in \text{dom } \xi\}$. Let $\Delta > 0$ and take any maximal solution ξ to (11), (12) such that $|\xi(0, 0)| \leq \Delta$. According to (19), there exists $\bar{\Delta} > 0$ (which depends on Δ) such that, for any $(t, k) \in \text{dom } \xi$, $\underline{\alpha}_U(|(z(t, k), v_1(t, k), v_2(t, k))|) \leq \bar{\alpha}_U(|(z(0, 0), v_1(0, 0), v_2(0, 0), \psi(z(0, 0)), \psi(\hat{z}(0, 0)))|) \leq \bar{\Delta}$, which implies

$$|z(t, k)| \leq |(z(t, k), v_1(t, k), v_2(t, k))| \leq \underline{\alpha}_U^{-1}(\bar{\Delta}). \quad (26)$$

On the other hand, when $k \geq 1$, the time between two jumps of any solution to (11), (12) corresponds to the time it takes for ϕ to decrease from b to a . This time interval is lower bounded by the time it takes for θ the solution to the differential equation below

$$\dot{\theta} = -\frac{1}{\sigma} \left(1 + \theta^2 \max_{z \text{ s.t. } |z| \leq \underline{\alpha}_U^{-1}(\bar{\Delta})} (\nabla \psi(z))^2 \right) \quad (27)$$

to decrease from b to a , in view of (26) and according to the comparison principle (see Lemma 3.4 in [12]). Note that the maximum in (27) is well-defined since $\nabla \psi$ is continuous and since it is taken over a compact set. The aforementioned time interval² is obviously a strictly positive constant $\tau(a, b, \bar{\Delta})$ in view of (27). This implies that ξ has a dwell-time which is larger than $\tau(a, b, \bar{\Delta})$; we have proved that the solutions to (11), (12) have a uniform semiglobal dwell-time (see Definition 1). We note that by using similar arguments, the time between two jumps of ξ is upper-bounded by the time it takes for $\dot{\theta} = -\frac{1}{\sigma}$ to decrease from b to a , *i.e.* $\sigma(b - a)$, which means that ξ has an unbounded hybrid time domain both in continuous and discrete times. The latter point is not necessary to apply the invariance principle but it will be useful in the following.

²In fact, the rate of change of θ is upper and lower bounded as follows

$$-\frac{1}{\sigma} \left(1 + b^2 \max_{z \text{ s.t. } |z| \leq \underline{\alpha}_U^{-1}(\bar{\Delta})} \{(\nabla \psi(z))^2\} \right) \leq \dot{\theta} \leq -\frac{1}{\sigma}.$$

5) *Application of the hybrid invariance principle:* We apply Corollary 4.4 in [26] to conclude that any maximal solution to (11), (12) approaches the largest weakly invariant subset \mathcal{S} of

$$U^{-1}(r) \cap \mathcal{U} \cap \overline{u_c^{-1}(0)} \quad (28)$$

for some $r \in U(\mathcal{U})$ where $\mathcal{U} = \mathbb{R}^5 \times [a, b]$. The set $\overline{u_c^{-1}(0)}$ is equal to $\{q : v_1 = v_2 = 0\}$. Note that \mathcal{S} is non-empty according to Lemma 3.3 in [26]. Let $q \in \mathcal{S}$ and ξ be a precompact solution such that $\xi(0, 0) = q$ and $\xi(t, k) \in \mathcal{S}$ for any $(t, k) \in \text{dom } \xi$. This implies that, in view of (10) and using the fact that $\mathcal{S} \subset \overline{u_c^{-1}(0)}$, for almost all $t \in \mathcal{I}^k$, where $\mathcal{I}^k = \{t : (t, k) \in \text{dom } \xi\}$ with $k \in \mathbb{Z}_{\geq 0}$,

$$\begin{cases} \dot{z}(t, k) = 0 \\ \dot{\hat{z}}(t, k) = 0 \\ 0 = 0 + \psi(\hat{z}(t, k)) \\ 0 = 0 - \psi(\hat{z}(t, k)) \end{cases} \quad (29)$$

and for any $(t, k) \in \text{dom } \xi$ such that $(t, k+1) \in \text{dom } \xi$,

$$\begin{cases} z(t, k+1) = z(t, k) \\ \hat{z}(t, k+1) = \hat{z}(t, k) \\ 0 = 0 \\ 0 = 0. \end{cases} \quad (30)$$

The function ψ only cancels at the origin according to Section III, hence $\hat{z}(t, k) = 0$ for almost $t \in \mathcal{I}_k$ and thus for all $t \in \mathcal{I}_k$ by continuity of $\hat{z}(\cdot, k)$ on \mathcal{I}^k . Moreover, we have seen that the solutions to (11), (12) have an unbounded hybrid time domain both in the continuous and in the discrete times. Consequently, $z(t, k) = \hat{z}(t, k) = 0$ for any $(t, k) \in \text{dom } \xi$ in view of (30). As a result, the point $q = \xi(0, 0)$ verifies $z(0, 0) = \hat{z}(0, 0) = v_1(0, 0) = v_2(0, 0) = 0$, which implies that $\mathcal{S} \subset \{q : (z, \hat{z}, v_1, v_2) = 0 \text{ and } \phi \in [a, b]\}$. We deduce that any maximal solution to (11), (12) approaches the set $\{q : (z, \hat{z}, v_1, v_2) = 0 \text{ and } \phi \in [a, b]\}$. This concludes the proof.

V. SELF-TRIGGERED CONTROL

The event-triggering condition proposed in Section IV requires continuous measures of $p_2 - p_1$ to trigger control updates. This may not be achievable in practice. We propose to *emulate* the event-triggering policy as a self-triggering rule to overcome this potential issue.

At each transmission instant, the two agents are now assumed to communicate to each other not only their relative position but also their relative velocity, *i.e.* $v_2 - v_1$. We can then estimate the value of $p_2 - p_1$ at any time between two successive transmission instants and use this estimate to determine when ϕ reaches a . We can exactly estimate $p_2 - p_1$ at time (t, k) based on the value of the state at the last jump instant (t_k, k) . Indeed, let ξ be a solution to (11), (12) and denote $\text{dom } \xi = \bigcup_{k \in \mathbb{Z}_{\geq 0}} [(t_k, t_{k+1}], k)$. For $(t, k) \in \text{dom } \xi$

$$p_1(t, k) = p_1(t_k, k) + \int_{t_k}^t v_1(s, k) ds, \quad (31)$$

since $v_1(t, k) = \exp(-(t - t_k))v_1(t_k, k) + (1 - \exp(-(t - t_k)))\psi(p_2(t_k, k) - p_1(t_k, k))$, we can compute analytically

$p_1(t, k)$ based on $p_1(t_k, k), p_2(t_k, k), v_1(t_k, k)$. We can similarly derive the expression of $p_2(t, k)$. Hence, using the variable $z = p_2 - p_1$ for the sake of simplicity, one obtains

$$\begin{aligned} z(t, k) &= z(t_k, k) + (1 - \exp(-(t - t_k))) \\ &\quad \times [v_2(t_k, k) - v_1(t_k, k) + 2\psi(z(t_k, k))] \\ &\quad - 2\psi(z(t_k, k))(t - t_k). \end{aligned} \quad (32)$$

We then need to analytically solve the differential equation below

$$\frac{d}{dt}\phi(t, k) = -\frac{1}{\sigma} \left(1 + \phi^2(\nabla\psi(z(t, k)))^2\right), \quad \phi(t_k, k) = b \quad (33)$$

to determine the next time ϕ reaches a . Notice that the solution exists locally and is unique. As an alternative to the analytic solution of the equation above, one can assume that enough computational power is available so that the solution of the differential equation can be computed instantaneously (implementation issues that arise with the introduction of this assumption are not tackled here). Then at time (t_k, k) , it is possible to exactly predict the next time (t_{k+1}, k) at which $\phi(t_{k+1}, k) = b$.

Remark 1: From (33), we notice that the triggering instants are periodic when $\nabla\psi$ is constant, that is when ψ is linear. This particular case is treated in more detail in Section VI. \square

We model the system as follows

$$\left. \begin{aligned} \dot{p}_1 &= v_1 \\ \dot{v}_1 &= -v_1 + \psi(\hat{z}) \\ \dot{p}_2 &= v_2 \\ \dot{v}_2 &= -v_2 - \psi(\hat{z}) \\ \dot{\hat{z}} &= 0 \\ \dot{\tau} &= -1 \end{aligned} \right\} \tau \geq 0, \quad (34)$$

$$\left. \begin{aligned} p_1^+ &= p_1 \\ v_1^+ &= v_1 \\ p_2^+ &= p_2 \\ v_2^+ &= v_2 \\ \hat{z}^+ &= p_2 - p_1 \\ \tau^+ &= T(a, b, p_1 - p_2, v_1 - v_2) \end{aligned} \right\} \tau = 0.$$

where $\tau \geq 0$ is a clock used to trigger transmissions and $T(a, b, p_1 - p_2, v_1 - v_2)$ represents the time it takes from the solution ϕ to the differential equation

$$\frac{d}{dt}\phi(t) = -\frac{1}{\sigma} \left(1 + \phi^2(\nabla\psi(z(t)))^2\right), \quad \phi(0) = b \quad (35)$$

to be equal to a , with

$$\begin{aligned} z(t) &= p_2 - p_1 + (1 - \exp(-t))[v_2 - v_1 + 2\psi(p_2 - p_1)] \\ &\quad - 2\psi(p_2 - p_1)t. \end{aligned} \quad (36)$$

The result below is obtained by following similar lines as in the proof of Theorem 1 in view of the developments above.

Theorem 2: Consider system (34). The solutions have a uniform semiglobal dwell-time and the maximal solutions are precompact and approach the set $\{(p_1, v_1, p_2, v_2, \hat{z}, \tau) : p_1 = p_2, v_1 = v_2 = \hat{z} = 0, \text{ and } \tau \in [0, \sigma(b - a)]\}$. \square

| | | $(a, b) = (1, 10)$ | | $(a, b) = (0.1, 10)$ | | $(a, b) = (0.1, 50)$ | |
|---------------------|-----|--------------------|------|----------------------|------|----------------------|------|
| | | (i) | (ii) | (i) | (ii) | (i) | (ii) |
| Average # of events | STC | 116 | 73 | 58 | 36 | 55 | 9 |
| | TTC | idem | 116 | idem | 58 | idem | 55 |
| Average t^* | STC | 10.4 | 13.4 | 13.0 | 16.0 | 13.1 | > 20 |
| | TTC | idem | 11.2 | idem | 11.8 | idem | 13.6 |

TABLE I

AVERAGE NUMBER OF EVENTS AND AVERAGE VALUE OF t^* FOR THE SELF-TRIGGERED CONTROL (#: NUMBER, STC: SELF-TRIGGERED CONTROL, TTC: TIME-TRIGGERED CONTROL).

VI. TIME-TRIGGERED CONTROL

It is also possible to emulate the event-triggering condition in Section IV as a time-triggered rule. In this case, the transmission instants are periodic, and we denote T this period. In view of the analysis of the dwell-times in Section IV-B.4, it is always possible to compute a lower bound on the inter-jump instants of system (9) and to use this bound as the sampling period T . This constant depends on the ball of initial conditions in general, see Section IV-B.4. In this section, we focus on the case where a global bound can be computed for the sake of simplicity. We make the assumption below for that purpose.

Assumption 1: There exists $M \geq 0$ such that $|\nabla\psi(z)| \leq M$ for any $z \in \mathbb{R}$. \square

Assumption 1 is verified when ψ is a linear function and more generally when it is globally Lipschitz. We define the period T as the time it takes from the solution θ to the differential equation

$$\dot{\theta} = -\frac{1}{\sigma}(1 + M^2\theta^2), \quad \theta(0) = b, \quad (37)$$

to decrease to a , like in [14]. The solution to the differential equation given the initial condition $\theta(0) = b$ verifies, for $t \geq 0$,

$$\arctan(M\phi(t)) = \arctan(Mb) - \frac{M}{\sigma}t, \quad (38)$$

from which it is inferred that

$$T := \frac{\sigma}{M}(\arctan(Mb) - \arctan(Ma)). \quad (39)$$

Since a, b can be chosen arbitrarily, the sampling interval can be changed, although it can never be larger than $\frac{\sigma}{M} \frac{\pi}{2}$ in view of (39). However, this choice might affect the speed of convergence of the system the evolution of the velocities depend on the sampled control input. We represent the system using the hybrid model below, like in [14],

$$\left. \begin{array}{l} \dot{p}_1 = v_1 \\ \dot{v}_1 = -v_1 + \psi(\hat{z}) \\ \dot{p}_2 = v_2 \\ \dot{v}_2 = -v_2 - \psi(\hat{z}) \\ \dot{\hat{z}} = 0 \\ \dot{\tau} = 1 \end{array} \right\} \tau \in [0, T] \quad (40)$$

$$\left. \begin{array}{l} p_1^+ = p_1 \\ v_1^+ = v_1 \\ p_2^+ = p_2 \\ v_2^+ = v_2 \\ \hat{z}^+ = p_2 - p_1 \\ \tau^+ = 0 \end{array} \right\} \tau = T.$$

where τ is the time elapsed since the last transmission and T is defined in (39). The theorem below is obtained by following similar lines as in the proof of Theorem 1.

Theorem 3: Consider system (40) and suppose Assumption 1 holds. The solutions have a uniform global dwell-time and the maximal solutions are precompact and approach the set $\{(p_1, v_1, p_2, v_2, \hat{z}, \tau) : p_1 = p_2, v_1 = v_2 = \hat{z} = 0 \text{ and } \tau \in [0, T]\}$. \square

VII. SIMULATION RESULTS

Simulations³ have been run for two choices of the function ψ : (i) $\psi(z) = z$, (ii) $\psi(z) = \arctan z$ for $z \in \mathbb{R}$. Both satisfy Assumption 1 with $M = 1$. We have implemented the self-triggering and the time-triggering rules respectively presented in Sections V and VI; recall that the results are equivalent for the event-triggered and the self-triggered policies as explained in Section V. Figure 1 shows the trajectories of p_1, p_2, v_1 and v_2 for each triggering rule in case (ii) when $p_1(0, 0) = 10, p_2(0, 0) = -5, v_1(0, 0) = v_2(0, 0) = 0, \hat{z}(0, 0) = 0, \phi(0, 0) = b, \tau(0, 0) = 0, \sigma = 1/4$ and $(a, b) = (1, 10)$. We see that the rendez-vous is achieved and that the velocities converge to zero as expected. We now consider 100 initial conditions uniformly distributed such that $p_1(0, 0), v_1(0, 0) \in [-10, 10], p_2(0, 0), v_2(0, 0) \in [-15, 15]$, with the same values as above for $\hat{z}(0, 0), \phi(0, 0), \tau(0, 0)$ and we have selected $\sigma = 1/4$. Table I provides the average number of events for a simulation time of 20s as well as the time t^* it takes for the absolute value of the relative position to become smaller than 0.1, which we use as a measure of the time of convergence. We see that the average number of events is the same for (i) and (ii) for time-triggered control, which is in agreement with the analysis as the constant T in (39) is the same in both cases.

We notice that the time-triggered rule generates more events than self-triggered control; except for (i) in which case these are equivalent, see Remark 1. That is not surprising as the time-triggering condition underestimates the next time at which the self-triggering law is violated. That observation justifies the interest of self-triggered control to reduce the number of events. On the other hand, there is obviously a trade-off between this reduction and the performances of the system. We see in Table I that the fewer the number of events, the bigger the time of convergence. The simulation results also confirm the impact of the pair (a, b) on the system, namely the smaller a (equivalently the bigger b), the

³We have used the HyEQ Matlab toolbox ([25]).

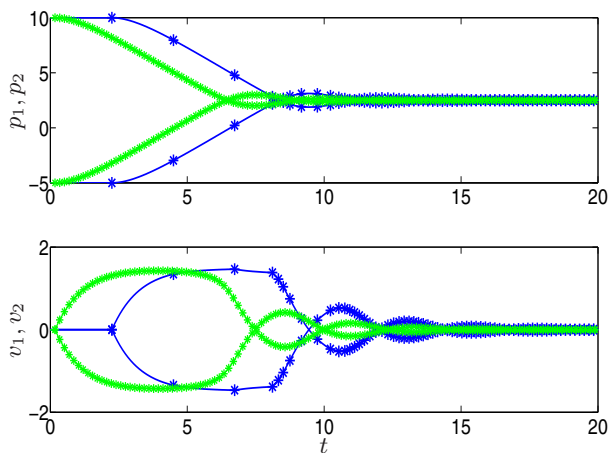


Fig. 1. Trajectories of p_1 , p_2 , v_1 and v_2 (self-triggered control (blue) / time-triggered control (green) / crosses indicate triggering instants).

fewer the number of events. This point is interesting as these parameters can be used to heuristically adapt the number of events to the desired performances.

VIII. CONCLUSIONS

We have proposed a new type of triggering rules to ensure the event-based rendez-vous of two identical systems. Solutions have been presented for event-triggered control, self-triggered control and time-triggered control. In all cases, the existence of a (uniform) minimum amount of time is guaranteed. The extension of these results to the case where an arbitrary finite number of agents are interconnected over a time-invariant connected graph is investigated in [19].

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