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# The weighted Weiss conjecture for admissible observation operators

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***Abstract*— The weighted Weiss conjecture states that the system theoretic property of weighted admissibility can be characterised by a resolvent growth condition. For positive weights, it is known that the conjecture is true if the system is governed by a normal operator; however, the conjecture fails if the system operator is the unilateral shift on the Hardy space $H^2(\mathbb{D})$ (discrete time) or the right-shift semigroup on $L^2(\mathbb{R}_+)$ (continuous time). To contrast and complement these counterexamples, in this talk positive results are presented characterising weighted admissibility of linear systems governed by shift operators and shift semigroups. These results are shown to be equivalent to the question of whether certain generalized Hankel operators satisfy a reproducing kernel thesis.**

## I. INTRODUCTION

Consider an infinite dimensional control system

$$\dot{x}(t) = Ax(t), \quad y(t) = Cx(t), \quad t \geq 0,$$
$$x(0) = x_0 \in X$$

where $A$ is the generator of a $C_0$-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space $X$ and the observation operator satisfies $C \in \mathcal{L}(D(A), \mathbb{C})$. For the system to be well-posed, in the sense of [9], a necessary condition is that $C$ is *admissible* for $A$, that is, there exists $k > 0$ such that

$$\|CT(\cdot)x_0\|_{L^2(\mathbb{R}_+)} \leq k\|x_0\|_X, \qquad x_0 \in D(A).$$

An important consequence of admissibility is that the output $y$ can be well defined even in the case that $C$ is unbounded. In particular, admissibility implies that the map $x_0 \mapsto CT(\cdot)x_0 \in L^2(\mathbb{R}_+)$, defined initially on $D(A)$, has a continuous extension to the whole space $X$, meaning that the output is well defined for any initial condition $x_0 \in X$.

A generalization of admissibility, first considered in [1], is to require that the output is an element of a weighted $L^2$-space. For $\beta > -1$, $C$ is said to be $\beta$-admissible for $A$ if there exists a constant $k > 0$ such that

$$\int_0^\infty t^\beta |CT(t)x_0|^2 dt \leq k^2\|x_0\|^2, \qquad x_0 \in D(A). \qquad (1)$$

To test whether a given system is $\beta$-admissible, a frequency-domain characterization is convenient and, to this end, it is not difficult to show that $\beta$-admissibility implies the resolvent growth condition

$$\sup_{\lambda\in\mathbb{C}_+} (\mathrm{Re}\lambda)^{\frac{1+\beta}{2}} \|CR(\lambda, A)^{-(1+\beta)}\|_{X^*} < \infty, \qquad (2)$$

where $R(\lambda, A) := (\lambda I - A)^{-1}$ denotes the resolvent of the semigroup generator $A$, and $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}$ is the right-half plane. The question of whether the converse statement (2) $\Rightarrow$ (1) holds, commonly referred to as a *(weighted) Weiss conjecture*, is much more subtle. Existing results concerning the conjecture are discussed below, but we first describe a discrete time version of the Weiss conjecture, introduced in [2], which will also be studied in this talk.

A discrete-time linear control system on a Hilbert space $X$ has the form

$$x_{n+1} = Tx_n, \quad y_n = Cx_n, \quad x_0 \in X, \qquad n \in \mathbb{N},$$

where $T \in \mathcal{L}(X)$ and $C \in X^*$. In this case, for $\beta > -1$, the observation functional $C$ is said to be (discrete) $\beta$-admissible for $T$ if there exists $k > 0$ such that

$$\sum_{n=0}^\infty (1+n)^\beta |CT^n x|^2 \leq k^2\|x\|_X^2, \qquad x \in X. \qquad (3)$$

Analogous to continuous time systems, the resolvent condition

$$\sup_{\omega\in\mathbb{D}} (1 - |\omega|^2)^{\frac{1+\beta}{2}} \|C(I - \bar{\omega}T)^{-(1+\beta)}\|_{X^*} < \infty \qquad (4)$$

is necessary for (3) and the discrete time form of the weighted Weiss conjecture is to ask when the converse implication is true. The Weiss conjecture is superficially easier to study in discrete time due to the boundedness of the operators involved. However, it should be noted that it is sometimes possible to translate positive and negative results concerning the conjecture via the Cayley transform [2], [10].

The continuous time conjecture (2) $\Rightarrow$ (1) was originally posed [7] in the unweighted case $\beta = 0$. In this situation, the conjecture is true if $A$ generates a $C_0$-semigroup of contractions [3], which extends the results that the conjecture holds if $A$ is normal [8] and if $A$ is the generator of the right-shift semigroup on $L^2(\mathbb{R}_+)$ [5]. The discrete time version (4) $\Rightarrow$ (3) for $\beta = 0$ and $T$ a contraction was shown in [2].

For $\beta \neq 0$, the behaviour of the conjecture is more complicated. In the case that $A$ is normal, the continuous time conjecture (2) $\Rightarrow$ (1) is true [12] for positive weight exponents $\beta \in (0, 1)$, but false [11] in the case that $\beta \in (-1, 0)$. Analogous results also hold for the discrete time conjecture conjecture when $T$ is normal [11], [12]. Furthermore, both continuous and discrete time conjectures are not true for general contraction operators when $\beta \in (0, 1)$: in continuous time, the right-shift semigroup on $L^2(\mathbb{R}_+)$ provides the counterexample [10]; while in discrete time (4) $\Rightarrow$ (3) fails if $T$ is the unilateral shift on the Hardy space $H^2(\mathbb{D})$ [11].

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It should be noted that the restriction $\beta \in (-1,1)$ in the above discussion arises from the fact that the growth bound $\sup_{\lambda\in\mathbb{C}_+}(\mathrm{Re}\lambda)^{\frac{1-\beta}{2}}\|CR(\lambda,A)\| < \infty$ (respectively, the condition $\sup_{\omega\in\mathbb{D}}(1-|\omega|^2)^{\frac{1-\beta}{2}}\|C(I-\bar{\omega}T)^{-1}\| < \infty$ in discrete time) was considered in the cited literature, i.e. a condition involving only the first power of the resolvent. In this situation, the restriction $\beta < 1$ is natural. However, as shown for example in [10], the truth of the weighted conjecture is not affected by considering instead the resolvent growth bound (2) and in this situation the natural range of weights is $\beta > -1$. Thus, the resolvent condition (2) is considered in the remainder of this talk.

The importance of determining the truth of the conjecture for the right-shift semigroup (or, in discrete time, the unilateral shift) is due to the Sz.Nagy-Foiaş model theory for contractions [6]. This states that a general contraction operator can be decomposed as a sum of operators, one of which is unitarily equivalent to a part of a shift operator. In [3] this decomposition was used in the case $\beta = 0$ (in discrete time, see [2]) to extend the truth of the conjecture for normal semigroups and the right-shift semigroup to general contraction semigroups. Thus, it is disappointing that neither the right-shift semigroup on $L^2(\mathbb{R}_+)$ nor the unilateral shift on $H^2(\mathbb{D})$ satisfy the weighted Weiss conjecture in the case $\beta \in (0,1)$.

The main results of this talk are to obtain positive results characterising $\beta$-admissibility for shift operators and semigroups. Results are proven in discrete time for the unilateral shift and in continuous time for the right-shift semigroup. For technical simplicity we first describe results in the discrete time setting. Two approaches are taken. The first is to consider the unilateral shift $(Sf)(z) = zf(z)$ acting on a different space to $H^2(\mathbb{D})$. $\beta$-admissibility of the shift $S : X \to X$ is considered in the case that $X$ is a weighted Bergman space $\mathcal{A}^2_\alpha(\mathbb{D}), \alpha > -1$, which contains analytic functions $f : \mathbb{D} \to \mathbb{C}$ for which

$$\|f\|^2_{\mathcal{A}^2_\alpha(\mathbb{D})} = \int_{\mathbb{D}} |f(z)|^2 dA_\alpha(z) < \infty,$$

where $dA_\alpha(z) = (1+\alpha)(1-|z|^2)^\alpha dA(z)$ and $dA(z) := \frac{1}{\pi}dxdy$ is area measure on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$, for $z = x+iy$. Since the norm $\|f\|_{\mathcal{A}^2_\alpha(\mathbb{D})}$ is equivalent to

$$\left(\sum_{n=0}^\infty |f_n|^2(1+n)^{-(1+\alpha)}\right)^{\frac{1}{2}}, \tag{5}$$

where $f_n$ are the Taylor coefficients of $f$, naively, the Hardy space $H^2(\mathbb{D})$ may be thought of as the 'corner' of the family of weighted Bergman spaces as $\alpha \to -1^+$. However, the behaviour of the weighted Weiss conjecture changes at this corner: it is shownthat for $\beta > 0$ the resolvent bound characterisation (4) $\Rightarrow$ (3) of $\beta$-admissibility holds for the shift $S : \mathcal{A}^2_\alpha(\mathbb{D}) \to \mathcal{A}^2_\alpha(\mathbb{D})$, for any $\alpha > -1$. The second approach is to derive a modified resolvent growth bound characterisation of $\beta$-admissibility for the shift $S : H^2(\mathbb{D}) \to H^2(\mathbb{D})$. In this case, it is shown that $\beta$-admissibility is characterised by

$$\sup_{\omega\in\mathbb{D}}(1-|\omega|^2)^{\frac{1}{2}}\left\|C(I-\bar{\omega}S)^{-1}\right\|_{\mathcal{A}^2_{\beta-1}(\mathbb{D})^*} < \infty. \tag{6}$$

The difference between this condition and (4), which *does not* characterise $\beta$-admissibility of $S : H^2(\mathbb{D}) \to H^2(\mathbb{D})$, is that the weight $\beta$ appears in the space in which the norm of the operator $C(I-\bar{\omega}S)^{-1}$ is tested, rather than as a power of the resolvent and the required growth rate.

That (6) is in some sense the 'correct' resolvent growth condition with which to test weighted admissibility of $S : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is intrinsically related to the notion of a Reproducing Kernel Thesis (RKT). A Reproducing Kernel Hilbert Space $H$ is a space of analytic functions on a set $\Omega$ (in this talk, either $\Omega = \mathbb{D}$ or $\Omega = \mathbb{C}_+ = \{\lambda : \mathrm{Re}\lambda > 0\}$) containing functions $(k_\omega)_{\omega\in\Omega} \subset H$, known as the *reproducing kernels*, which satisfy $f(\omega) = \langle f, k_\omega\rangle_H$ for any $f \in H, \omega \in \Omega$. A class $\mathcal{T}$ which contains linear operators acting from $H$ to a second Hilbert space $K$ is said to satisfy a Reproducing Kernel Thesis if boundedness of each operator $T \in \mathcal{T}$ is characterized by

$$\sup_{\omega\in\Omega}\frac{\|Tk_\omega\|_K}{\|k_\omega\|_H} < \infty. \tag{7}$$

That is, for each operator $T \in \mathcal{T}$, boundedness of $T : H \to K$ can be tested just by considering the behaviour of $T$ on the reproducing kernels.

The question of which classes of operator satisfy a RKT has received much attention and it is known that many important classes of operator do satisfy a RKT (see, e.g. [4, p. 131] for a brief overview). Of particular relevance to the study of the Weiss conjecture for shifts is the fact that the class of little Hankel operators $h_{\bar{c}}(f) := \overline{P}(\bar{c}f)$ with symbol $c \in H^2(\mathbb{D})$, mapping from $H^2(\mathbb{D})$ to $\overline{H^2(\mathbb{D})} = \overline{P}H^2(\mathbb{D})$, satisfy a RKT. Here, $\overline{P}$ denotes the projection onto anti-analytic functions. In the case $\beta = 0$, it was shown in [2] that if $T = S$ is the unilateral shift on $H^2(\mathbb{D})$ and $Cf = \langle f, c\rangle_{H^2}$ for $c \in H^2(\mathbb{D})$, then (3) holds if and only if $h_{\bar{c}}$ is bounded on $H^2(\mathbb{D})$. On the other hand, since the reproducing kernels for $H^2(\mathbb{D})$ are $k_w(z) = (1-\bar{\omega}z)^{-1}$ and $\|k_\omega\|_{H^2(\mathbb{D})} = (1-|\omega|^2)^{-\frac{1}{2}}$, it is not difficult to show that $h_{\bar{c}} : H^2(\mathbb{D}) \to \overline{H^2(\mathbb{D})}$ satisfies (7) if and only if the resolvent condition (4) holds for $\beta = 0$. Hence, the truth of the discrete Weiss conjecture for the shift $S$ in the unweighted case $\beta = 0$ is equivalent to the fact that the Hankel operators $h_{\bar{c}}$, for $c \in H^2(\mathbb{D})$, satisfy a RKT.

In the weighted case $\beta > 0$, it is shown that $\beta$-admissibility of the shift $S : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is equivalent to boundedness of one/both of the *generalized* Hankel operators $h^{\beta/2,0}_{\bar{c}}$ or $h^{0,\beta/2}_{\bar{c}}$. It turns out that whether $h^{\beta/2,0}_{\bar{c}}$ satisfies (7) is equivalent to the modified resolvent condition (6); while whether $h^{0,\beta/2}_{\bar{c}}$ satisfies (7) is equivalent to the original resolvent condition (4). Consequently, the characterization (3) $\Leftrightarrow$ (6) of $\beta$-admissibility follows from the fact that the generalized Hankel operators $h^{\beta/2,0}_{\bar{c}}$, for $c \in H^2(\mathbb{D})$, satisfy a RKT ; while the failure of the original conjecture (4) $\not\Rightarrow$ (3) can now be explained by the fact that the operators $h^{0,\beta/2}_{\bar{c}}$, for $c \in H^2(\mathbb{D})$, do not. The technical reason for this result is that the inclusion $D^{-\beta/2}BMOA \subset \Lambda^+_{\beta/2}$ between two certain classes of operator symbols is strict.

Analogous results to the ones described above are proven for the continuous time case. It is shown that for $\beta > 0$ the weighted Weiss conjecture (2) $\Rightarrow$ (1) holds for the right-shift semigroup acting on any of the weighted spaces $L^2_\alpha(\mathbb{R}_+), \alpha > 0$, where

$$L^2_\alpha(\mathbb{R}_+) = \{f : \mathbb{R}_+ \to \mathbb{C} : \int_0^\infty t^{-\alpha}|f(t)|^2 dt < \infty\}.$$

The 'corner' case of the right-shift semigroup on $L^2(\mathbb{R}_+)$ is discussed, it is shown that $\beta$-admissibility, $\beta > 0$, is characterised by the modified resolvent growth condition

$$\sup_{\lambda \in \mathbb{C}_+} (\mathrm{Re}\lambda)^{\frac{1}{2}} \|CR(\lambda, A)\|_{L^2_{\beta/2}(\mathbb{R}_+)^*} < \infty, \tag{8}$$

where $A$ is the generator of the right-shift semigroup. In the continuous time setting, the characterisation of weighted admissibility is related to whether certain generalised Hankel operators satisfy a RKT on the Hardy space $H^2(\mathbb{C}_+)$.

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