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Observability and controllability for linear neutral type systems

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Abstract— **For a large class of linear neutral type systems which include distributed delays we give the duality relation between exact controllability and exact observability. The characterization of exact observability is given.**

I. INTRODUCTION

Our purpose is to study the problem of exact observability of a large class of linear neutral type system. The cases of approximate and spectral controllability and the corresponding dual notions of observability were widely investigated at the end of the last century (see books by [1] and [2] and references therein). The duality between these notions is not so simple. The main reason is that the dual or adjoint system is not obtained directly by simple transposition. It is necessary to consider the duality using some hereditary product proposed first for retarded systems and later for neutral type systems (see [3], [4], [5] and [2] for example). An important tool in this context is the structural operator. It enables some explicit expression for duality between approximate controllability, spectral controllability and approximate observability. Moreover, there are several interesting results concerning these notions of observability and controllability and about duality.

The cases of exact controllability and exact observability for neutral type systems are less developed. The infinite dimensional setting has been developed essentially for exact controllability and often for neutral type systems without distributed delays. In [6], [7] and [8] an approach is described based on the reconstruction of a part of the state for the case of a neutral type system with discrete delays. A duality condition with null controllability is given. The time of controllability (or of possible reconstruction of a part of the state) is estimated “sufficiently large”.

The present paper is concerned with exact observability which is related to the notion of exact controllability developed in [9]. The semigroup approach used there is based on the model introduced in [10]. In the infinite dimensional setting described in [9], exact controllability means reachability of the state space operator domain because reachability of all the state space is not possible by finite dimensional control. Hence, the dual notion of observability is also adapted. The approach using the structural operator is not used. Considering the adjoint systems in the infinite dimensional framework, we construct a transposed neutral type system corresponding to this adjoint system. This relation between the semigroup and the neutral type system is different from that of the model given in [10].

These notions are important because they imply exponential stabilizability or exponential convergences for possible estimators.

The approach developed in [9] uses the theory of moments problem and allows the minimal time of exact controllability to be determined. We specify here how duality may be used and then we give not only the conditions for exact observability but also the minimal time of observability.

We consider the following neutral type system

$$\dot z(t) = A_{-1}\dot z(t-1) + \int_{-1}^{0} [A_2(\theta)\dot z(t+\theta) + A_3(\theta) z(t+\theta)]\,\mathrm{d}\theta \qquad (1)$$

where A_{-1} is a constant $n\times n$-matrix, and A_2, A_3 are $n\times n$-matrices whose elements belong to $L_2(-1,0)$.

If we introduce the linear operator L defined by

$$Lf = \int_{-1}^{0} A_2(\theta) f'(\theta) + A_3(\theta) f(\theta)\mathrm{d}\theta \qquad (2)$$

then the system may be written concisely as

$$\dot z(t) = A_{-1}\dot z(t-1) + Lz_t, \qquad z_t(\theta) = z(t+\theta).$$

This system may be represented, following the approach developed in [10], by an operator model in Hilbert space given by the equation

$$\dot x = \mathscr{A} x, \quad x(t) = \begin{pmatrix} v(t) \\ z_t(\cdot)\end{pmatrix}, \qquad (3)$$

where $\mathscr{A}$ is the infinitesimal generator of a C_0-semigroup given in the product space

$$M_2 = M_2(-1,0;\mathbb{R}^n) \stackrel{\text{def}}{=} \mathbb{R}^n \times L_2(-1,0;\mathbb{R}^n)$$

and defined by

$$\mathscr{A} x(t) = \mathscr{A}\begin{pmatrix} v(t) \\ z_t(\cdot)\end{pmatrix} = \begin{pmatrix} Lz_t(\cdot) \\ \mathrm{d}z_t(\theta)/\mathrm{d}\theta\end{pmatrix}, \qquad (4)$$

with the domain $\mathscr{D}(\mathscr{A}) \subset M_2$ given by

$$\left\{ \begin{pmatrix} v \\ \varphi(\cdot)\end{pmatrix} : \varphi(\cdot) \in H^1, v = \varphi(0) - A_{-1}\varphi(-1)\right\}, \qquad (5)$$

where $H^1 = H^1([-1,0];\mathbb{R}^n)$.

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The output of the system. We consider the finite dimensional observation

$$y(t)=\mathscr{C}x(t) \tag{6}$$

where $\mathscr{C}$ is a linear operator and $y(t)\in\mathbb{R}^p$ is a finite dimensional output. There are several ways to design the output operator $\mathscr{C}$ [11], [2], [6]. One of our goal in this paper is to investigate how to design a minimal output operator like

$$\mathscr{C}x(t)=Cz(t) \qquad \text{or} \qquad \mathscr{C}x(t)=Cz(t-1), \tag{7}$$

where C is a $p\times n$ matrix. More general output, for example with several and/or distributed delays are not considered in this paper. We want to use some result on exact controllability in order to analyze, by duality, the exact observability property in the infinite dimensional setting like, for example, in [12].

The operator $\mathscr{C}$ defined in (7) is linear but not bounded in M_2. However, in both cases it is admissible in the following sense:

$$\int_0^T \|\mathscr{C}S(t)x_0\|^2_{\mathbb{R}^n}\,\mathrm{d}t\le\kappa^2\|x_0\|^2_{M^2}, \qquad \forall x_0\in\mathscr{D}(\mathscr{A}),$$

because it is bounded on $\mathscr{D}(\mathscr{A})$. We recall that if $x_0\in\mathscr{D}(\mathscr{A})$ then $S(t)x_0\in\mathscr{D}(\mathscr{A})$, $t\ge 0$ (see for example [13]). In fact, $\mathscr{C}$ is admissible in the resolvent norm: $\|x_0\|_{-1}=\left\|(\lambda I-\mathscr{A})^{-1}x_0\right\|=\|R(\lambda,\mathscr{A})x_0\|$, $\lambda\in\rho(\mathscr{A})$. This is a consequence of the fact that $\mathscr{C}$ is a closed operator and takes value in a finite dimensional space (see [12, Definition 4.3.1] and comments on this Definition).

Definition 1.1: Let $\mathscr{K}$ be the output operator

$$\mathscr{K}:M_2\longrightarrow L_2(0,T;\mathbb{R}^p), \quad x_0\longmapsto\mathscr{K}x_0=\mathscr{C}S(t)x_0.$$

The system (1) is said to be observable (or approximately observable) if $\ker\mathscr{K}=\{0\}$ and exactly observable if

$$\int_0^T \|\mathscr{C}S(t)x_0\|^2_{\mathbb{R}^p}\,\mathrm{d}t\ge\delta^2\,\|x_0\|^2, \tag{8}$$

for some constant δ.

This is the classic definition. In the case of a neutral type system with a finite dimensional output (7) the exact observability in this sense is not possible. It may be possible if we consider another topology for the initial states x_0.
Unlike approximate observability, which does not depend on the topology, exact observability depends essentially on the topology in the space. We can show that, the given neutral type system is not exactly observable if we consider $x_0\in\mathscr{D}(\mathscr{A})$, with the norm of the graph and no longer in the topology of M_2. Moreover, in our case the relation (8) must be changed by taking a weaker norm for x_0, namely the resolvent norm $\|R(\lambda,\mathscr{A})x_0\|$ and considering the extension of the operator $\mathscr{K}$ to the completion of the space with this norm.

Exact observability can be investigated directly, but another way is to use the duality between exact observability and exact controllability. In [9] the conditions of exact controllability were given for the controlled system (33). In order to use the duality between observability and controllability, we need to compute the adjoint operator $\mathscr{K}^*$ in the duality with respect to the pivot space M_2 in the embedding

$$X_1\subset X=M_2\subset X_{-1}, \tag{9}$$

where $X_1=\mathscr{D}(\mathscr{A})$ with the graph norm noted $\|x\|_1$ and X_{-1} is the completion of the space M_2 with respect to the resolvent norm $\|x\|_{-1}=\|(\lambda I-\mathscr{A})^{-1}x\|$. The duality relation is

$$\langle\mathscr{K}x_0,u(\cdot)\rangle_{L_2(0,T;\mathbb{R}^p)}=\left\langle x_0,\mathscr{K}^*u(\cdot)\right\rangle_{X_1,X^{\mathrm{d}}_{-1}}, \tag{10}$$

where X^{d}_{-1} is constructed as X_{-1} with $\mathscr{A}^*$ instead of $\mathscr{A}$ (see [12] for example). Our purpose is to compute the adjoint operator

$$\mathscr{K}^*:L_2(0,T;\mathbb{R}^p)\to X^{\mathrm{d}}_{-1}.$$

The abstract formulation is well known. Exact controllability is dual with exact observability in the corresponding spaces with the corresponding topologies. It is expected that the operator $\mathscr{K}^*$ corresponds to a control operator for some adjoint system.

We then need to calculate the adjoint state operator $\mathscr{A}^*$ and the corresponding adjoint system in the same class: the class of neutral type systems. We shall see that the situation is not so simple.

This paper is organized as follows. First we give the duality relation in Section 2. Section 3 is concerned with the calculation of the adjoint system of neutral type . Then we return to the duality relation with the explicit expression of the adjoint system after formulation of exact controllability results. As the adjoint neutral type system has a slghtly different structure, we give an explicit relation between the new neutral type system and the original one. After that we can give the expression of duality between exact controllability and exact observability.

II. The adjoint operator $\mathscr{A}^*$

We now give the expression of the adjoint operator for $\mathscr{A}$ obtained by [14]. Of special interest is the characterization of the eigenvectors of $\mathscr{A}^*$ to be used in the next sections.

Theorem 2.1: The adjoint operator $\mathscr{A}^*$ is given by

$$\mathscr{A}^*\begin{pmatrix} w\\ \psi(\cdot)\end{pmatrix}=\begin{pmatrix} (A_2^*(0)\,w+\psi(0)\\ -\frac{\mathrm{d}[\psi(\theta)+A_2^*(\theta)\,w]}{\mathrm{d}\theta}+A_3^*(\theta)\,w\end{pmatrix}, \tag{11}$$

with the domain $\mathscr{D}(\mathscr{A}^*)$:

$$\left\{\left(w,\psi(\cdot)\right):\psi(\theta)+A_2^*(\theta)\,w\in H^1,\right.$$
$$\left.A_{-1}^*\left(A_2^*(0)\,w+\psi(0)\right)=\psi(-1)+A_2^*(-1)\,w\right\}. \tag{12}$$

The eigenvectors of $\mathscr{A}^*$ are given by

$$\begin{pmatrix} w\\ E^*(\lambda,\theta)\,w\end{pmatrix}, \qquad w\in\operatorname{Ker}\Delta^*(\lambda) \tag{13}$$

where $E^*(\lambda,\theta)$ and $\Delta^*(\lambda)$ are the matrices

$$E^*(\lambda,\theta) = \lambda e^{-\lambda\theta} - A_2^*(\theta) + \int_0^\theta e^{\lambda(s-\theta)}\left[A_3^*(s) + \lambda A_2^*(s)\right] ds,$$
$$\Delta^*(\lambda) = \lambda I - \lambda e^{-\lambda} A_{-1}^* - \int_{-1}^0 e^{\lambda s}\left[A_3^*(s) + \lambda A_2^*(s)\right] ds.$$

The eigenvalues are the roots of the equation $\det\Delta^*(\lambda) = 0$.

III. The adjoint system

In this section we give the expression of the adjoint system corresponding to the adjoint operator $\mathscr{A}^*$ as the operator $\mathscr{A}$ corresponds to the system (1).

Theorem 3.1: Let x be a solution of the abstract equation

$$\dot{x} = \mathscr{A}^* x, \quad x(t) = \begin{pmatrix} w(t) \\ \psi_t(\theta) \end{pmatrix}. \tag{14}$$

Then the function $w(t)$ is the solution of the neutral type equation

$$\dot{w}(t+1) = A_{-1}^*\dot{w}(t) + \int_{-1}^0 \left[A_2^*(\tau)\dot{w}(t+1+\tau) + A_3^*(\tau)w(t+1+\tau)\right] d\tau. \tag{15}$$

We want to find the corresponding neutral type equation in $\mathbb{R}^n$. Equation (14) may be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} w(t) \\ \psi_t(\theta)) \end{pmatrix} = \begin{pmatrix} A_2^*(0)w(t) + \psi_t(0) \\ -\frac{\partial[\psi_t(\theta) + A_2^*(\theta)w(t)]}{\partial\theta} + A_3^*(\theta)w(t) \end{pmatrix}$$

Let us denote by $r(\theta) = A_2^*(\theta)w + \psi(\theta)$ and

$$r(t,\theta) = A_2^*(\theta)w(t) + \psi_t(\theta) = A_2^*(\theta)w(t) + \psi_t(\theta). \tag{16}$$

Then $\mathscr{A}^*$ may be rewritten as

$$\mathscr{A}^*\begin{pmatrix} w \\ r(\theta) - A_2^*(\theta)w \end{pmatrix} = \begin{pmatrix} r(0) \\ -\frac{dr(\theta)}{d\theta} + A_3^*(\theta)w \end{pmatrix}, \tag{17}$$

and the system $\dot{x} = \mathscr{A}^* x$ as

$$\frac{\partial}{\partial t}\begin{pmatrix} w(t) \\ r(t,\theta) - A_2^*(\theta)w(t) \end{pmatrix} = \begin{pmatrix} r(t,0) \\ -\frac{\partial r(t,\theta)}{\partial\theta} + A_3^*(\theta)w(t) \end{pmatrix}. \tag{18}$$

The second line of this equation gives

$$\frac{\partial}{\partial t}r(t,\theta) + \frac{\partial}{\partial\theta}r(t,\theta) = A_2^*(\theta)\dot{w}(t) + A_3^*(\theta)w(t). \tag{19}$$

The general solution of this partial differential equation is

$$r(t,\theta) = f(t-\theta) + \int_0^\theta \left[A_2^*(\tau)\dot{w}(t-\theta+\tau) + A_3^*(\tau)w(t-\theta+\tau)\right] d\tau, \tag{20}$$

where $f(t-\theta)$ is the solution of the homogenous equation obtained from (19):

$$\frac{\partial}{\partial t}r(t,\theta) + \frac{\partial}{\partial\theta}r(t,\theta) = 0.$$

and the second term is a particular solution of (19).

The first line of the equation (18) gives

$$\dot{w}(t) = r(t,0). \tag{21}$$

From (20) (obtained from the second line), putting $\theta = 0$, we get with (21)

$$\dot{w}(t) = r(t,0) = f(t). \tag{22}$$

Then (20) and (22) allow $r(t,\theta)$ to be written as follows:

$$r(t,\theta) = \dot{w}(t-\theta) + \int_0^\theta \left[A_2^*(\tau)\dot{w}(t-\theta+\tau) + A_3^*(\tau)w(t-\theta+\tau)\right] d\tau. \tag{23}$$

From the definition of the domain $\mathscr{D}(\mathscr{A}^*)$ we obtain $A_{-1}^* r(0) = r(-1)$, or

$$A_{-1}^* r(0) - r(-1) = 0.$$

For the function $r(t,\theta)$, this condition reads

$$r(t,-1) = A_{-1}^* r(t,0) = A_{-1}^*\dot{w}(t) \tag{24}$$

and by (23) we have

$$r(t,-1) = \dot{w}(t+1) - \int_{-1}^0 \left[A_2^*(\tau)\dot{w}(t+1+\tau) + A_3^*(\tau)w(t+1+\tau)\right] d\tau. \tag{25}$$

Finally, from (24) and (25), we obtain the dual equation

$$\dot{w}(t+1) = A_{-1}^*\dot{w}(t) + \int_{-1}^0 \left[A_2^*(\tau)\dot{w}(t+1+\tau) + A_3^*(\tau)w(t+1+\tau)\right] d\tau. \tag{26}$$

On the other hand the solution of equation (18) is

$$e^{\mathscr{A}^* t}x_0 = \begin{pmatrix} w(t) \\ \psi_t(\theta) \end{pmatrix} = \begin{pmatrix} w(t) \\ r(t,\theta) - A_2^*(\theta)w(t) \end{pmatrix}, \tag{27}$$

where $w(t)$ is the solution of equation (26). If $x_0 \in X$ then it is a mild solution.

This result may also be formulated in the following way.

Theorem 3.2: Let x be a solution of the abstract equation

$$\dot{x} = \tilde{\mathscr{A}}x, \quad x(t) = \begin{pmatrix} w(t) \\ \psi_t(\theta) \end{pmatrix},$$

where the operator $\tilde{\mathscr{A}}$ is defined by

$$\tilde{\mathscr{A}}\begin{pmatrix} w \\ \psi(\cdot) \end{pmatrix} = \begin{pmatrix} A_2(0)y + \psi(0) \\ -\frac{d[\psi(\theta) + A_2(\theta)y]}{d\theta} + A_3(\theta)w \end{pmatrix},$$

with the domain

$$\mathscr{D}(\tilde{\mathscr{A}}) = \left\{(w,\psi(\cdot)) : \psi(\theta) + A_2(\theta)w \in H^1, \right.$$
$$\left. (A_{-1}A_2(0) - A_2(-1))\, w = \psi(-1) - A_{-1}\psi(0)\right\}.$$

Then the function $w(t)$ is the solution of the neutral type equation

$$\dot{w}(t+1) = A_{-1}\dot{w}(t) + \int_{-1}^0 [A_2(\tau)\dot{w}(t+1+\tau) + A_3(\tau)w(t+1+\tau)]\, d\tau. \tag{28}$$

Let us now specify the relation between the solutions of neutral type equations (28) and (1). Let us put

$$\begin{pmatrix} w(t) \\ \psi_t(\theta) \end{pmatrix} = e^{\tilde{\mathscr{A}}t}\tilde{x}_0 = e^{\tilde{\mathscr{A}}t}\begin{pmatrix} w(0) \\ \psi_0(\theta) \end{pmatrix},$$

and

$$\begin{pmatrix} v(t) \\ z_t(\theta) \end{pmatrix} = \begin{pmatrix} w(t+1) - A_{-1}w(t) \\ w(t+1+\theta) \end{pmatrix} = e^{\mathscr{A}t}\begin{pmatrix} v(0) \\ z_0(\theta) \end{pmatrix} = e^{\mathscr{A}t}\xi_0,$$

where $z_0(\theta) = w(t+1)$ and $v(0) = z_0(0) - A_{-1}z_0(-1)$. Our purpose is to give the explicit relation between the initial conditions $\tilde{x}_0$ and ξ_0:

$$\tilde{x}_0 = \begin{pmatrix} w(0) \\ \psi_0(\theta) \end{pmatrix}, \qquad \xi_0 = \begin{pmatrix} v(0) \\ z_0(\theta) \end{pmatrix}.$$

The formal relation between these vectors is

$$\tilde{x}_0 = \begin{pmatrix} w(0) \\ \psi_0(\theta) \end{pmatrix}) = F\xi_0 = F\begin{pmatrix} w(1) - A_{-1}w(0) \\ w(\theta+1) \end{pmatrix}.$$

Let us calculate the explicit expression for the linear operator F. From (23) and (16) we obtain

$$\begin{aligned} r(0,\theta) &= \psi_0(\theta) + A_2(\theta)w(0) = \\ &\dot{w}(-\theta) + \textstyle\int_0^\theta [A_2(\tau)\dot{w}(\tau-\theta) + A_3(\tau)w(\tau-\theta)]\,\mathrm{d}\tau, \end{aligned} \tag{29}$$

which can be written as

$$r(0,\theta) = \dot{w}(-\theta) + \int_0^\theta [A_2(\theta-s)\dot{w}(-s) + A_3(\theta-s)w(-s)]\,\mathrm{d}s.$$

Putting $w(-s) = \int_0^s \dot{w}(-\sigma)\mathrm{d}\sigma + w(0)$, we get

$$\begin{aligned} r(0,\theta) - \int_0^\theta A_3(\theta-s)\mathrm{d}s\cdot w(0) = \dot{w}(-\theta) + \\ \int_0^\theta \left[A_2(\theta-s)\dot{w}(-s) + A_3(\theta-s)\int_0^s \dot{w}(-\sigma)\mathrm{d}\sigma\right]\mathrm{d}s. \end{aligned} \tag{30}$$

This may be represented by the expression

$$r(0,\theta) - \int_0^\theta A_3(\theta-s)\mathrm{d}s\cdot w(0) = (I+\mathcal{V})\dot{w}(-s), \tag{31}$$

where $\mathcal{V}$ is the Volterra operator defined by

$$\mathcal{V}\mu(\cdot) = \int_0^\theta \left[A_2(\theta-s)\mu(s) + A_3(\theta-s)\int_0^s \mu(\sigma)\mathrm{d}\sigma\right]\mathrm{d}s.$$

The operator $\mathcal{V}$ is a compact linear operator from $L_2(-1,0;\mathbb{R}^n)$ to $L_2(-1,0;\mathbb{R}^n)$ with a spectrum $\sigma(\mathcal{V}) = \{0\}$. This implies that the operator $I+\mathcal{V}$ is bounded invertible on $L_2(-1,0;\mathbb{R}^n)$.

Let us now represent the operator F as a composition of operators according to the following commutative diagram

$$\begin{array}{ccc} \begin{pmatrix} v(0) \\ z_0(\theta) \end{pmatrix} & \xrightarrow{P} & \begin{pmatrix} w(0) \\ \dot{w}(-\theta) \end{pmatrix} \\ F\big\downarrow & & \big\downarrow Q \\ \begin{pmatrix} w(0) \\ \psi_0(\theta) \end{pmatrix} & \xleftarrow[R]{} & \begin{pmatrix} w(0) \\ (I+\mathcal{V})\dot{w}(-s) \end{pmatrix} \end{array}$$

where, as explained above, $\begin{pmatrix} v(0) \\ z_0(\theta) \end{pmatrix} = \begin{pmatrix} w(1) - A_{-1}w(0) \\ w(\theta+1) \end{pmatrix}$ and

$$(I+\mathcal{V})\dot{w}(-s) = r(0,\theta) + \int_0^{-\theta} A_3(\theta+s)\mathrm{d}s\cdot w(0).$$

The operators $P: X_1 \to M_2$ and $R: M_2 \to M_2$ are bounded invertible. Moreover, as $I+\mathcal{V}: L_2(-1,0;\mathbb{R}^n) \to L_2(-1,0;\mathbb{R}^n)$ is bounded invertible, then $Q: M_2 \to M_2$ is also bounded invertible.

We can summarize these considerations in the following theorem.

Theorem 3.3: The operator F representing the relation between initial conditions $\tilde{x}_0$ and ξ_0 corresponding to the neutral type systems (1) and (28) is linear bounded and bounded invertible from X_1 to M_2.

We also need the following property of the bounded operator F^{-1}.

Proposition 3.4: For $\lambda \neq \sigma(\tilde{\mathscr{A}})$, the operator

$$F^{-a} = F^{-1}(\lambda I - \tilde{\mathscr{A}})$$

can be extended to a bounded (and bounded invertible) operator from M_2 to M_2.

Proof: We need to prove that

$$\|F^{-1}(\lambda I - \tilde{\mathscr{A}})\tilde{x}_0\| \le C\|\tilde{x}_0\|, \qquad \tilde{x}_0 \in \mathscr{D}(\tilde{\mathscr{A}}), C > 0, \tag{32}$$

where $\|\cdot\|$ is the initial norm in M_2. Let L_0 and $\mathscr{D}_0$ be the subspaces

$$L_0 = \{(0,\psi(\cdot) : \psi(\cdot) \in L_2(-1,0;\mathbb{R}^n)\}, \qquad \mathscr{D}_0 = L_0 \cap \mathscr{D}(\tilde{\mathscr{A}}).$$

It is clear that $\mathscr{D}_0$ is of finite co-dimension n, and this implies that it is enough to prove the relation (32) for $\tilde{x}_0 \in \mathscr{D}_0$. Let $\tilde{x}_0 = (0, \psi_0(\cdot) \in \mathscr{D}_0$. The action of the operator $F^{-a} = F^{-1}(\lambda I - \tilde{\mathscr{A}})$ may be decomposed according to the following diagram

$$\begin{array}{ccccc} \begin{pmatrix} 0 \\ \psi_0(\cdot) \end{pmatrix} & \xrightarrow{(\lambda I - \tilde{\mathscr{A}})} & \begin{pmatrix} \psi_0(0) \\ \lambda\psi_0(\cdot) - \dot{\psi}_0(\cdot) \end{pmatrix} & \xrightarrow{R^{-1}} & R^{-1}\begin{pmatrix} \psi_0(0) \\ \lambda\psi_0(\cdot) - \dot{\psi}_0(\cdot) \end{pmatrix} \\ \big\downarrow F^{-a} & & & & \big\downarrow Q^{-1} \\ \begin{pmatrix} w(1) - A_{-1}\psi_0(0) \\ w(\theta+1) \end{pmatrix} & & \xleftarrow[P^{-1}]{} & & \begin{pmatrix} \psi_0(0) \\ \dot{w}(-\theta) \end{pmatrix} \end{array}$$

where

$$R^{-1}\begin{pmatrix} \psi_0(0) \\ \lambda\psi_0(\cdot) - \dot{\psi}_0(\cdot) \end{pmatrix} = \begin{pmatrix} \psi_0(0) \\ \lambda\psi_0(\cdot) - \dot{\psi}_0(\cdot) + \int\limits_0^{-\theta} A_3(\theta+s)\mathrm{d}s\psi_0(0) \end{pmatrix}$$

and the function $w(\cdot)$ is determinated from the equation (obtained from (29))

$$\begin{aligned} \lambda\psi_0(\theta) - \dot{\psi}_0(\theta) + A_2(\theta)\psi_0(0) = \\ \dot{w}(-\theta) + \int_0^\theta [A_2(\tau)\dot{w}(\tau-\theta) + A_3(\tau)w(\tau-\theta)]\,\mathrm{d}\tau \end{aligned}$$

with the initial condition $w(0) = \psi_0(0)$. Integrating the last equation from 0 to $-1-\theta$ and taking in account the initial condition, we obtain

$$\begin{aligned} -\psi(-1-\theta) - \lambda\int_0^{-1-\theta}\psi(\tau)\mathrm{d}\tau + \int_0^{-1-\theta} A_2(\tau)\mathrm{d}\tau\cdot\psi_0(0) = \\ -w(1+\theta) + \int_0^{-1-\theta}\left(\int_0^\tau A_2(s)\dot{w}(s-\tau) + A_3(s)w(s-\tau)\mathrm{d}s\right)\mathrm{d}\tau. \end{aligned}$$

After a transformation in the double integration, and with the initial condition, we get

$$\psi(-1-\theta)+\lambda\int_0^{-1-\theta}\psi(\tau)\mathrm{d}\tau = \int_0^{-1-\theta}\left(A_2(s)\dot{w}(1+\theta+\tau)-\int_0^{\tau}A_3(s)w(s-\tau)\mathrm{d}s\right)\mathrm{d}\tau,$$

and this can be written as

$$(I+\mathcal{V}_1)\psi_0(-1-\theta)=(I+\mathcal{V}_2)w(1+\theta),\quad \theta\in[-1,0],$$

where $\mathcal{V}_1$ and $\mathcal{V}_2$ are Volterra operators from $L_2(-1,0;\mathbb{R}^n$ to $L_2(-1,0;\mathbb{R}^n$. Both operators have spectra concentrated at $\{0\}$. Then

$$w(1+\theta)=(I+\mathcal{V}_2)^{-1}(I+\mathcal{V}_1)\psi_0(-1-\theta)=\mathcal{W}\psi_0(-1-\theta),$$

where $\mathcal{W}$ is a bounded invertible operator. Putting $\theta=-1$ in the relation between ψ_0 and w we get $\psi_0(-1)=w(1)$. This enables the final expression for the operator $F^{-a}=F^{-1}(\lambda I-\tilde{\mathscr{A}})$ on the set $\mathscr{D}_0$ to be obtained:

$$F^{-a}\begin{pmatrix}0\\ \psi_0(\theta)\end{pmatrix}=\begin{pmatrix}\psi_0(-1)-A_{-1}\psi_0(0)\\ \mathcal{W}\psi_0(-1-\theta)\end{pmatrix}=\begin{pmatrix}0\\ \mathcal{W}\psi_0(-1-\theta)\end{pmatrix},$$

because, by the definition of $\mathscr{D}_0=L_0\cap\mathscr{D}(\tilde{\mathscr{A}})$ we have

$$\psi_0(-1)-A_{-1}\psi_0(0)=0.$$

This means that the operator $F^{-1}(\lambda I-\tilde{\mathscr{A}})$ is continuous with continuous inverse from L_0 to L_0 (in the norm of M_2). Hence, $F^{-1}(\lambda I-\tilde{\mathscr{A}})$ may be extended as a bi-continuous operator on all M_2 because

$$F^{-1}(\lambda I-\tilde{\mathscr{A}})\mathscr{D}(\tilde{\mathscr{A}})=\mathscr{D}(\mathscr{A}).$$

The proof of the proposition is complete. ■

A direct consequence of this proposition is the following corollary.

Corollary 3.5: For all $x\in\mathscr{D}(\tilde{\mathscr{A}})$, for $\lambda\neq\sigma(\mathscr{A})$, we have

$$c\|x\|\leq\|(\lambda I-\tilde{\mathscr{A}})^{-1}Fx\|\leq C\|x\|.$$

IV. The control system and duality

Consider the controlled neutral type system

$$\dot{z}(t)=A_{-1}\dot{z}(t-1)+Lz_t+Bu(t), \tag{33}$$

where $u(t)\in L_2(0,T;\mathbb{R}^m)$ is a m-dimensional control vector-function. This system may be represented by an operator model in Hilbert space given by the equation

$$\dot{x}=\mathscr{A}x+\mathscr{B}u(t),\quad x(t)=\begin{pmatrix}v(t)\\ z_t(\cdot)\end{pmatrix}, \tag{34}$$

where $\mathscr{B}u=(Bu,0)$ is linear and bounded from $\mathbb{R}$ to M_2. We can note that $\mathscr{B}$ is not bounded from M_2 to X_1 (M_2 with the norm of the graph of $\mathscr{A}$) because $\mathscr{B}u=(Bu,0)\notin\mathscr{D}(\mathscr{A})$ if $Bu\neq 0$.

A. Exact controllability

Let us denote by $\mathscr{R}_T\subset M_2$ the reachable subspace of the system (34):

$$\mathscr{R}_T=\left\{R_Tu(\cdot)=\int_0^T \mathrm{e}^{\mathscr{A}t}\mathscr{B}u(t)\mathrm{d}t : u(t)\in L_2(0,T;\mathbb{R}^m)\right\},$$

where $R_T:L_2\mapsto M_2$ is a linear bounded operator.

As was pointed out in [15] and [9], $\mathscr{R}_T\subset\mathscr{D}(\mathscr{A})$ for all $T>0$. This implies that exact controllability may be defined as follows.

Definition 4.1: The system (34) is exactly controllable if $\mathscr{R}_T=\mathscr{D}(\mathscr{A})$.

The abstract condition of exact controllability is (see [16] for example)

$$\int_0^T\|\mathscr{B}^*\mathrm{e}^{\mathscr{A}^*t}x\|^2_{\mathbb{R}^m}\mathrm{d}t\geq\delta^2\|x\|^2_{X^{\mathrm{d}}_{-1}},\qquad\forall x\in\mathscr{D}(\mathscr{A}^*), \tag{35}$$

which means that the operator $R_T:L_2\mapsto X_1$ is onto. Here the space X^{d}_{-1} is the completion of the space $X=M_2$ with respect to the norm

$$\|x\|_{X^{\mathrm{d}}_{-1}}=\|(\lambda I-\mathscr{A}^*)^{-1}x\|,\quad\lambda\notin\sigma(\mathscr{A}^*).$$

For the system (33) the condition of exact controllability is given by the following theorem (see [9]).

Theorem 4.2: The system (33) is exactly controllable at time T if and only if, for all $\lambda\in\mathbb{C}$, the following two conditions are verified

i) *There is no vector $z\neq 0$ such that $B^*z=0$ and*

$$\Delta^*(\lambda)z=0,$$

ii) *There is no eigenvector z of the matrix A^*_{-1} such that $B^*z=0$.*

The time of controllability is $T>n_1(A_{-1},B)$.

The integer $n_1(A_{-1},B)$ is the controllability index of the pair (A_{-1},B) (see [17]). If the delay is h, then the critical time is n_1h.

Let us now consider the dual notion of observability for the adjoint system. The condition (39) is equivalent to the exact observability of the observable system

$$\begin{cases}\dot{x} &= \mathscr{A}^*x,\\ y &= \mathscr{B}^*x\end{cases} \tag{36}$$

and the corresponding neutral type system is the system (15). Then the conditions (i)–(ii) of Theorem 4.2 are necessary and sufficient for the exact controllability of the adjoint system (36). But what is the corresponding property for the associate neutral type system (26)? This will be shown in the following paragraph.

B. Duality

Consider the transposed controlled neutral type system

$$\dot{z}(t)=A^*_{-1}\dot{z}(t-1)+L^*z_t+C^*u(t), \tag{37}$$

where $L^*f=\int_{-1}^0 A_2^*(\theta)f'(\theta)+A_3^*(\theta)f(\theta)\mathrm{d}\theta$. Let $\mathscr{A}^\dagger$ be the generator of the semigroup $\mathrm{e}^{\mathscr{A}^\dagger t}$ generated by this equation (37). We cannot consider $\mathscr{A}^*$ for this system because this operator does not correspond directly to this system as the generator of the semigroup of solutions. The domain $\mathscr{D}(\mathscr{A}^\dagger)$ of the operator $\mathscr{A}^\dagger$ is given by

$$\left\{\begin{pmatrix} v \\ z(\cdot)\end{pmatrix} : z \in H^1([-1,0];\mathbb{C}^n), v = z(0) - A_{-1}^* z(-1)\right\}.$$

The spectrum of $\mathscr{A}^\dagger$ is $\sigma(\mathscr{A}^\dagger) = \{\lambda : \Delta^*(\lambda) = 0\} = \sigma(\mathscr{A}^*)$. Let $X_1^\dagger$ be $\mathscr{D}(\mathscr{A}^\dagger)$ with the norm

$$\|x\|_{X_1^\dagger} = \|(\lambda I - \mathscr{A}^\dagger)x\|, \quad \lambda \notin \sigma(\mathscr{A}^\dagger),$$

which is equivalent to the graph norm. Consider now the reachability operator for this system

$$R_T^\dagger u(t) = \int_0^T \mathrm{e}^{\mathscr{A}^\dagger t}\begin{pmatrix} C^* \\ 0\end{pmatrix} u(t)\mathrm{d}t.$$

From the properties of the operator R_T, we can deduce that $R_T^\dagger$ is linear, bounded from $L_2(0,T;\mathbb{R}^)$ to $X_1^\dagger$. The exact controllabilty for the system (37) can be formulated as

$$\mathscr{R}_T^\dagger = \operatorname{Im} R_T^\dagger = X_1^\dagger.$$

The conditions of exact controllability for this system (37) may be obtained directly from Theorem 4.2.

Let us now consider the corresponding space $X_{-1}^\dagger$ of linear functionals on $X_1^\dagger$ as a the completion of the space $X = M_2$ with respect to the norm

$$\|x\|_{X_{-1}^\dagger} = \|(\lambda I - \mathscr{A}^\dagger)^{-1}x\|, \quad \lambda \notin \sigma(\mathscr{A}^\dagger).$$

We then have the embedding

$$X_1^\dagger \subset X = M_2 \subset X_{-1}^\dagger. \tag{38}$$

Then, for $x \in X_1^\dagger$ and $y \in X_{-1}^\dagger$, the functional acts as

$$\langle x, y\rangle_{X_1^\dagger, X_{-1}^{\dagger d}} = \left\langle (\lambda I - \mathscr{A}^\dagger)^{-1}x, (\lambda I - \mathscr{A}^{\dagger *})^{-1}y\right\rangle_X,$$

where $\mathscr{A}^{\dagger *}$ is the adjoint of the operator $\mathscr{A}^\dagger$ in M_2, and the space $X_{-1}^{\dagger d}$ is constructed as $X_{-1}^{\dagger d}$ with $\mathscr{A}^{\dagger *}$ instead of $\mathscr{A}^\dagger$ (see [12] for example).

Let us note that the operator $\mathscr{A}^{\dagger *}$ is in fact the operator $\tilde{\mathscr{A}}$ defined in Section III (see Theorem 3.2 and later). We shall use the properties obtained for this operator.

Let us now consider the adjoint $R_T^{\dagger *}$ of $R_T^\dagger$ with respect to the duality induced by $X_1^\dagger$ and $X_{-1}^{\dagger d}$ with the pivot space $X = M_2$. Let $x_0 \in X$, then

$$\begin{aligned}\left\langle R_T^\dagger u(\cdot), x_0\right\rangle_{X_1, X_{-1}^{d}} &= \left\langle R_T^\dagger u(\cdot), x_0\right\rangle_X \\ &= \left\langle \int_0^T \mathrm{e}^{\mathscr{A}^\dagger t}\begin{pmatrix} C^* \\ 0\end{pmatrix} u(t)\mathrm{d}t, x_0\right\rangle_X \\ &= \int_0^T \left\langle \begin{pmatrix} C^* \\ 0\end{pmatrix} u(t), \mathrm{e}^{\mathscr{A}^{\dagger *} t}x_0\right\rangle_X \mathrm{d}t.\end{aligned}$$

Suppose now that $x_0 = \begin{pmatrix} w(0) \\ \psi_0(\theta)\end{pmatrix} \in \mathscr{D}(\mathscr{A}^{\dagger *})$.

Then, as a consequence of the results in Section III, namely from (27) but for the operator $\mathscr{A}^{\dagger *} = \tilde{\mathscr{A}}$, we obtain

$$\mathrm{e}^{\mathscr{A}^{\dagger *} t}x_0 = \begin{pmatrix} w(t) \\ \psi_t(\theta)\end{pmatrix} = \begin{pmatrix} w(t) \\ r(t, \theta - A_2(\theta)w(t)\end{pmatrix}, \quad t \geq 0.$$

Hence,

$$\begin{aligned}\left\langle R_T^\dagger u(\cdot), x_0\right\rangle_X &= \textstyle\int_0^T \langle u(t), Cw(t)\rangle_{\mathbb{R}^p}\,\mathrm{d}t \\ &= \left\langle u(\cdot), R_T^{\dagger *}x_0\right\rangle_{L_2}.\end{aligned}$$

On the other hand, we can write $x_0 = F\xi_0$ (cf. Proposition 3.4), where

$$\xi_0 = \begin{pmatrix} v(0) \\ z_0(\theta)\end{pmatrix} = \begin{pmatrix} z_0(0) - A_{-1}z_0(-1) \\ z_0(\theta)\end{pmatrix} = \begin{pmatrix} w(1) - A_{-1}w(0) \\ w(\theta+1)\end{pmatrix},$$

and then

$$\mathrm{e}^{\mathscr{A} t}\xi_O = \begin{pmatrix} w(t+1) - A_{-1}w(t) \\ w(t+1+\theta)\end{pmatrix}.$$

Let $\mathscr{K}$ be the output operator introduced in Definition 1.1. Then

$$\langle u(\cdot), \mathscr{K}\xi_0\rangle_{L_2} = \left\langle u(\cdot), \mathscr{C}\mathrm{e}^{\mathscr{A} t}\xi_0\right\rangle_{L_2} = \begin{cases} \int_0^T \langle u(t), Cw(t)\rangle_{\mathbb{R}^p}\,\mathrm{d}t & \text{if} \quad \mathscr{C}x(t) = Cz(t-1) \\ \int_0^T \langle u(t), Cw(t+1)\rangle_{\mathbb{R}^p}\,\mathrm{d}t & \text{if} \quad \mathscr{C}x(t) = Cz(t)\end{cases},$$

which implies for all $x_0 \in X$:

$$\mathscr{K}F^{-1}x_0 = \begin{cases} \mathscr{R}_T^{\dagger *}x_0 & \text{if} \quad \mathscr{C}x(t) = Cz(t-1), \\ \mathrm{e}^{\mathscr{A}^{\dagger *}}\mathscr{R}_T^{\dagger *}x_0 & \text{if} \quad \mathscr{C}x(t) = Cz(t).\end{cases}$$

We can now formulate our main result on duality.

Theorem 4.3: 1. The system (1) with the output

$$y(t) = \mathscr{C}x(t) = Cz(t-1)$$

is exactly observable in the interval $[0,T]$, *i.e.*

$$\|\mathscr{K}x_0\|^2 = \int_0^T \left\|\mathscr{C}\mathrm{e}^{\mathscr{A} t}x_0\right\|^2 \geq \delta^2\|x_0\|^2$$

if and only if the adjoint system (37) is exactly controllable a time T, i.e.

$$\mathscr{R}_T = R_T(L_2(0,T;\mathbb{R}^p) = X_1^\dagger = \mathscr{D}(\mathscr{A}^\dagger).$$

2. If $\det A_{-1} \neq 0$, *assertion 1 of the theorem is verified for the output*

$$y(t) = \mathscr{C}x(t) = Cz(t),$$

and for the same time T.

Proof: The condition of exact controllability is (see [16] for example)

$$\|R_T^{\dagger *}x_0\|_{L_2} \geq \delta\left\|(\lambda I - \mathscr{A}^{\dagger *})^{-1}x_0\right\|, \qquad \forall x \in X. \tag{39}$$

Let $\xi_0 = F^{-1}x_0 \in \mathscr{D}(\mathscr{A})$, where $F : \mathscr{D}(\mathscr{A}) \to M_2$ is the bounded invertible operator defined in Section III. Let us remember that

$$R_T^{\dagger *}x_0 = \mathscr{K}F^{-1}x_0 = \mathscr{K}\xi_0.$$

Then the inequality (39) is equivalent to

$$\|\mathscr{K}\xi_0\|_{L_2} \geq \delta\left\|(\lambda I - \mathscr{A}^{\dagger *})^{-1}F\xi_0\right\|, \qquad \xi_0 \in \mathscr{D}(\mathscr{A}). \tag{40}$$

Suppose now that the relation (40) is verified for all $\xi_0 \in \mathscr{D}(\mathscr{A})$, then from Corollary 3.5 we obtain

$$\|\mathscr{K}\xi_0\|_{L_2} \geq \delta\left\|(\lambda I - \mathscr{A}^{\dagger *})^{-1}F\xi_0\right\| \geq \underbrace{\delta c}_{=\delta_1}\|\xi_0\|,\ \xi_0 \in \mathscr{D}(\mathscr{A}).$$

This inequality can be extended by continuity to $\xi_0 \in M_2$:

$$\|\mathscr{K}\xi_0\|_{L_2} \geq \delta_1 \|\xi_0\|, \quad \forall \xi_0 \in M_2$$

Conversely, suppose that the preceding relation is verified. For $\xi_0 \in \mathscr{D}(\mathscr{A})$, and from Corollary 3.5, we get

$$\|\xi_0\| \geq \frac{1}{C}\left\|\left(\lambda I - \mathscr{A}^{\dagger *}\right)^{-1} F\xi_0\right\|,$$

and then

$$\|\mathscr{K}\xi_0\|_{L_2} \geq \frac{\delta_1}{C}\left\|\left(\lambda I - \mathscr{A}^{\dagger *}\right)^{-1} F\xi_0\right\|.$$

This is the relation (40) with $\delta = \delta_1/C$. As the relations (39) and (40) are equivalent, the first assertion of the theorem is proved.
To prove item 2 of the theorem, it is sufficient to remark that the condition $\det A_{-1} \neq 0$ is equivalent to the fact that the operator $e^{\mathscr{A}}$ is bounded invertible ($e^{\mathscr{A}t}$ is a group), and then the relations (39) and (40) are equivalent. ■
From this result and from Theorem 4.2 we can formulate the condition of exact observability.

Theorem 4.4: 1. The system (1) with the output $y = Cz(t-1)$ *is exactly observable over* $[0, T]$ *if and only if*

i) *For all* $\lambda \in \mathbb{C}$, $\operatorname{rank}\begin{pmatrix}\Delta^*(\lambda) & C^*\end{pmatrix} = n$,
ii) *For all* $\lambda \in \mathbb{C}$, $\operatorname{rank}\begin{pmatrix}\lambda I - A_{-1}^* & C^*\end{pmatrix} = n$,
iii) $T > n_1(A_{-1}^*, C^*)$, *where* n_1 *is the first index of controllability for the par* (A_{-1}^*, C^*).

2. If $\det A_{-1} \neq 0$, *then assertion 1 is verified for the output* $y(t) = Cz(t)$.

V. Examples

Let us give some simple examples to illustrate our results.
Example 1. Consider the system

$$\dot{z}(t) = \dot{z}(t-1),$$

where $z(t) \in \mathbb{R}^n, n > 1$, with two possible outputs

$$y_0(t) = \mathscr{C}_0 x(t) = z(t), \qquad y_1(t) = \mathscr{C}_1 x(t) = z(t-1).$$

The conditions of observability are verified, and the system is exactly observable for the output y_0 or y_1.
Example 2. Consider the system

$$\dot{z}(t) = 0,$$

where $z(t) \in \mathbb{R}^n, n > 1$, with two possible output

$$y_0(t) = \mathscr{C}_0 x(t) = z(t), \qquad y_0(t) = \mathscr{C}_1 x(t) = z(t-1).$$

The system with the output y_1 is exactly observable for the time $T > 1$ and not observable for $T = 1$. The system with the output y_0 is not observable for any time $T > 0$.

VI. Conclusion

Duality between exact controllability and observability for a wide class of neutral type systems is analyzed by an infinite dimensional approach. We give an explicit neutral type system wich corresponds to the abstract adjoint system.

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