

Modeling and stabilization of current-actuated piezoelectric beams

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Abstract—Current controlled piezoelectric beams have been very popular recently due to the substantial reduction of hysteresis between the input and output. Even though there are many different PDE models for the voltage and charge controlled piezoelectric beams, there is no thoroughly derived PDE model in the literature for current controlled piezoelectric beams. A variational approach is utilized to derive the boundary value problem which models a single piezoelectric beam actuated by a current source at the electrodes. Magnetic effects are included. In addition to the Euler-Bernoulli displacement assumptions for the mechanical part, we assume that electrical and magnetic vector potential terms are quadratic-through thickness. Since current controller at the electrodes can control only the stretching equations, only the stretching equations are considered. Unlike the voltage or the charge actuation cases, a bounded control operator in the natural energy space is obtained.

I. INTRODUCTION

Piezoelectric beams are elastic beams covered by electrodes at its top and bottom surfaces, insulated at the edges (to prevent fringing effects), and connected to an external electric circuit (see Fig. 1). They are widely used in civil, aeronautic and space structures due to their small size and high power density. They convert mechanical energy to electrico-magnetic energy, and vice versa. Therefore they can be used as both sensors and actuators.

There are mainly three ways to electrically actuate piezoelectric materials: voltage, current or charge. Piezoelectric materials have been traditionally activated by a voltage source [6], [21], [22], [25], [26]. The control operator is unbounded in the energy space if the piezoelectric structure is controlled by a voltage or a charge source, for instance see [2], [11], [16], [22]. Another complication is there is hysteresis between voltage and strain. This can be avoided by only applying low voltage, but this approach prevents these actuators from being used at their full potential. Controller design needs to consider hysteresis in order to obtain maximum accuracy and effectiveness. Some approaches are passivity [9] and inverse compensation [23].

Using current or charge actuation substantially reduces the hysteresis; see for instance, [3], [8], [10], [13], [14], [19]. A model of current actuated piezo-electric beams is considered in this talk. No assumption is made about the magnetic field and a fully dynamic version of Maxwell's

equation is considered. After the electromagnetic field is described in terms of scalar electric potential and magnetic vector potential, application of Hamilton's principle yields a PDE model where the control operator is bounded in the energy space. Finally, it is shown that the current-actuated beams can be strongly stabilized by an appropriately chosen feedback controller.

II. MODEL DESCRIPTION

Consider a piezoelectric beam of dimensions $\Omega = [0, L] \times [-r, r] \times [-\frac{h}{2}, \frac{h}{2}]$ where $L \gg h$. Let x, y be the longitudinal directions, z the transverse direction and $\partial\Omega$ the boundary. Since $L \gg h$ we assume that there is no force acting in the y direction and use beam theory.

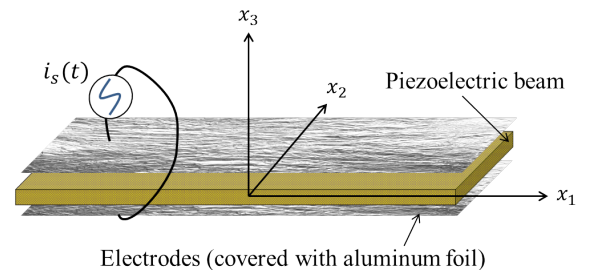


Fig. 1. When the current density $i_s(t)$ is prescribed to the electrodes, an electric field is created between the electrodes, and therefore the beam/plate either shrinks or extends. Unlike the voltage or charge actuation, the input-output hysteresis reduces substantially.

We use Hamilton's principle to derive the constitutive equations of the beam. Let $\mathbf{K}, \mathbf{P}, \mathbf{E}$ and \mathbf{B} be kinetic, potential, electrical, and magnetic energies of the beam, respectively, and let \mathbf{W} be the work done by the external forces. To model current (or charge) actuated piezoelectric beams we use the Lagrangian [12]

$$\mathbf{L} = \int_0^T [\mathbf{K} - (\mathbf{P} - \mathbf{E} + \mathbf{B}) + \mathbf{W}] dt. \quad (1)$$

The term $\mathbf{P} - \mathbf{E} + \mathbf{B}$ is the electrical enthalpy. This Lagrangian is different from the Lagrangian used for modeling voltage actuated piezoelectric beams [16]. Since $L \gg h$, beam theory is appropriate and Euler-Bernoulli small displacement assumptions are used. To model the electromagnetic properties, there are mainly three approaches [25]: electrostatic, quasi-static, and dynamic. Since the piezoelectric materials are not perfectly insulated, the electric field E causes currents to flow when conductivity occurs. Therefore the time-dependent equation of the continuity of electric charge must be employed. We follow a dynamic approach

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and include all of the magnetic effects. Gauss's law of magnetism

$$\nabla \cdot B = 0$$

implies that there exists a magnetic potential vector A such that

$$B = \nabla \times A.$$

Substituting B into Faraday's law

$$\dot{B} = -\nabla \times E$$

yields the existence of a scalar electric potential ϕ such that

$$E + \dot{A} = -\nabla\phi. \quad (2)$$

With current actuation, the work term is

$$\mathbf{W} = \int_{\Omega} i_s \cdot A \, dX$$

where i_s is the current density at the electrodes. If the magnetic effects are neglected, a variational approach cannot be used since $A \equiv 0$ and so $\mathbf{W} \equiv 0$. This is very different from the charge and voltage actuation cases since for charge and voltage actuation \mathbf{W} is not a function of A .

In this paper, the electric and magnetic potential distributions is assumed to vary quadratically through the thickness h [24]. Consistent with beam theory, it is assumed that the magnetic vector potential A has no component in y direction. Therefore electric potential ϕ and magnetic vector potential A have the form

$$\phi(x, z) = \phi^0(x) + z\phi^1(x) + \frac{z^2}{2}\phi^2(x) \quad (3)$$

$$A(x, z) = \begin{pmatrix} A_1^0(x) + zA_1^1(x) + \frac{z^2}{2}A_1^2(x) \\ 0 \\ A_3^0(x) + zA_3^1(x) + \frac{z^2}{2}A_3^2(x) \end{pmatrix}. \quad (4)$$

The application of Hamilton's principle to \mathbf{L} with respect to all possible admissible displacements yields two sets of equations: one for describing the stretching motion of the centerline, and the other one for describing the bending of the beam [17]. The applied current at the electrodes affects only the stretching motion, and so only the stretching equations are considered for the rest of the paper. We assume that that the beam is free at both ends. Denoting v to be the stretching of the centerline of the beam, under various assumptions such as polarization in transverse direction, the equations of motion are

$$\left\{ \begin{array}{l} \rho h \ddot{v} - \alpha h v_{xx} - \gamma h \left(\phi^1 + \dot{A}_3^0 + \frac{h^2}{24} \dot{A}_3^2 \right)_x = 0 \\ -\frac{\varepsilon_1 h^3}{12} \left((\phi^1)_{xx} + (\dot{A}_1^1)_x \right) \\ \quad + \varepsilon_3 h \left(\dot{A}_3^0 + \frac{h^2}{24} \dot{A}_3^2 + \phi^1 \right) - \gamma h v_x = 0 \\ \frac{\varepsilon_1 h^3}{12} \ddot{A}_1^1 + \frac{\varepsilon_1 h^3}{12} (\dot{\phi}^1)_x \\ -\mu h \left((A_3^0)_x + \frac{h^2}{24} (A_3^2)_x - A_1^1 \right) = i_s(t) \\ \varepsilon_3 h \left(\ddot{A}_3^0 + \frac{h^2}{24} \ddot{A}_3^2 + \dot{\phi}^1 \right) - \gamma h \dot{v}_x \\ \quad - \mu h \left((A_3^0)_{xx} + \frac{h^2}{24} (A_3^2)_{xx} - (A_1^1)_x \right) = 0 \\ \frac{\varepsilon_3 h^3}{24} \left(\ddot{A}_3^0 + \frac{h^2}{24} \ddot{A}_3^2 + \dot{\phi}^1 \right) - \frac{\gamma h^3}{24} \dot{v}_x \\ \quad - \frac{\mu h^3}{24} \left((A_3^0)_{xx} + \frac{h^2}{24} (A_3^2)_{xx} - (A_1^1)_x \right) \\ \quad + \left(\frac{\varepsilon_3 h^5}{720} \ddot{A}_3^2 - \frac{\mu h^5}{720} (A_3^2)_{xx} \right) = 0 \end{array} \right. \quad (5)$$

with the natural boundary conditions at $x = 0, L$

$$\left\{ \begin{array}{l} \alpha h v_x + \gamma h \left(\phi^1 + \dot{A}_3^0 + \frac{h^2}{24} \dot{A}_3^2 \right) = 0 \quad (\text{Lateral force}) \\ \frac{\varepsilon_1 h^3}{12} \left(\dot{A}_1^1 + (\phi^1)_x \right) = 0 \quad (\text{First charge moment}) \\ \mu h \left(A_1^1 - (A_3^0)_x - \frac{h^2}{24} (A_3^2)_x \right) = 0 \quad (\text{Current}) \\ \mu h^3 \left(\frac{1}{12} \dot{A}_1^1 - \frac{1}{12} (A_3^0)_x \right. \\ \quad \left. - \frac{h^2}{160} (A_3^2)_x \right) = 0 \quad (\text{Second current moment}) \end{array} \right. \quad (6)$$

where $\rho, \alpha, \gamma, \varepsilon, \mu$ denote the mass density per unit volume, elastic stiffness, piezoelectric coupling coefficient, permittivity, and permeability of the piezoelectric beam, respectively, and $i_s(t)$ denotes the applied current density at the electrodes.

Notice that the magnetic potential vector A and the electric potential ϕ are not uniquely defined by (2). This can be used to simplify the equations. The following theorem states that the Lagrangian \mathbf{L} (1) is invariant under certain transformations.

Theorem 2.1: [17] For an arbitrary scalar function $\chi = \chi(x, z, t)$, the Lagrangian \mathbf{L} is invariant under the mapping

$$\begin{aligned} A &\mapsto \tilde{A} := A + \nabla\chi \\ \phi &\mapsto \tilde{\phi} := \phi - \dot{\chi}. \end{aligned} \quad (7)$$

The invariance of \mathbf{W} under the mapping (7) is not obvious, but follows from the electric continuity equations [7, Section 3.9].

Let

$$\xi = \frac{\varepsilon_1 h^2}{12\varepsilon_3}$$

and choose the particular gauge known as *Coulomb-type*

$$-\xi(A_1^1)_x + \left(A_3^0 + \frac{h^2}{24} A_3^2 \right) = 0 \quad (8)$$

together with the boundary conditions

$$(A_1^1)(0) = (A_1^1)(L) = 0. \quad (9)$$

This choice of gauge is convenient in electrodynamics since the magnetic terms in the charge equation are eliminated.

Using (8)-(9), the equations of motion (5)-(6) reduce to

$$\rho \ddot{v} - \alpha v_{xx} - \gamma \left(\phi^1 + \dot{A}_3^0 + \frac{h^2}{24} \dot{A}_3^2 \right)_x = 0 \quad (10a)$$

$$-\frac{\varepsilon_1 h^2}{12} (\phi^1)_{xx} + \varepsilon_3 \phi^1 - \gamma v_x = 0 \quad (10b)$$

$$\frac{\varepsilon_1 h^2}{12} \ddot{A}_1^1 + \mu A_1^1 - \mu \left(A_3^0 + \frac{h^2}{24} A_3^2 \right)_x \\ + \frac{\varepsilon_1 h^2}{12} (\dot{\phi}^1)_x = \frac{i_s(t)}{h} \quad (10c)$$

$$\varepsilon_3 \ddot{A}_3^0 - \mu (A_3^0)_{xx} + \mu (A_1^1)_x + \varepsilon_3 \dot{\phi}^1 - \gamma \dot{v}_x = 0 \quad (10d)$$

$$\varepsilon_3 \ddot{A}_3^2 - \mu (A_3^2)_{xx} = 0, \quad (10e)$$

with the boundary conditions

$$\left. \alpha v_x + \gamma \left(\phi^1 + \dot{A}_3^0 + \frac{h^2}{24} \dot{A}_3^2 \right) \right|_{x=0,L} = 0 \quad (10f)$$

$$\left. (\phi^1)_x = A_1^1 = (A_3^0)_x = (A_3^2)_x \right|_{x=0,L} = 0. \quad (10g)$$

Equation (10b) is elliptic and so electric potential can be eliminated. The elliptic equation (10b) and related boundary conditions are

$$-\xi\phi_{xx}^1 + \phi^1 = \frac{\gamma}{\varepsilon_3}v_x, \quad (\phi^1)_x(0) = (\phi^1)_x(L) = 0. \quad (11)$$

By the Lax-Milgram theorem, for any $v \in H^1(0, L)$ the equation (11) has a unique solution for ϕ . Defining the compact positive definite operator on $\mathcal{L}_2(0, L)$

$$P_\xi := (-\xi D_x^2 + I)^{-1}, \quad (12)$$

$$\phi^1 = \frac{\gamma}{\varepsilon_3} P_\xi v_x. \quad (13)$$

Letting $\theta = A_1^1$ and $\eta = A_3^0 + \frac{h^2}{24}A_3^2$, and eliminating ϕ^1 in (10b), the stretching equations (10) are rewritten as

$$\begin{cases} \rho\ddot{v} - \alpha v_{xx} - \frac{\gamma^2}{\varepsilon_3}(P_\xi v_x)_x - \gamma\dot{\eta}_x = 0 & (14a) \end{cases}$$

$$\begin{cases} \frac{\varepsilon_1 h^2}{12}\ddot{\theta} + \mu\theta - \mu\eta_x + \frac{\varepsilon_1 h^2}{12}\frac{\gamma}{\varepsilon_3}(P_\xi \dot{v}_x)_x = \frac{i_s^1(t)}{h} & (14b) \end{cases}$$

$$\begin{cases} \varepsilon_3\dot{\eta} - \mu\eta_{xx} + \mu\theta_x - \gamma(\dot{v}_x - (P_\xi \dot{v}_x)) = 0 & (14c) \end{cases}$$

with the boundary conditions

$$\left| \alpha v_x + \frac{\gamma^2}{\varepsilon_3}P_\xi v_x + \gamma\dot{\eta} = \theta = \eta_x \right|_{x=0,L} = 0 \quad (15a)$$

III. WELL-POSEDNESS

Let $\mathbf{y} = [y_1, y_2, \dots, y_6]^T$, $\mathbf{z} = [z_1, z_2, \dots, z_6]^T$, and define the functional

$$\begin{aligned} E(\mathbf{y}) = & \frac{1}{2} \int_0^L \left\{ \rho y_4^2 + \frac{\varepsilon_1 h^2}{12} y_5^2 + \varepsilon_3 y_6^2 \right. \\ & \left. + \alpha y_1^2 + \frac{\gamma^2}{\varepsilon_3} (P_\xi y_1)_x y_1 + \mu (y_2 - y_3)_x^2 \right\} dx, \quad (16) \end{aligned}$$

and the Hilbert space

$$\begin{aligned} H = & \{ \mathbf{y} \in \mathcal{L}_2(0, L) \times H_0^1(0, L) \times H^1(0, L) \times \mathcal{L}_2(0, L) \\ & \times H_0^1(0, L) \times \mathcal{L}_2(0, L) : -\xi(y_2)_x + y_3 = 0, \\ & -\xi(y_5)_x + y_6 = 0 \} \end{aligned} \quad (17)$$

with the following bilinear form:

$$\begin{aligned} \langle \mathbf{y}, \mathbf{z} \rangle_H = & \int_0^L \left\{ \rho y_4 \bar{z}_4 + \frac{\varepsilon_1 h^2}{12} y_5 \bar{z}_5 + \varepsilon_3 y_6 \bar{z}_6 \right. \\ & + \alpha y_1 \bar{z}_1 + \frac{\gamma^2}{\varepsilon_3} (P_\xi y_1)_x \bar{z}_1 + \mu y_2 \bar{z}_2 \\ & \left. + \mu (y_3)_x (\bar{z}_3)_x - \mu y_2 (\bar{z}_3)_x - \mu (y_3)_x \bar{z}_2 \right\} dx. \quad (18) \end{aligned}$$

We claim that (18) defines an inner product on H. The main problem is to show that this bilinear form (18) is coercive on H. This follows since P_ξ is a positive operator on $\mathcal{L}_2(0, L)$, and

$$\begin{aligned} - \int_0^L y_2 (\bar{y}_3)_x dx &= - \int_0^L \frac{\varepsilon_1 h^2}{12 \varepsilon_3} y_2 (\bar{y}_2)_{xx} dx \\ &= \xi \int_0^L |(y_2)_x|^2 dx \\ &\geq C \xi \int_0^L |y_2|^2 dx \end{aligned}$$

where we used the gauge condition in (17), and Poincaré's inequality with the Poincaré constant C . Therefore, (18) is a valid inner product on H, and the norm on H can be determined by (16). It can also easily be shown that H with this norm is complete.

Let $D_x^j = \frac{\partial^j}{\partial x^j}$ for $j = 1, 2$, and choose $\mathbf{y} = [v_x, \theta, \eta, \dot{v}, \dot{\theta}, \dot{\eta}]^T$. Now $E(\mathbf{y})$ is the energy corresponding to the system (14). Defining

$$\begin{aligned} A_1 &= \begin{pmatrix} \frac{\alpha}{\rho} D_x + \frac{\gamma^2}{\varepsilon_3 \rho} D_x P_\xi & 0 & 0 \\ 0 & -\frac{12\mu}{\varepsilon_1 h^2} I & \frac{12\mu}{\varepsilon_1 h^2} D_x \\ 0 & -\frac{\mu}{\varepsilon_3} D_x & \frac{\mu}{\varepsilon_3} D_x^2 \end{pmatrix}, \\ A_2 &= \begin{pmatrix} 0 & 0 & \frac{\gamma}{\rho} D_x \\ -\frac{\gamma}{\varepsilon_3} D_x P_\xi D_x & 0 & 0 \\ \frac{12\gamma}{\varepsilon_3 h^2} (I - P_\xi) D_x & 0 & 0 \end{pmatrix}, \\ B &= \left(0, 0, 0, 0, \frac{12}{\varepsilon_1 h^3}, 0 \right)^T, \end{aligned} \quad (19)$$

the system of partial differential equations (14) can be written in the abstract form

$$\begin{aligned} \dot{\mathbf{y}} &= A\mathbf{y} + B i_s^1(t) \\ &= \begin{pmatrix} 0 & I_{3 \times 3} \\ A_1 & A_2 \end{pmatrix} \mathbf{y} + B i_s^1(t), \quad \mathbf{y}(x, 0) = \mathbf{y}^0. \end{aligned} \quad (20)$$

The control operator B is clearly a bounded operator on H; that is $B \in \mathcal{L}(\mathbb{C}, H)$. The operator $\mathcal{A} : \text{Dom}(\mathcal{A}) \subset H \rightarrow H$ with

$$\begin{aligned} \text{Dom}(\mathcal{A}) = & ((H^1(0, L) \times H_0^1(0, L) \times H^2(0, L) \\ & \times H^1(0, L) \times H_0^1(0, L) \times (H^1(0, L))) \cap \\ & \left\{ Y \in H : \left(\alpha I + \frac{\gamma^2}{\varepsilon_3} P_\xi \right) y_1 + \gamma y_6 = (y_3)_x \Big|_{x=0,L} = 0 \right\} \end{aligned} \quad (21)$$

is densely defined in H. This is quite different from the situation where voltage or charge actuation is used. For instance, in the case of voltage actuation, the control operator is of the form

$$B = c \begin{pmatrix} 0_{3 \times 1} \\ \delta(x - L) - \delta(x) \end{pmatrix}$$

where c is a parameter that depends on the material constants and the thickness of the beam [16]. The control operator B is not bounded on the state space.

Proposition 3.1: [17] The operator $\mathcal{A} : \text{Dom}(\mathcal{A}) \subset H \rightarrow H$ defined by (20) is the generator of a unitary semigroup $\{e^{A t}\}_{t \geq 0}$ on H. The energy $E(t)$ is conserved along solution trajectories of (20) if $i_s^1(t) \equiv 0$.

Theorem 3.1: [17] For any $T > 0$, current $i_s^1(t) \in \mathcal{L}_2(0, L)(0, T)$ and $\mathbf{y}^0 \in H$, there is a unique solution W of (20) with $\mathbf{y} \in C([0, T]; H)$. Moreover, there exists a constant $c > 0$ such that

$$\|\mathbf{y}\|_H \leq c \{ \|\mathbf{y}^0\|_H + \|i_s^1(t)\|_{\mathcal{L}_2(0, T)} \}. \quad (23)$$

IV. STABILIZATION

The uncontrolled system has an infinite number of eigenvalues on the imaginary axis. Since the control operator is bounded and rank 1, it is compact and it is not possible to exponentially stabilize the piezo-electric beam; see for instance [4]. It can however be strongly stabilized.

The dual of the control operator B is

$$B^*y = (0 \ 0 \ 0 \ 0 \ 1 \ 0)^T y = y_5.$$

Based on duality, choose the partial state feedback control $i_s^1(t) = -k_1 \dot{\theta}$ for some arbitrary $k_1 > 0$. Then (20) becomes

$$\dot{y} = \tilde{A}y = \begin{pmatrix} 0 & I_{3 \times 3} \\ A_1 & A_2 \end{pmatrix} y - \frac{12k_1}{\varepsilon_1 h^3} BB^*y, \\ y(x, 0) = y^0. \quad (24)$$

The operator $\tilde{A} : \text{Dom}(\tilde{A}) \subset H \rightarrow H$ defined by (24) with domain $\text{Dom}(\tilde{A}) = \text{Dom}(A)$ is densely defined in H .

Theorem 4.1: [17] $\tilde{A} : \text{Dom}(\tilde{A}) \rightarrow H$ is the infinitesimal generator of a C_0 -semigroup of contractions. Therefore for every $T \geq 0$, and $y^0 \in \text{Dom}(\tilde{A})$ solves (24) and we have $y \in C([0, T]; \text{Dom}(\tilde{A})) \cap C^1([0, T]; H)$. Moreover, the spectrum $\sigma(\tilde{A})$ of \tilde{A} has all isolated eigenvalues.

Theorem 4.2: [17] For $y^0 \in H$, the semigroup $\{e^{\tilde{A}t}\}_{t \geq 0}$ corresponding to (24) is strongly stable in H , i.e. $\|e^{\tilde{A}t}y^0\|_H \rightarrow 0, \quad t \rightarrow \infty$.

Proof: It can be shown that along the trajectories of (24), $\frac{dE(t)}{dt} = -k_1 \|z_5\|_{\mathcal{L}_2(0, L)}^2$. The remainder of the proof uses Arendt-Batty's stability theorem [1]. \square

V. CONCLUSIONS

Voltage or charge-control of a piezo-electric beam lead to a control operator that is unbounded on the energy space. Also, previous work of the authors ([16], [18], [20]) showed that a voltage-controlled beam cannot be exponentially stabilized in the energy space. Moreover, even strong stability by a state feedback is not achieved for many material parameters.

On the other hand, not only does current control lead to a reduction in the amount of hysteresis in the response, but if magnetic effects are included in the model, the control operator is bounded. It is still not possible to exponentially stabilize an undamped system, but strong stability is easily achieved.

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