21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

Controllability of Bilinear Interconnected Systems

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***Abstract*— We analyze controllability properties for two different classes of bilinear interconnected systems. The first class consists of networks of single-input single-output (SISO) linear systems, where coupling parameters act as control variables. We characterize the system Lie algebra of the resulting bilinear control system. Necessary and sufficient conditions for accessibility are derived in terms of the underlying interconnection graph. Similar results are stated for the multi-input multi-output (MIMO) case. Such problems are relevant for analyzing controllability of switching networks of linear systems. In the second part, motivated by applications in quantum control, we investigate parallel connections of bilinear systems. We deduce from a more general accessibility result on bilinear systems on semisimple Lie groups a necessary and sufficient accessibility/controllability condition for parallel connected bilinear systems.**

I. INTRODUCTION

Networks of dynamical systems have been an active and challenging field of interdisciplinary research. While in the past electrical engineering certainly provided the major application area of "network theory", nowadays various new players as biology, medicine, economics, computer science etc. enter the stage and inspire new ideas and concepts: flock and formation control, distributed control and synchronization are just a few aspects of the emerging field of interconnected dynamical systems.

In applications where the network structure is not fixed apriori and can be considered therefore as a control parameter one is faced with the fundamental question whether a given initial formation of the network can be steered to a desired final formation by switching or tuning interconnections appropriately. Such a scenario arises e.g. when linear systems are coupled by time-varying output feedback according to an underlying feedback graph. The overall system then constitutes (in the SISO case) a bilinear system of the form

$$\dot{x}_i = A_i x_i + \sum_{(j,i)\in E} u_{ij}(t) b_i c_j x_j \,,$$

where E is the edge set of the underlying feedback graph. Hence, accessibility can be studied by means of the associated system Lie algebra.

In Section II, we determine all possible system Lie algebras resulting from feedback systems with strongly connected feedback graphs. From this we derive necessary and sufficient conditions for these networks to be accessible. Similar results for single linear output feedback systems were obtained by R. W. Brockett [4], [5]. Our arguments and techniques are based on his ideas in [4]. However, our results do not cover Brockett's work, as we require that our network has at least three subsystems.

In Section III we analyze a different scenario. Here, the individual subsystems are actually not internally coupled, but an external coupling results from the fact that all subsystems are subject to the same control addressing all of them simultaneously. Hence, the coupled system constitutes a parallel connection. For the linear case, parallel connections have been studied extensively in the literature see e.g. P. Fuhrmann [11]. The control of parallel connected bilinear systems is motivated by studies on ensemble controllability for quantum control systems [20]. Here, we focus on bilinear systems evolving on semisimple matrix Lie groups and derive a necessary and sufficient accessibility condition which can be verified easily once the automorphism group of the associated system Lie algebra is known. For the classical simple complex Lie algebras, this problem is well-studied and the automorphisms are completely classified in the literature, see [12], [16]. The real case can be found in [14], [22].

II. CONTROL OF BILINEAR NETWORKS OF SYSTEMS

In this section we study a finite number of linear SISO systems that are interconnected by time-varying couplings. In contrast to previous work we consider the coupling parameters as control variables. This results in a large bilinear control system whose controllability properties we analyze.

We begin with a brief summary of relevant notions from graph theory. A *directed graph* $\Gamma = (E, V)$ is a pair of a set of ordered edges E and a finite set of vertices V. Here, we always take $V := \{1, 2, \ldots, N\}$ as given and assume, that the graph Γ is *simple*, i.e., it has no parallel edges and no self-loops. The *adjacency matrix of a directed graph* Γ is defined by a matrix $\gamma \in \mathbb{R}^{N\times N}$ where

$$\gamma_{ij} := \begin{cases} 1 & \text{if } (i,j) \in V \\ 0 & else. \end{cases} \tag{1}$$

A *path* $i_0 i_1 \ldots i_S$ from vertex i to vertex j is a finite sequence of vertices, such that $(i_{k-1}, i_k) \in E$ for $k = 1, \ldots, S$ and $i_0 = i$, $i_S = j$. Two vertices i and j of a directed graph $\Gamma = (E, V)$ are called *strongly connected* if there exists a directed path in Γ from vertex i to vertex j and one from j to i. A directed graph $\Gamma = (E, V)$ is called *strongly connected*, if every pair of vertices (i, j) is strongly connected. A well-known characterization for strongly connected graphs is the following (see e.g. [3]).

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ISBN: 978-90-367-6321-9

Theorem 1: A directed graph $\Gamma = (E, V)$ is strongly connected if and only if its adjacency matrix γ is permutation-irreducible.

A. Controllability of Bilinear Systems

Next we summarize relevant facts about controllability of bilinear systems

$$\dot{x} = \Big(A + \sum_{j=1}^{m} u_j(t) B_j\Big) x \tag{2}$$

on $\mathbb{R}^n \setminus \{0\}$. The *system Lie algebra* $\mathfrak{g}$ is defined as the real Lie subalgebra of $n \times n$-matrices that is generated by $A, B_1, \ldots, B_m \in \mathbb{R}^{n \times n}$. Accessibility of (2) then is characterized by the property whether or not the associated *system Lie group* G with Lie algebra $\mathfrak{g}$ acts transitively on $\mathbb{R}^n \setminus \{0\}$. Therefore, one has the following accessibility criterion for bilinear systems, see [6], [7], [17].

Theorem 2: A bilinear system (2) on $\mathbb{R}^n \setminus \{0\}$ is accessible if and only if its system Lie algebra $\mathfrak{g}$ is conjugated to one of the following types:

(1) $\mathfrak{so}(n) \oplus \mathbb{R}$, if $n \geq 2$.
(2) $\mathfrak{su}(n/2) \oplus e^{i\alpha}\mathbb{R}$ or $\mathfrak{su}(n/2) \oplus \mathbb{C}$, if n is even and $n \geq 3$.
(3) $\mathfrak{sp}(n/4) \oplus e^{i\alpha}\mathbb{R}$, $\mathfrak{sp}(n/4) \oplus \mathbb{C}$ or $\mathfrak{sp}(n/4) \oplus \mathbb{H}$, if $n = 4k$.
(4) $\mathfrak{g}_2 \oplus \mathbb{R}$, if $n = 7$.
(5) $\mathfrak{spin}(7) \oplus \mathbb{R}$, if $n = 8$.
(6) $\mathfrak{spin}(9) \oplus \mathbb{R}$, if $n = 16$.
(7) $\mathfrak{sl}(n, \mathbb{R})$ or $\mathfrak{gl}(n, \mathbb{R})$, if $n \geq 2$.
(8) $\mathfrak{sl}(n/2, \mathbb{C})$, $\mathfrak{sl}(n/2, \mathbb{C}) \oplus e^{i\beta}\mathbb{R}$ or $\mathfrak{gl}(n/2, \mathbb{C})$, if $n = 2k$.
(9) $\mathfrak{sl}(n/4, \mathbb{H})$, $\mathfrak{sl}(n/4, \mathbb{H}) \oplus e^{i\beta}\mathbb{R}$ or $\mathfrak{sl}(n/4, \mathbb{H}) \oplus \mathbb{C}$, if $n = 4k$.
(10) $\mathfrak{sl}(n/4, \mathbb{H}) \oplus \mathfrak{sp}(1)$ or $\mathfrak{sl}(n/4, \mathbb{H}) \oplus \mathbb{H}$, if $n = 4k$.
(11) $\mathfrak{sp}(n/2, \mathbb{R})$ or $\mathfrak{sp}(n/2, \mathbb{R}) \oplus \mathbb{R}$, if n is even and $n \geq 3$.
(12) $\mathfrak{sp}(n/4, \mathbb{C})$, $\mathfrak{sp}(n/4, \mathbb{C}) \oplus e^{i\beta}\mathbb{R}$ or $\mathfrak{sp}(n/4, \mathbb{C}) \oplus \mathbb{C}$, if $n = 4k$.
(13) $\mathfrak{spin}(9, 1, \mathbb{R})$ or $\mathfrak{spin}(9, 1, \mathbb{R}) \oplus \mathbb{R}$, if $n = 16$.

Here, α and β have to satisfy $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

The preceding list of Lie algebras which guarantee accessibility on $\mathbb{R}^n \setminus \{0\}$ is complete, but impressively long. A detailed discussion is provided in [7] and [18]. For the Lie-theortic background we refer the reader to Ch. I and II in [16]. In special cases, the above list can be reduced considerably and more explicit characterizations for accessibility can be found. An important case is the one studied by R. W. Brockett in his early work on output feedback control systems.

Let (A, b, c) denote a minimal triple of a SISO system with $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^{1 \times n}$. Here and henceforth, a triple (A, b, c) is called *minimal* if (A, b) is controllable and (A, c) is observable. Then, a SISO system can be described by

$$\begin{aligned} \dot{x} &= Ax + bv(t), \\ y &= cx, \end{aligned} \tag{3}$$

where the associated transfer function is $g(s) = c(sI - A)^{-1}b$. By applying time-varying output feedback to (3) of the form $v(t) = u(t)cx$ we obtain the bilinear system

$$\dot{x} = \big(A + u(t)bc\big)x. \tag{4}$$

The associated system Lie algebra is the Lie algebra generated by the coefficient matrices A and bc. In [4], Brockett determines all possible systems Lie algebras for linear output feedback systems of the above form (4).

Theorem 3 ([4],[5]): Let (A, b, c) be minimal, $g(s) = c(sI - A)^{-1}b$. Then the system Lie-algebra $\mathfrak{g}$ of

$$\dot{x} = \big(A + u(t)bc\big)x$$

satisfies:

(i) $\mathfrak{g} \cong \mathfrak{sp}_{n/2}(\mathbb{R})$ if and only if $g(s) = g(-s)$
(ii) $\mathfrak{g} \cong \mathfrak{sp}_{n/2}(\mathbb{R}) \oplus \mathbb{R}I$ if and only if $g(s+\alpha) = g(-s+\alpha)$ with $\alpha \neq 0$ suitable.
(iii) $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$ if and only if $g(s+\alpha) \neq g(-s+\alpha)$ for all $\alpha \in \mathbb{R}$ and $cb = \operatorname{tr} A = 0$.
(iv) $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$ else.

Here, $\cong$ means that the respective Lie algebras are conjugated.

Furthermore, in [4] Brockett introduced the appropriate multivariable analogon to (4) as MIMO systems of the form

$$\dot{x} = \big(A + BK(t)C\big)x, \tag{5}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$ and $C \in \mathbb{R}^{p \times n}$. Here, $K(t)$ is a matrix of appropriate dimension. Then, the associated system Lie algebra is the smallest Lie algebra containing A and BKC for all real $K \in \mathbb{R}^{p \times p}$. For minimal (A, B, C), where minimality is defined as in the SISO case, Brockett proved the following result.

Theorem 4 ([4]): Let (A, B, C) be a minimal triple and suppose that $G(s) = C(sI - A)^{-1}B$ is of $\operatorname{rk} \mathrm{G} \geq 2$ for some s. Then the system Lie algebra $\mathfrak{g}$ of the output-feedback control system

$$\dot{x} = (A + BK(t)C)\,x$$

satisfies:

(i) $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$ if and only if $CB = 0, \operatorname{tr} A = 0$.
(ii) $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$ else.

B. Control by Interconnections

Networks of SISO systems: We next study networks of interconnected linear SISO systems. Thus let the state of each vertex i be a n-dimensional vector x_i in $\mathbb{R}^n$. Then, the dynamics of every single linear system in this network is described by

$$\begin{aligned} \dot{x}_i &= A_i x_i + b_i v_i(t), \\ y_i &= c_i x_i \end{aligned}$$

with $A_i \in \mathbb{R}^{n \times n}$, $b_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}^{1 \times n}$. We fix the interconnection structure via the adjacency matrix γ of a graph $\Gamma = (E, V)$. The time-varying interconnection parameters are then regarded as real valued control functions

$u_{ij}(t)$ wiht $(i,j) \in E$. By applying time-varying output feedback of the form

$$v_i(t) = \sum_{(j,i)\in E} u_{ij}(t) c_j x_j.$$

we obtain a coupled network of linear systems that is controlled by the coupling strengths $u_{ij}(t)$. Explicitly, the dynamics of the interconnected network with graph Γ is given as

$$\dot{x}_i = A_i x_i + \sum_{(j,i)\in E} u_{ij}(t) b_i c_j x_j$$

for $i = 1, \dots N$. By rewriting this in matrix form we arrive at the bilinear control system

$$\dot{x} = \big(\mathcal{A} + \mathcal{B}\gamma(u)\mathcal{C}\big)x, \tag{6}$$

where

$$\gamma(u) := \sum_{(i,j)\in E} u_{ij} E_{ij} \tag{7}$$

is the so-called *controlled adjacency matrix* and the coefficient matrices of (6) are given by

$$\mathcal{A} := \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_N \end{pmatrix}, \mathcal{B} := \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_N \end{pmatrix},$$

$$\text{and} \quad \mathcal{C} := \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_N \end{pmatrix}.$$

Thus, $\mathcal{A} \in \mathbb{R}^{nN\times nN}$, $\mathcal{B} \in \mathbb{R}^{nN\times N}$ and $\mathcal{C} \in \mathbb{R}^{N\times Nn}$. Note that (6) is of the same output feedback form as Brockett's system (5), with the important distinction that the matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are block-diagonal and the output feedback matrix $\gamma(u)$ has the structure of a weighted adjacency matrix of a graph (with no self-loops).

Similar to Theorem 3 we characterize the system Lie algebras for heterogeneous networks with more than two vertices.

Theorem 5: Let (A_i, b_i, c_i) be minimal for $i = 1, \dots N$ and

$$\gamma(u) := \sum_{(i,j)\in E} u_{ij} E_{ij}$$

be the controlled adjacency matrix of a strongly connected graph $\Gamma = (E, V)$ with $N > 2$ vertices. Then, the system Lie algebra $\mathfrak{G}$ of

$$\dot{x} = \big(\mathcal{A} + \mathcal{B}\gamma(u)\mathcal{C}\big)x$$

is either $\mathfrak{sl}_{nN}(\mathbb{R})$ or $\mathfrak{gl}_{nN}(\mathbb{R})$.

If all node systems (A_i, b_i, c_i) are equal the network is homogeneous. In this case we obtain the following result as a special case of Theorem 5.

Corollary 1: Let (A, b, c) be minimal and

$$\gamma(u) := \sum_{(i,j)\in E} u_{ij} E_{ij}$$

be the controlled adjacency matrix of a strongly connected graph $\Gamma = (E, V)$ with $N > 2$ vertices. Then, the system Lie algebra $\mathfrak{G}$ of the controlled network

$$\dot{x} = \big(I_N \otimes A + \gamma(u) \otimes bc\big)x$$

is either $\mathfrak{sl}_{nN}(\mathbb{R})$ or $\mathfrak{gl}_{nN}(\mathbb{R})$.

We briefly sketch a proof of Theorem 5 in the special case that all systems (A_i, b_i, c_i) are equal. Clearly, this is exactly the scenario of Corollary 1. The more general case is treated in the forthcoming PhD Thesis [23]. Moreover, we assume for simplicity $cb \neq 0$.

Proof: For $(i,j) \in E$ we obtain $E_{ij} \otimes bc \in \mathfrak{G}$ and by taking iteratively commutators with the matrix $I_N \otimes A$ it follows $E_{ij} \otimes \mathrm{ad}_A^k(bc) \in \mathfrak{G}$ for all $k \in \mathbb{N}$. Let $(i,j) \notin E$. Due to the strong connectedness of Γ we find a directed path in Γ from i to j. W.l.o.g., let us assume that the path consists of two edges, i.e., it exists a vertex m, such that (i,m) and $(m,j) \in E$. Thus,

$$[E_{im} \otimes bc, E_{mj} \otimes bc] = (cb) E_{ij} \otimes bc \in \mathfrak{G}.$$

Since $cb \neq 0$ it follows $E_{ij} \otimes bc \in \mathfrak{G}$ and $E_{ij} \otimes \mathrm{ad}_A^k(bc) \in \mathfrak{G}$ for all $i \neq j$ and all $k \in \mathbb{N}$. Furthermore, one has

$$[E_{im} \otimes \mathrm{ad}_A^k(bc), E_{mj} \otimes \mathrm{ad}_A^l(bc)] = E_{ij} \otimes \mathrm{ad}_A^k(bc) \cdot \mathrm{ad}_A^l(bc)$$

and it can be easily shown, that the real vector space spanned by elements of the form

$$X = \mathrm{ad}_A^k(bc) \cdot \mathrm{ad}_A^l(bc)$$

with $k, l \in \mathbb{N}_0$ is $\mathbb{R}^{n\times n}$. Hence, we obtain $E_{ij} \otimes \mathbb{R}^{n\times n} \subset \mathfrak{G}$ for $i \neq j$. Completing this to a Lie algebra yields

$$\mathfrak{sl}_{nN}(\mathbb{R}) \subset \mathfrak{G}$$

and the result follows. ■

As an immediate consequence of Theorem 2 and 5 we obtain the following result.

Corollary 2: Let $\Gamma = (E, V)$ be a directed graph with $N > 2$ vertices and the triples (A_i, b_i, c_i) be minimal for all $i = 1, \dots, N$. Then, the controlled network

$$\dot{x} = \big(\mathcal{A} + \mathcal{B}\gamma(u)\mathcal{C}\big)x. \tag{8}$$

is accessible on $\mathbb{R}^{nN} \setminus \{0\}$ if and only if Γ is strongly connected.

Networks of MIMO systems: Analogous to the SISO case, the dynamics of every single linear system in the network can be described by

$$\dot{x}_i = A_i x_i + B_i v_i(t),$$
$$y_i = C_i x_i,$$

where $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times p}$ and $C_i \in \mathbb{R}^{p\times n}$. Applying time-varying output feedback of the form

$$v_i(t) = \sum_{(j,i)\in E} K_{ij}(t)C_j x_j$$

we again obtain a coupled network of linear systems that is controlled by the coupling/feedback matrices $K_{ij}(t)$, i.e. by the matrix valued control functions $K_{ij}(t)$. Then, the dynamics of the network with interconnected graph Γ is given by the bilinear system

$$\dot{x} = \big(\mathcal{A} + \mathcal{B}\gamma(K)\mathcal{C}\big)x.$$

Here, we have

$$\gamma(K) := \sum_{(i,j)\in E} E_{ij} \otimes K_{ij}$$

and

$$\mathcal{A} := \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_N \end{pmatrix}, \mathcal{B} := \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_N \end{pmatrix}$$

$$\text{and} \quad \mathcal{C} := \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_N \end{pmatrix},$$

with $\mathcal{A} \in \mathbb{R}^{nN\times nN}$, $\mathcal{B} \in \mathbb{R}^{nN\times pN}$ and $\mathcal{C} \in \mathbb{R}^{pN\times Nn}$.

Similar to Theorem 5 we obtain the following result for networks of MIMO systems.

Theorem 6: Let (A_i, B_i, C_i) be minimal for $i = 1, \ldots, N$ and let $\Gamma = (E, V)$ be a strongly connected graph with $N > 2$ vertices. Moreover, let

$$\gamma(K) := \sum_{(i,j)\in E} E_{ij} \otimes K_{ij}$$

be the controlled adjacency matrix of the graph Γ. Then, the system Lie algebra $\mathfrak{G}$ of

$$\dot{x} = \big(\mathcal{A} + \mathcal{B}\gamma(K)\mathcal{C}\big)x$$

is either $\mathfrak{sl}_{nN}(\mathbb{R})$ or $\mathfrak{gl}_{nN}(\mathbb{R})$.

Again Theorem 2 yields the corresponding accessibility result for the MIMO case.

Corollary 3: Let $\Gamma = (E, V)$ be a directed graph with $N > 2$ vertices and (A_i, B_i, C_i) be minimal for $i = 1, \ldots, N$. Then, the controlled network

$$\dot{x} = (\mathcal{A} + \mathcal{B}\gamma(K)\mathcal{C})\, x$$

is accessible on $\mathbb{R}^{nN}\backslash\{0\}$ if and only if Γ is strongly connected.

Remark 1: Certainly, control systems arising from networks of linear systems were studied many times in the literature. But usually the resulting control systems are again linear. For instance, in [21] the authors consider linear networks where preassigned node systems act as input/output units, while the remaining node systems execute a consensus-like protocol. Thus the overall network constitutes again a linear control system. In our setting, however, the graph itself, or more precisely, the interconnection weights which serve as output feedback control are regarded as input parameters. This finally leads to a particular class of bilinear control systems which up to the authors' knowledge was not considered before.

III. Parallel Connections of Bilinear Systems

In the previous section, we investigated systems arising from networks of linear systems via feedback interconnections. Here, we focus on bilinear systems. But unfortunately, general networks of bilinear systems do not preserve bilinearity. An important exception are parallel connected ones which are in the focus of current research due to their relevance in quantum control [1], [20] and open loop control of parameter dependent systems [24]. In Subsection III-B, we will sketch two applications in this context.

A. Control of Parallel Connected Bilinear Systems

Our main result on parallel connections of bilinear systems rests on Theorem 7 which provides a general accessibility criterion for bilinear systems on semisimple matrix Lie groups. Here, the essential notion is the concept of Lie-relatedness which is defined as follows:

Given two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ as well as two finite sequences of Lie algebra elements $C_0, \ldots, C_m \in \mathfrak{g}$ and $C'_0, \ldots, C'_m \in \mathfrak{g}'$. Then $C_0, \ldots, C_m$ and $C'_0, \ldots, C'_m$ are *Lie-related* if there exists a Lie algebra isomorphism $\Phi : \mathfrak{g} \to \mathfrak{g}'$ such that $\Phi(C_l) = C'_l$ for $l = 0, \ldots, m$. Otherwise, they are *Lie-unrelated*. With this terminology, our result reads as follows.

Theorem 7: Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \subset \mathfrak{gl}_N(\mathbb{R})$ be a semisimple (matrix) Lie subalgebra with simple ideals $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ and let $G \subset GL_N(\mathbb{R})$ be the corresponding connected (matrix) Lie subgroup. Moreover, let $A, B_1, \ldots, B_m \in \mathfrak{g}$ and let $\pi_i : \mathfrak{g} \to \mathfrak{g}_i$ for $i = 1, \ldots, k$ denote the canonical projections onto the simple ideals $\mathfrak{g}_i$. Then the following statements are equivalent:

(a) The system

$$\dot{X}(t) = \Big(A + \sum_{l=1}^{m} u_l(t)B_l\Big)X(t)$$

with $u(t) \in \mathbb{R}^m$ is accessible on G.

(b) The Lie algebra elements $A, B_1, \ldots, B_m \in \mathfrak{g}$ satisfy the following conditions:

(1) For all $i = 1, \ldots, k$, the Lie algebra generated by $\pi_i(A), \pi_i(B_1), \ldots, \pi_i(B_m)$ coincides with $\mathfrak{g}_i$.

(2) For all $i, j = 1, \ldots, k$ and $i \neq j$ the projected elements $\pi_i(A), \pi_i(B_1), \ldots, \pi_i(B_m) \in \mathfrak{g}_i$ and $\pi_j(A), \pi_j(B_1), \ldots, \pi_j(B_m) \in \mathfrak{g}_j$ are Lie-unrelated.

A complete proof of Theorem 7 is beyond the scope of this contribution and will be given in a forthcoming paper. Here, we limit ourselves to show that Theorem 7 provides a simple accessibility criterion for parallel connections of bilinear system.

We consider k bilinear system of the form

$$\dot{X}_i(t) = \big(A_i + u(t)B_i\big)X_i(t)\,, \quad i = 1, \ldots, k \tag{9}$$

each evolving on the same (matrix) Lie group $G_0 \subset GL_n(\mathbb{R})$ with Lie algebra $\mathfrak{g}_0 \subset \mathfrak{gl}_n(\mathbb{R})$ and parallel connected via the simultaneous control $u(t) \in \mathbb{R}$. For simplicity, we restrict the number of controls to $m = 1$. Moreover, it is convenient to identify (9) with the following block diagonal system on $GL_{kn}(\mathbb{R})$:

$$\dot{X}(t) = \big(\mathcal{A} + u(t)\mathcal{B}\big)X(t)\,, \qquad u \in \mathbb{R}\,, \tag{10}$$

where $\mathcal{A}$ and $\mathcal{B}$ are defined by

$$\mathcal{A} := \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix} \quad \text{and} \quad \mathcal{B} := \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix}.$$

Then (10) evolves on the direct product $G_0 \times \cdots \times G_0$ regarded as set of block diagonal matrices of the form

$$X := \begin{pmatrix} X_1 & & \\ & \ddots & \\ & & X_k \end{pmatrix} \quad \text{with} \quad X_i \in G_0\,.$$

Thus, we are prepared to apply Theorem 7, which leads to the following result.

Corollary 4: Let $A_i, B_i \in \mathfrak{g}_0$ for $i = 1, \ldots, k$, where $\mathfrak{g}_0$ is a simple (matrix) Lie algebra, and let G_0 be the corresponding connected (matrix) Lie group. Then the following statements are equivalent:

(a) The parallel connection (9) or, equivalently, (10) is accessible on $G_0 \times \cdots \times G_0 \subset GL_{kn}(\mathbb{R})$.
(b) For all $i = 1, \ldots, k$ the Lie algebra generated by A_i, B_i coincides with $\mathfrak{g}_0$ and for $i \neq j$ the pairs (A_i, B_i) and (A_j, B_j) are Lie-unrelated.

Proof: Apply Theorem 7 to the special case $\mathfrak{g} := \bigoplus_{i=1}^k \mathfrak{g}_0 \subset \mathfrak{gl}_{kn}(\mathbb{R})$, i.e. $\mathfrak{g}_i := \mathfrak{g}_0$ for $i = 1, \ldots, k$. Note that the simplicity of $\mathfrak{g}_0$ guarantees that $\mathfrak{g}$ is semisimple. ■

As for compact groups accessibility and controllability are equivalent [15], the above result actually yields a necessary and sufficient condition for controllability of parallel connections if G_0 is compact.

Clearly, for applying Corollary 4 in practice, one has to know all possible Lie algebra automorphisms of $\mathfrak{g}_0$ which can be delicate in general. But for the classical simple (complex) Lie algebras and their real forms they are well-known, see e.g. [12], Part III, Ch. 1, Lemma 12, [16], Ch. VII, Thm. 7.8, [14], and [22].

B. Applications

In the following, we give two application of the above results. The first one is more of theoretical interest; it combines results from Section II and III. The second one outlines the significance of Corollary 4 to quantum control.

1) Parallel Connected Output Feedback Systems: From Theorem 3 and Corollary 4, we obtain the following result for parallel connected SISO feedback systems.

Corollary 5: Consider k minimal SISO systems (A_i, b_i, c_i) with corresponding transfer functions $g_i(s) = c_i(sI - A_i)^{-1}b_i$ satisfying the conditions:

(a) $\operatorname{tr} A_i = c_i b_i = 0$ for $i = 1, \ldots, k$.
(b) $g_i(s+\alpha) \neq g_i(-s+\alpha)$ for $i=1, \ldots, k$ and all $\alpha \in \mathbb{C}$.
(c) $g_i(s) \neq \pm g_j(\pm s)$ for $i, j = 1, \ldots, k$ and $i \neq j$.

Then the parallel connection of linear output feedback systems

$$\begin{aligned} \dot{x}_1 &= (A_1 + u(t)b_1c_1)x_1 \\ &\vdots \\ \dot{x}_k &= (A_k + u(t)b_kc_k)x_k \end{aligned}$$

is accessible on $\big(\mathbb{R}^n \setminus \{0\}\big)^k$.

Proof: By Theorem 3, the system Lie algebras generated by A_i and b_ic_i for $i = 1, \ldots, k$ coincide with $\mathfrak{sl}_n(\mathbb{R})$ which is simple and acts transitively on $\mathbb{R}^n \setminus \{0\}$. The automorphisms of $\mathfrak{sl}_n(\mathbb{R})$ are either conjugation $\Theta \mapsto \Theta X \Theta^{-1}$ by an element $\Theta \in GL_n(\mathbb{R})$ or $X \mapsto -X^\top$, cf. [22]. Therefore, by Corollary 4, accessibility on $\big(SL_n(\mathbb{R})\big)^k$ and thus on $\big(\mathbb{R}^n \setminus \{0\}\big)^k$ holds, provided (A_i, b_i, c_i) is neither similar to (A_j, b_j, c_j) nor to $(-A_j^\top, -c_j^\top, b_j^\top)$ for $i \neq j$. But this follows immediately from our assumptions on the transfer functions. ■

2) Quantum Control: Here, we briefly describe a typical scenario in quantum control which leads to parallel connected bilinear systems. Consider an ensemble of coupled spin-$\frac{1}{2}$ systems – such systems naturally arise in NMR spectroscopy, see e.g. [8], [9]. Its time-evolution is governed by the Liouville – von Neumann equation

$$\dot{\rho}(t) = -\mathrm{i}[H_u(t), \rho(t)] \tag{11}$$

where $\rho(t)$ denotes the density matrix representing the state of the ensemble at time $t \geq 0$, or, equivalently, by the lifted equation

$$\dot{U}(t) = -\mathrm{i}H_u(t)U(t) \tag{12}$$

where $U(t)$ is the corresponding unitary propagator at time $t \geq 0$. Moreover, H_u is a controllable Hamiltonian, which usually reads as

$$H_u(t) := H_0 + \sum_{k=1}^{m} u_k(t)H_k\,. \tag{13}$$

Here, $H_0, H_1 \ldots, H_m$ are traceless Hermitian matrices modeling the uncontrollable and, respectively, controllable part of the system's dynamics.

In many experimental settings, H_u will depend on additional known or unknown parameters. For instance in NMR spectroscopy, the involved magnetic fields often exhibit undesired inhomogeneities and chemical shieldings cause unknown shifts in the Lamor frequencies [10], [20]. Both effects result in a parameter dependent form of (12) and (13), respectively:

$$\dot{U}(t,\theta) = -\mathrm{i}H_u(t,\theta)U(t,\theta) \tag{14}$$

and

$$H_u(t,\theta) := H_0(\theta) + \sum_{k=1}^{m} u_k(t)H_k(\theta)\,. \tag{15}$$

Note that $u_1(t),\ldots,u_m(t) \in \mathbb{R}$ are assumed to be independent of θ. Therefore, the major challenge is to simultaneously steer a given initial value ρ_0 which may depend on θ as well as on a common final value ρ_* by means of controls $u_1(t),\ldots,u_m(t)$ which are independent of θ.

After discretizing the in general continuous parameter θ and assuming $m = 1$ for notational convenience, one arrives at the following finite family of bilinear control system

$$\dot{U}_j(t) = -\mathrm{i}H_{u,j}(t)U_j(t)\,, \quad j = 1,\ldots,k \tag{16}$$

with

$$H_{u,j}(t) := H_{0,j} + u(t)H_{1,j}\,, \quad j = 1,\ldots,k\,. \tag{17}$$

As the control $u(t)$ is independent of j the above system (16) actually constitutes a parallel connection of bilinear systems each evolving on the same special unitary group. Now, Corollary 4 which generalizes Theorem 4 in [1] guarantees simultaneous controllability of (16) under very mild conditions on the pairs $(H_{0,j}, H_{1,j})$.

The future challenge is to provide necessary and sufficient conditions for (approximate) controllability of (14) which represents the continuous version of (16). First results in this direction are given in [2] and [20]. There the authors completely focus on the Bloch equation evolving on the orthogonal group $SO(3)$. A general theory for simultaneous controllability of distributed bilinear systems, however, is missing. For the linear case, we refer to e.g. [13] and [19].

References

[1] C. Altafini, Controllability and simultaneous controllability of isospectral bilinear control systems on complex flag manifolds, *System and Control Letters (58)*, pp. 213–216, 2009.
[2] K. Beauchard, J.-M. Coron, and P. Rouchon, Controllability Issues for Continuous-Spectrum Systems and Ensemble Controllability of the Bloch Equation. *Commun. Math. Phys* (296), pp. 525–557, 2010.
[3] A. Berman and R.J. Plemmons, Nonnegative matrices in the mathematical sciences, Philadelphia, PA: SIAM, 1994.
[4] R.W. Brockett, The Lie groups of simple feedback systems, IEEE Conference on Decision and Control including the 15th Symposium on Adaptive Processes, pp. 1189–1193, 1976.
[5] R.W. Brockett, Linear feedback systems and the groups of Galois and Lie, *Linear Algebra Appl.* (50), pp. 45–60, 1983.
[6] W.M. Boothby, A transitivity problem from control theory, *J. Diff. Eqn* (17), pp. 296–307, 1975.
[7] W.M. Boothby and E. Wilson, Determination of the transitivity of bilinear systems, *SICON* 17(2), pp. 212–221, 1979.
[8] D. D'Alessandro, *Introduction to Quantum Control and Dynamics*, chapman & Hall/CRC, Boca Raton, 2008.
[9] G. Dirr and U. Helmke, Lie Theory for quantum control, *GAMM Mitteilungen* 31(1), pp. 59–93, 2008.
[10] R. R. Ernst, G. Bodenhausen, and A. Wokaun, *Principles of Nuclear Magnetic Resonance in One and Two Dimensions*, Oxford University Press, New York, 1987.
[11] P. Fuhrmann, On controllability and observability of systems connected in parallel, IEEE Trans. *Circuits and Systems* 22(1), pp. 57, 1975.
[12] M. Hauser and J.T. Schwartz, *Lie Groups; Lie Algebras*, Gordon and Breach, New York, 1968.
[13] U. Helmke and M. Schönlein. Uniform ensemble controllability for one-parameter families of time-invariant linear systems. *Systems & Control Letters*, provisionally accepted, 2014.
[14] N. Jacobson, Structure and automorphisms of semi-simple Lie groups in the large, *Annals of Mathematics* 40(4), pp. 755–763, 1939.
[15] V. Jurdjevic, *Geometric Control Theory*, Cambridge University Press, New York, 1997
[16] A.W. Knapp, *Lie Groups Beyond an Introduction*, Progress in Mathematics, Birkhäuser Boston, 2002.
[17] L. Kramer, Two-transitive Lie groups, *J. Reine Ang. Math.* (563), pp. 83–113, 2003.
[18] I. Kurniawan, G. Dirr, and U. Helmke, Controllability Aspects of Quantum Dynamics: A Unified Approach to Closed and Open Systems, *IEEE Trans. Autom. Contr.* 57(8), pp. 1984–1996, 2012.
[19] J.-S. Li, Ensemble Control of Finite-Dimensional Time-Varying Linear Systems, *IEEE Trans. Autom. Control* (56), pp. 345–357, 2011.
[20] J.-S. Li and N. Khaneja, Ensemble Control of Bloch Equations, *IEEE Trans. Autom. Control* (54), pp. 528–536, 2009.
[21] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*, Princeton Series in Applied Mathematics, Princeton University Press, 2010.
[22] S. Murakami, On the automorphisms of a real semisimple Lie algebra, *J. Math. Soc. Japan*, pp. 103–133, 1952.
[23] F. Rüppel, *Accessibility of Bilinear Interconnected Systems*, PhD Thesis, Würzburg, 2014.
[24] F. Ticozzi, A. Ferrante, and M. Pavon, Robust steering of n-level quantum systems, *IEEE Trans. Autom. Contr.* 49(10), pp. 1742–1745, 2004.