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WELL-POSEDNESS OF INFINITE-DIMENSIONAL NON-AUTONOMOUS PORT-HAMILTONIAN SYSTEMS*

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Abstract— **We study well-posedness of linear non-autonomous evolutionary Cauchy-problem of the form**

$$\left\{\begin{array}{ll} \dot{u}(t) + A(t)\mathcal{H}(t)u(t) & = f(t) \quad t\textbf{-a.e. on } [0,\tau] \\ u(0) & = 0, \end{array}\right.$$

where the family $(A(t), D(A(t)))_{t\in[0,\tau]}$ arises from a time depending sesquilinear form and $(\mathcal{H}(t))_{t\in[0,\tau]}$ are bounded operators on a Hilbert space H. This class of evolution equations is motivated by non-autonomous port-Hamiltonian systems with dissipation.

I. PROBLEM FORMULATION

We study autonomous port-Hamiltonian systems given by the partial differential equation

$$\frac{\partial x}{\partial t}(\zeta,t) = (P_1\frac{\partial}{\partial \zeta} + P_0)(\mathcal{H}x)(\zeta,t) \tag{1}$$

with the boundary conditions

$$0 = W_B \begin{pmatrix} (\mathcal{H}x)(0) \\ (\mathcal{H}x)(1) \end{pmatrix} \tag{2}$$

where $P_1 \in \mathbb{C}^{n\times n}$ is invertible and self-adjoint, $P_0 \in \mathbb{C}^{n\times n}$ is skew-adjoint, and $\mathcal{H} \in L^\infty(0,1;\mathbb{C}^{n\times n})$ such that $\mathcal{H}(\zeta)^* = \mathcal{H}(\zeta)$, $mI \le \mathcal{H}(\zeta) \le MI$ for a.e. $\zeta \in [0,1]$ and $W_B \in \mathbb{C}^{n\times 2n}$.

The port-Hamiltonian system (1)-(2) can be seen as a bounded multiplicative perturbation of an m-dissipative operator A :

$$\begin{aligned} \dot{x}(t) &= A\mathcal{H}x(t) && (3) \\ x(0) &= x_0 && (4) \end{aligned}$$

where A is given by $Ax := (P_1\frac{\partial}{\partial\zeta} + P_0)x$ with domain

$$D(A) = \left\{ x \in H^1(0,1;\mathbb{C}^n), W_B \begin{pmatrix} x(0) \\ x(1) \end{pmatrix} = 0 \right\}.$$

It is well-known [11], [14] that the operator $A\mathcal{H}$ generates a C_0-semigroup on $H := L^2(0,1;\mathbb{C}^n)$ if and only if W_B has full rank and $W_B\Sigma W_B^* \ge 0$ with $\Sigma := \begin{pmatrix} P_1^{-1} & 0 \\ 0 & -P_1^{-1} \end{pmatrix}$.

The proof of [11, Theorem 7.2.4] is based on the fact, that A generates a contraction C_0-semigroup on $L^2(0,1;\mathbb{C}^n)$ if and only if $A\mathcal{H}$ generates a contraction C_0-semigroup on $L^2(0,1;\mathbb{C}^n)$ with the inner product $(f \mid g)_{\mathcal{H}} := (f \mid \mathcal{H}g)_{L^2}$, see [11, Lemma 7.2.3].

Now we add to (1) a dissipation term, and we obtain the following partial differential equation

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) \;=\; & \frac{\partial}{\partial\zeta}\left(GS\frac{\partial}{\partial\zeta}G^*\mathcal{H}x + P_1\mathcal{H}x\right)(\zeta,t) && (5) \\ & +P_0(\mathcal{H}x)(\zeta,t) \end{aligned}$$

where $S \in L^\infty(0,1;\mathbb{C}^{k\times k})$ such that S is coercive operator on $L^2(0,1;\mathbb{C}^k)$, whereas $G \in \mathbb{C}^{n\times k}$ and $P_1, P_0 \in L^\infty(0,1;\mathbb{C}^{n\times n})$.

We may write (5) as an abstract Cauchy problem

$$\begin{aligned} \dot{x}(t) + A\mathcal{H}x(t) &= 0 && (6) \\ x(0) &= x_0 && (7) \end{aligned}$$

where the operator A is given by

$$A := -\frac{\partial}{\partial\zeta}\left(GS\frac{\partial}{\partial\zeta}G^* + P_1\right) - P_0 \tag{8}$$

on a domain $D(A)$ which include appropriate boundary conditions. We aim to characterize boundary conditions such that A generates a holomorphic C_0-semigroup on H. Further, the question whether $A\mathcal{H}$ generates a holomorphic C_0-semigroup if and only if A generates a holomorphic C_0-semigroup will be investigated.

If S and $\mathcal{H}$ also depend on the time variable t, the problem (6)-(7) becomes a non-autonomous Cauchy problem

$$\begin{aligned} \dot{x}(t) + A(t)\mathcal{H}(t)x(t) &= 0 && (9) \\ x(0) &= x_0. && (10) \end{aligned}$$

Our aim is to prove well-posedness of (9)-(10).

II. EVOLUTION EQUATION GOVERNED BY NON-AUTONOMOUS FORMS

In this section we study non-autonomous evolutionary linear Cauchy-problems of the form

$$\begin{aligned} \dot{x}(t) + A(t)\mathcal{H}(t)x(t) &= f(t) \text{ a.e on } (0,\tau), && (11) \\ x(0) &= x_0 && (12) \end{aligned}$$

where the operators $A(t)$, $t \in [0,\tau]$, arise from sesquilinear forms on Hilbert spaces. Forms methods give a very efficient tool to solve evolution equations on Hilbert space. They were developed by T. Kato [12] and in different but equivalent language by J. L. Lions [15].

Let V, H be two separable Hilbert spaces over $\mathbb{C}$. Their scalar products and the corresponding norms will be denoted by $(\cdot \mid \cdot)$, $(\cdot \mid \cdot)_V$, $\|\cdot\|$ and $\|\cdot\|_V$, respectively. We assume that

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$V \underset{d}{\hookrightarrow} H$ i.e., V is a dense subspace of H such that for some constant $c_H > 0$,

$$\|u\| \le c_H \|u\|_V \quad (u \in V). \tag{13}$$

Let V' denote the antidual of V. The duality between V' and V is denoted by $\langle \cdot, \cdot \rangle$. As usual, we identify H with H'. It follows that $V \hookrightarrow H \cong H' \hookrightarrow V'$ and so V is identified with a subspace of V'. These embeddings are continuous and

$$\|f\|_{V'} \le c_H \|f\| \quad (f \in V') \tag{14}$$

with the same constant c_H as in (13) (see e.g., [5]).

Let $\tau > 0$ and

$$\mathfrak{a} : [0,\tau] \times V \times V \to \mathbb{C}$$

be a function such that $\mathfrak{a}(t,\cdot,\cdot) : V \times V \to \mathbb{C}$ is sesquilinear for all $t \in [0,\tau]$ and $\mathfrak{a}(\cdot,u,v) : [0,\tau] \to \mathbb{C}$ is measurable for all $u, v \in V$. We assume that $\mathfrak{a}$ is V-bounded and H-elliptic, i.e.

$$|\mathfrak{a}(t,u,v)| \le M \|u\|_V \|v\|_V \quad (t \in [0,\tau], u, v \in V) \tag{15}$$

and

$$\text{Re } \mathfrak{a}(t,u,u) + \omega \|u\|^2 \ge \alpha \|u\|_V^2 \quad (t \in [0,\tau], u \in V) \tag{16}$$

where $M \ge 0$ and $\alpha > 0$ and $\omega \in \mathbb{R}$.
We denote by $\mathcal{A}(t)$ the operator associated with $\mathfrak{a}$ given by $\langle \mathcal{A}(t)u, v \rangle = \mathfrak{a}(t,u,v)$ and by $A(t)$ we denote the part of $\mathcal{A}(t)$ in H, i.e.,

$$\begin{aligned} D(A(t)) &:= \{u \in V : \mathcal{A}(t)u \in H\} \\ A(t)u &= \mathcal{A}(t)u. \end{aligned}$$

It is a known fact that $-A(t)$ generates a holomorphic C_0-semigroup on H, see [12] and [16]. Let $\mathcal{H} : [0,\tau] \to \mathcal{L}(H)$ strongly measurable. The notion of L^2-maximal regularity of (11)-(12) is defined as follows.

Definition 1. We say that (11)-(12) has L^2-maximal regularity if for each $f \in L^2(0,\tau;H)$ and $x_0 \in V$ there exists a unique function $x \in H^1(0,\tau;H)$ such that $x(t) \in D(A(t)\mathcal{H}(t))$ for almost every $t \in (0,\tau)$ and x satisfies (11)-(12).

In the case where $\mathcal{H} \equiv I$ we recall the following results. Lions proved L^2-maximal regularity of (11)-(12) for $x_0 = 0$ (respectively for $x_0 \in D(A(0))$) provided $\mathfrak{a}(\cdot,u,v) \in C^1[0,\tau]$ (respectively $\mathfrak{a}(\cdot,u,v) \in C^2[0,\tau]$) for all $u, v \in V$, [15, p. 68 and p. 94]. Moreover, a combination of [15, Theorem 1.1, p. 129] and [15, Theorem 5.1, p. 138] shows that if $\mathfrak{a}(\cdot,u,v) \in C^1[0,\tau]$ for all $u, v \in V$, then (11)-(12) has L^2-maximal regularity. Bardos [4] proved also L^2-maximal regularity under the assumptions that the domains of both $A(t)^{1/2}$ and $A(t)^{*1/2}$ coincide with V and that $\mathcal{A}(\cdot)^{1/2}$ is continuously differentiable with values in $\mathcal{L}(V,V')$. We mention also a result of Ouhabaz and Spina [17] and Ouhabaz and Haak [8]. They proved L^2-maximal regularity for (possibly non-symmetric) forms such that $\mathfrak{a}(\cdot,u,v) \in C^\alpha[0,\tau]$ for all $u, v \in V$ and some $\alpha > \frac{1}{2}$. The result in [17] concerns the case $x_0 = 0$ and the one in [8] concerns the case x_0 in the real-interpolation space $(H, D(A(0)))_{1/2,2}$. Left multiplicative perturbation by $\mathcal{H}$ was recently investigated in [3]. Arendt et al. [3] prove L^2-maximal regularity for

$$\begin{aligned} \dot{x}(t) + \mathcal{H}(t)A(t)x(t) &= f(t) \text{ a.e on } (0,\tau), \\ x(0) &= x_0 \in V \end{aligned}$$

assuming that the sesquilinear form $\mathfrak{a}$ can be written as $\mathfrak{a}(t,u,v) = \mathfrak{a}_1(t,u,v) + \mathfrak{a}_2(t,u,v)$ where $\mathfrak{a}_1$ is symmetric, bounded (i.e $\mathfrak{a}_1(t,u,v) \leqslant M_1 \|u\| \|v\|$, $M_1 \ge 0$) and coercive and piecewise Lipschitz-continuous on $[0,\tau]$, whereas $\mathfrak{a}_2 \colon [0,\tau] \times V \times H \to \mathbb{C}$ satisfies $|\mathfrak{a}_2(t,u,v)| \le M_2 \|u\|_V \|v\|$ and $\mathfrak{a}_2(\cdot,u,v)$ is measurable for all $u \in V$, $v \in H$. Furthermore, they assume that $\mathcal{H} \colon [0,\tau] \to \mathcal{L}(H)$ is strongly measurable such that $\|\mathcal{H}(t)\|_{\mathcal{L}(H)} \le \beta_1$ for all $t \in (0,\tau)$ and $0 < \beta_0 \le (\mathcal{H}(t)g \mid g)_H$ for $g \in H$, $\|g\|_H = 1$, $t \in [0,\tau]$.

In order the prove L^2-maximal regularity for (11)-(12), we assume in addition that $\mathfrak{a}$ is *symmetric*; i.e.,

$$\mathfrak{a}(t,u,v) = \overline{\mathfrak{a}(t,v,u)} \quad (t \in [0,\tau], u, v \in V),$$

and that $\mathfrak{a}$ is *Lipschitz continuous*; i.e., there exists a positive constant L_1 such that

$$|\mathfrak{a}(t,u,v) - \mathfrak{a}(s,u,v)| \le L_1 |t-s| \|u\|_V \|v\|_V \tag{17}$$

for $t, s \in [0,\tau]$ and $u, v \in V$. We assume also on $\mathcal{H} : [0,\tau] \to \mathcal{L}(H)$ the following assumptions

$$\begin{gathered} \|\mathcal{H}(t)v - \mathcal{H}(s)v\| \le L_2 |t-s| \|v\| \\ \mathcal{H}(t)^* = \mathcal{H}(t) \\ (\mathcal{H}(t)v \mid v) \ge \beta \|v\|^2. \end{gathered}$$

for some constant $L_2, \beta > 0$ and for all $t \in [0,\tau], v \in H$. Then following an idea given in [20] and using the result given in [3] we deduce L^2-maximal regularity for (11)-(12) from the one of

$$\begin{aligned} \dot{v}(t) + (\mathcal{H}(t)A(t) - \dot{\mathcal{H}}(t)\mathcal{H}^{-1}(t))v(t) &= \mathcal{H}(t)f(t) && (18) \\ v(0) &= 0 && (19) \end{aligned}$$

In fact, observe that v satisfies (18)-(19) if and only if $x := \mathcal{H}^{-1}v$ satisfies (11)-(12). One of our main results reads as follows.

Theorem 2. *Let $x_0 \in V$, $f \in L^2(0,\tau;H)$. Then there exists a unique $x \in H^1(0,\tau;H) \cap L^2(0,\tau;V)$ satisfying*

$$\begin{gathered} \dot{x}(t) + A(t)\mathcal{H}(t)x(t) = f(t) \quad \textit{a.e.} \\ \mathcal{H}(0)x(0) = x_0. \end{gathered}$$

Moreover, $\mathcal{H}(\cdot)u(\cdot) \in C([0,\tau];V)$ and

$$\begin{aligned} \|x\|^2_{L^2(0,\tau;V)} + \|\dot{x}\|^2_{L^2(0,\tau;H)} + \|\mathcal{A}(\cdot)\mathcal{H}(\cdot)x(\cdot)\|^2_{L^2(0,\tau;H)} \\ \le c\Big[\|x_0\|_V + \|f\|_{L^2(0,\tau;H)}\Big], \end{aligned}$$

where the constant $c = c(L_1, c_H, \tau, L_2, M, \alpha, \beta)$.

III. APPLICATION

This section is devoted to an application of our results on L^2-maximal regularity to non-autonomous port-Hamiltonian with dissipation described by (5) with time depending coefficients.

Consider on $H := L^2(0,1;\mathbb{C}^n)$ the operator

$$A := -\frac{\partial}{\partial\zeta}\left(GS\frac{\partial}{\partial\zeta}G^* + P_1\right) \tag{20}$$

where $S \in L^\infty(0,1;\mathbb{C}^{k\times k})$ such that S is coercive operator on $L^2(0,1;\mathbb{C}^k)$, whereas $G \in \mathbb{C}^{n\times k}$ and $P_1 \in L^\infty(0,1;\mathbb{C}^{n\times n})$. Consider the sesquilinear form $\mathfrak{a} : V\times V \to \mathbb{C}$ defined by

$$\begin{aligned}\mathfrak{a}(u,v) := &\int_0^1 \Big(\frac{\partial}{\partial\zeta}v\Big)^*\Big(G^*GS\frac{\partial}{\partial\zeta}G^*u(\zeta) - G^*P_1u\Big)(\zeta)d\zeta\\ &+\int_0^1 v^*(\zeta)\frac{\partial}{\partial\zeta}\Big((I-GG^*)P_1u\Big)(\zeta)d\zeta\end{aligned}$$

and

$$\begin{aligned}V := \Big\{&v\in H : G^*v \in H^1(0,1;\mathbb{C}^k), (I-GG^*)P_1v \in\\ &H^1(0,1;\mathbb{C}^n) \text{ and } W\begin{pmatrix}(G^*v)(1)\\(G^*v)(0)\end{pmatrix} = 0\Big\}\end{aligned}$$

where W is a $k \times 2k$ matrix with full rank r and V is equipped with the norm

$$\|v\|_V^2 := \|v\|_{L^2}^2 + \|G^*v\|_{H^1}^2 \quad \text{for } v\in V.$$

Moreover, we assume that $GG^* = (GG^*)^2$ and there exists a positive constant $\kappa > 0$ such that

$$\|(I-GG^*)Pv\|_{H^1} \leq \kappa(\|v\|_{L^2} + \|G^*v\|_{H^1})$$

for all $u \in V$. Under these assumptions $\mathfrak{a}$ is a V-bounded and H-elliptic form with dense domain V. The operator associated with $\mathfrak{a}$ in H is the operator A with domain

$$\begin{aligned}D(A) := \Big\{&u\in H \text{ such that } G^*u \in H^1(0,1;\mathbb{C}^k),\\ &GS\frac{\partial}{\partial\zeta}G^*u + P_1u \in H^1(0,1;\mathbb{C}^n) \text{ and}\\ &R^*\begin{pmatrix}G^*(GS\frac{\partial}{\partial\zeta}G^*u - G^*P_1u)(1)\\ G^*(GS\frac{\partial}{\partial\zeta}G^*u - G^*P_1u)(0)\end{pmatrix} = 0\Big\}\end{aligned}$$

where R is $2k\times k$ matrix with $\mathrm{rang}R = \ker W$. Let now $\mathcal{H}(\cdot) \in L^\infty(0,1;\mathbb{C}^{n\times n})$ such that $\mathcal{H}(\zeta)^* = \mathcal{H}(\zeta)$, $mI \leq \mathcal{H}(\zeta) \leq MI$ for a.e. $\zeta \in [0,1]$ and constants $m, M > 0$ independent of ζ.

Proposition 3. Under the hypothesis above, the operator $-A\mathcal{H}$ with domain

$$D(A\mathcal{H}) = \{u\in H \text{ such that } \mathcal{H}u \in D(A)\}$$

generates a holomorphic C_0-semigroup on H.

We remark, that in [24], [25] and [23] closed relations methods are used to shows that $-A\mathcal{H}$ generates a contraction semigroup for suitable boundary condition.

Assume now that S and $\mathcal{H}$ depend on the time variable $t\in[0,\tau]$ such that $S : [0,1]\times[0,\tau] \longrightarrow \mathbb{C}^{k\times k}$ and $\mathcal{H} : [0,1]\times[0,\tau] \longrightarrow \mathbb{C}^{n\times n}$ are Lipschitz continuous w.r.t. the second variable. Thus by Theorem 2 we have the following maximal regularity result.

Theorem 4. The non-autonomous port-Hamiltonian system

$$\frac{\partial x}{\partial t}(t) - G\frac{\partial}{\partial\zeta}\left(S(t)\frac{\partial}{\partial\zeta}G^*\mathcal{H}(t)x(t) - P_1\mathcal{H}(t)x(t)\right) = f(t)$$
$$\mathcal{H}(0)x(0) = x_0,$$

with boundary condition

$$R^*\begin{pmatrix}G^*(GS(t)\frac{\partial}{\partial\zeta}G^*\mathcal{H}(t)x(t) + P_1\mathcal{H}(t)x(t))(1)\\ -G^*(GS(t)\frac{\partial}{\partial\zeta}G^*\mathcal{H}(t)x(t) + P_1\mathcal{H}(t)x(t))(0)\end{pmatrix} = 0$$

has a unique solution $x\in H^1(0,\tau;H)\cap L^2(0,\tau;V)$ whenever $x_0 \in V$ and $f\in L^2(0,\tau;H)$.

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