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A Generalized Existence Theorem in Impulsive Control Problems

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Abstract— **Some optimal control problems do not have solution in the class of classic controls. This suggests the need of a relaxation or extension of the control problem ensuring the existence of a solution in some enlarged class of controls. This work aims at the development of an extension for optimal control problems with nonlinear control dynamics and control functions which take values in some closed, but not necessarily bounded, set. To achieve this goal, we exploit the approach of R.V. Gamkrelidze based on the concept of generalized controls. However it is adapted for the case of discontinuous arcs. This leads to the notion of generalized impulsive control.**

I. INTRODUCTION

The aim of the present research is to give a strict mathematical meaning to discontinuous solutions that may arise in the following optimal control problem:

$$\begin{aligned} &\text{Minimize} && \int_{t_0}^{t_1} g_0(x,v,t)dt, \\ &\text{subject to} && \dot{x} = g(x,v,t), \;\; t \in [t_0,t_1], \\ & && x(t_0) \in A, \; x(t_1) \in B, \\ & && v(t) \in V \text{ a.a.}\, t \in [t_0,t_1], \\ & && \varphi(x(t),t) \le 0 \;\; \forall\, t \in [t_0,t_1], \end{aligned} \tag{1}$$

where $g_0 : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^1 \to \mathbb{R}^1$, $g : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^1 \to \mathbb{R}^n$, and $\varphi : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^l$ are given continuous maps, A, and B are given closed subsets of $\mathbb{R}^n$, V is a given closed subset of $\mathbb{R}^k$, *not necessarily bounded*, and $v(t)$ is a control function. The function φ defines the so-called state constraints.

In what follows, we associate the control problem (1) with some a priori given scalar function $\omega(\xi) : \mathbb{R}_+ \to \mathbb{R}_+$ which is nonnegative, increasing and continuous. The purpose of introducing this function is to characterize the growth of the dynamics at infinity. Simple examples of $\omega(\xi)$ are: ξ (linear growth), ξ^2 (quadratic growth), ξ^p, e^ξ, etc.

Once ω is chosen, assume that the admissible control function $v(t)$ in (1) is such that the function $\omega(|v|)$ is integrable. So, when $w(\xi) = \xi$, v is a $\mathbb{L}_1$-function, when $w(\xi) = \xi^2$, v is a $\mathbb{L}_2$-function, etc. Thus, the function $\omega(\xi)$ determines the class of admissible controls in (1).

To achieve the above goal, consisting in extension of (1), we exploit the approach to extension by R.V. Gamkrelidze based on generalized controls, see [1]. Our extension is a certain upgrade of this extension, however it is related to *discontinuous arcs* unlike continuous arcs considered in [1]. Ultimately, the proposed extension links the approach developed by R.V. Gamkrelidze with the approaches of R.W. Rishel, J. Warga, R.T. Rockafellar in [2], [3], [4], and of some other authors (see [5], [6], [7]). Moreover, it also generalizes the extension from [8], [9], [10] performed for v-linear dynamics with separated control variables.

II. PRELIMINARIES

To perform the extension, it is necessary to compactify the space $\mathbb{R}^k$ by adding a set S_∞^{k-1}, called sphere at infinity. "Sphere at infinity" means that there is a homeomorphism $\Pi : S_\infty^{k-1} \to S^{k-1}$, where S^{k-1} is the unit sphere in $\mathbb{R}^k$. The compactified space $\bar{\mathbb{R}}^k := \mathbb{R}^k \cup S_\infty^{k-1}$ is endowed with a natural topology in which any sequence of points $v_i \in \mathbb{R}^k$ converges to the point $l \in S_\infty^{k-1}$ if and only if $|v_i| \to \infty$ and

$$v_i = |v_i| \cdot \Pi(l) + o(|v_i|).$$

Note that the compact space $\bar{\mathbb{R}}^k$ is topologically equivalent to the closed unit ball $B_{\mathbb{R}^k}$ in $\mathbb{R}^k$ due to the following homeomorphism Θ:

$$\Theta(v) = \frac{v}{1+|v|}, \;\; v \in \mathbb{R}^k,$$

and $\Theta(v) = \Pi(v)$ when $v \in S_\infty^{k-1}$.

Denote

$$V_\infty := \Pi^{-1}\left(\operatorname*{Limsup}_{|v| \xrightarrow{V} \infty} \frac{v}{|v|} \right).$$

Here the notation "Limsup" signifies the sequential Painlev-Kuratowski upper/outer limit, (see its definition, for example, in [11], p. 3); the notation $|v| \xrightarrow{V} \infty$ means that $|v| \to \infty$ and the limit is taken by the points $v \in V$ only.

Now, the set $\bar{V} := V \cup V_\infty$ is compact.

Let us introduce our main hypothesis about the right-hand side g and function g_0. Assume that there exists a continuous function

$$g^\infty : \mathbb{R}^n \times S^{k-1} \times \mathbb{R}^1 \to \mathbb{R}^n,$$

such that, $\forall\, e \in S^{k-1}$,

$$\lim_{v \to \Pi^{-1}(e)} \frac{g(x,v,t)}{\omega(|v|)} = g^\infty(x,e,t) \;\; \forall\, x,t \in \mathbb{R}^n \times \mathbb{R}^1.$$

Then, there exists a continuous map $\bar{g} : \mathbb{R}^n \times \bar{\mathbb{R}}^k \times \mathbb{R}^1 \to \mathbb{R}^n$ defined by

$$\bar{g}(x,v,t) = \begin{cases} \dfrac{g(x,v,t)}{1+\omega(|v|)} & \text{if } v \in \mathbb{R}^k, \\ g^\infty(x,\Pi(v),t) & \text{if } v \in S_\infty^{k-1}. \end{cases}$$

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In a similar way, assume that there exist functions $g_0^\infty : \mathbb{R}^n \times S^{k-1} \times \mathbb{R}^1 \to \mathbb{R}^1$, and $\bar{g}_0 : \mathbb{R}^n \times \bar{\mathbb{R}}^k \times \mathbb{R}^1 \to \mathbb{R}^1$ which are defined just as above but using g_0 instead of g.

We shall use the following assumptions.

H1) Functions $\bar{g}$, and $\bar{g}_0$ introduced above do exist. Moreover the function $\bar{g}$ is such that:

$$|\bar{g}(x,v,t)| \le m(t)\kappa(|x|)\ \ \forall\,(x,v,t) \in \mathbb{R}^n \times \bar{V} \times \mathbb{R}^1,$$

where m is some locally integrable function and $\kappa : \mathbb{R}^1 \to \mathbb{R}^1$ is such that

$$\frac{\kappa(|x|)}{1+|x|} \le \text{const}\ \ \forall x.$$

H2) Functions $\bar{g}$, $\bar{g}_0$ are continuously differentiable in x, and t for all $v \in \bar{\mathbb{R}}^k$.

Definition 1: The control problem (1) is said to allow the impulsive extension of order ω provided the hypothesis H1) is satisfied and, at least one of the functions g^∞ or g_0^∞ is not a zero function.

Consider a scalar Borel measure $\mu : \mathcal{B}(T) \to [0,+\infty)$, $T = [t_0, t_1]$. Here, $\mathcal{B}(T)$ stands for the σ-algebra of Borel subsets of T.

Denote by $\mathcal{D}(t;\mu)$ the Radon-Nikodym derivative:

$$\mathcal{D}(t;\mu) := \frac{d\ell}{d(\ell+\mu)},$$

where ℓ is the Lebesgue measure on the real line (length).

Note that $\mathcal{D}(\cdot;\mu)$ is $(\ell+\mu)$-measurable and takes values in $[0,1]$.

In what follows, the principal role will be played by the so called discontinuous time-variable change introduced first by H. Lebesgue. This variable change is given by the function $\pi : T \to [0,c]$,

$$\pi(t) = t - t_0 + \mu([t_0,t]),\ t \in (t_0,t_1],\ \ \pi(t_0) = 0, \qquad (2)$$

where $c = t_1 - t_0 + \|\mu\|$. Here $\|\mu\| = \mu([t_0,t_1])$ is the total variation of the Borel measure.

It is a straightforward task to derive that there exists an inverse function $\theta(s)$: $[0,c] \to T$ such that

a) $\theta(s)$ is monotonically increasing;
b) $\theta(s)$ is absolutely continuous and $|\theta(s)-\theta(t)| \le |s-t|$ $\forall s,t$;
c) $\theta(s) = \tau$, $\forall s \in \Gamma_\tau$, $\forall \tau \in T$, where $\Gamma_\tau = [\pi(\tau^-), \pi(\tau^+)]$.

Note that the function $\pi(t)$ maps $(\ell+\mu)$-measurable sets into ℓ-measurable sets. Indeed, it follows directly from the definition of π and from the representation of set as a union of Borel and zero measure sets. Therefore, if the set E is $(\ell+\mu)$-measurable, then $\theta^{-1}(E)$ is measurable. This implies that $v(\theta(s))$ is measurable provided that $v(t)$ is $(\ell+\mu)$-measurable. So, the following variable change in the integral is possible:

$$\int_{t_0}^{t_1} v(t)d\mu = \int_0^c v(\theta(s))m(\theta(s))ds,$$

where $m(t)$ is the Radon-Nikodym derivative of measure μ with respect to $\ell+\mu$.

By $\mathrm{Ds}(\mu)$ we designate the atomic set of μ, i.e.,

$$\mathrm{Ds}(\mu) := \{\tau \in T : \mu(\tau) > 0\}.$$

Recall that, due to Gamkrelidze, [1], the generalized control is a weakly measurable family of probabilistic Radon measures $\nu_t : \mathcal{B}(V) \to [0,1]$, $t \in [t_0,t_1]$. "Weakly measurable" means that, for any continuous scalar function $h(v,t)$, the function

$$\langle h(v,t), \nu_t \rangle := \int_V h(v,t)d\nu_t$$

is measurable in t. Similarly, "weakly μ-measurable" means that the above function is μ-measurable.

If the set V is compact, then the set of generalized controls is weakly sequentially compact, [1]. However, when the set V is unbounded this is not the case anymore. In what follows, we shall widely use generalized controls and their properties. There is extensive literature on the issue of generalized controls, aside from [1], see, for example, [12], [13], [14] and the bibliography cited therein.

III. EXTENSION CONCEPT

The extension for problem (1) acquires the form:

$$\begin{aligned}
&\text{Minimize} && \int_{t_0}^{t_1} \langle \bar{g}_0(x,v,t), d\mathfrak{c} \rangle, && (3)\\
&\text{subject to} && dx = \langle \bar{g}(x,v,t), d\mathfrak{c} \rangle, && (4)\\
& && x(t_0) \in A,\ x(t_1) \in B, && (5)\\
& && \varphi(x,t) \le 0, && (6)\\
& && \mathfrak{c} = \{\mu, \nu_t, \nu_s^\tau\},\ \ \mathrm{supp}(\mathfrak{c}) \subseteq \bar{V}. &&
\end{aligned}$$

The above formulas and notations require clarification. Below, we consistently give the necessary definitions.

The symbol $\mathfrak{c}$ designates *generalized impulsive control*. By definition, it consists of three components:

- $\mu : \mathcal{B}(T) \to [0,+\infty)$ is a non-negative scalar Borel measure;
- $\nu_t : \mathcal{B}(\bar{V}) \to [0,1]$ is a weakly $(\ell+\mu)$-measurable family of Radon probabilistic measures concentrated on $\bar{V}$, depending on $t \in [t_0,t_1]$, such that

$$\mathcal{D}(t;\mu) + \int_{\bar{V}} \frac{\omega(|v|)}{1+\omega(|v|)} \cdot d\nu_t = 1 \qquad (7)$$

a.a. t w.r.t. $\ell+\mu$; and
- $\nu_s^\tau : \mathcal{B}(V_\infty) \to [0,1]$ is a family of Radon probabilistic measures concentrated on V_∞, depending on $s \in [0,1]$, and on $\tau \in \mathrm{Ds}(\mu)$, weakly ℓ-measurable in s for each $\tau \in \mathrm{Ds}(\mu)$.

Above, we identify the Borel measure μ with its unique Lebesgue-Stieltjes extension. In the sequel, "μ-measurable" means measurability in the sense of Lebesgue.

Thus, the families of measures $\{\nu_t\}_{t\in T}$, and $\{\nu_s^\tau\}_{s\in[0,1]}$, for each $\tau \in \mathrm{Ds}(\mu)$, are generalized controls in the sense of R.V. Gamkrelidze. The family of generalized controls $\{\nu_s^\tau\}_{\tau\in\mathrm{Ds}(\mu)}$ is called *attached* to the control measure μ.

The symbolic notation $\operatorname{supp}(\mathfrak{c}) \subseteq \bar{V}$ refers to the above conditions imposed on the supports of ν_t, ν_s^τ.

Let us now proceed to the concept of trajectory. Denote by $x_\tau(s)$ the solution to the attached differential system:

$$\begin{cases} \dot{x}_\tau(s) = \Delta_\tau \cdot \langle \bar{g}(x_\tau(s), v, \tau), \nu_s^\tau \rangle, \ s \in [0,1], \\ x_\tau(0) = x(\tau^-), \end{cases}$$

where $\Delta_\tau := \mu(\{\tau\})$.

The function of bounded variation $x(t)$ is said to be solution to the differential equation (4) corresponding to the initial value $x_A \in A$, if and only if

$$\begin{aligned} x(t) \ &= \ x_A + \int_{t_0}^{t} \langle \bar{g}(x, v, \varsigma), \nu_\varsigma \rangle d(\ell + \mu_c) \\ &\qquad + \sum_{\tau \in \mathrm{Ds}(\mu)\colon \tau \le t} (x_\tau(1) - x_\tau(0)). \end{aligned}$$

for all $t \in (t_0, t_1]$, and $x(t_0) = x_A$.

By using this definition, it is easy to clarify the notion of integral in (3). According to what has been already introduced:

$$\begin{aligned} \int_{t_0}^{t_1} \langle \bar{g}_0(x, v, t), d\mathfrak{c} \rangle \ &= \ \int_{t_0}^{t_1} \langle \bar{g}_0(x, v, \varsigma), \nu_\varsigma \rangle d(\ell + \mu_c) \\ &\quad + \sum_{\tau \in \mathrm{Ds}(\mu)\colon \tau \le t} \int_0^1 \Delta_\tau \langle \bar{g}_0(x_\tau(s), v, \tau), \nu_s^\tau \rangle ds. \end{aligned}$$

It remains to give a meaning to the state constraints (6). Note that the state constraints $\varphi(x,t) \le 0$ here should be understood in a wider sense than that of a conventional inequality. This is due to the presence of the attached family of trajectories $x_\tau(s)$.

Namely, $\varphi(x,t) \le 0 \ \Leftrightarrow$

$$\begin{cases} \varphi(x(t), t) \le 0, & \forall\, t \in T, \\ \varphi(x_\tau(s), \tau) \le 0, & \forall\, s \in [0,1], \ \forall\, \tau \in \mathrm{Ds}(\mu). \end{cases}$$

Problem (3)–(6) indeed represents an extension of problem (1) since, for any admissible control v of problem (1), there is a control $\mathfrak{c}$ of the problem (3)–(6) such that the corresponding trajectories $x(t)$ and the values of the cost functional are equal. Indeed, let $v(t)$ be a control function. Consider the absolutely continuous Borel measure

$$\mu(C) = \int_C \omega(|v(t)|) dt, \ \ C \in \mathcal{B}(T),$$

and take $\nu_t = \delta_{v(t)}$ a.a. $t \in T$, where δ_r designates the Dirac's measure concentrated at point $r \in \mathbb{R}^m$. This is clear that $\mathcal{D}(t;\mu) = \dfrac{1}{1 + \omega(|v(t)|)}$, and hence ν_t satisfies (7).

By taking into account definition of $\bar{g}$:

$$\begin{aligned} x(t) \ &= \ x_A + \int_{t_0}^{t} \langle \bar{g}(x, v, \varsigma), d\mathfrak{c} \rangle \\ &= \ x_A + \int_{t_0}^{t} \int_{v \in V} \left\langle \frac{g(x(\varsigma), v, \varsigma)}{1 + \omega(|v(\varsigma)|)}, d\delta_{v(\varsigma)} \right\rangle \\ &\qquad\qquad (1 + \omega(|v(\varsigma)|)) d\varsigma \\ &= \ x_A + \int_{t_0}^{t} g(x(\varsigma), v(\varsigma), \varsigma) d\varsigma. \end{aligned}$$

The pair $(x, \mathfrak{c})$ is called control process if (4) holds true. A control process is said to be admissible if the endpoint constraints (5) and state constraints (6) are satisfied. Let us denote the set of all admissible processes by $\mathcal{P}$. An admissible process $(x^*, \mathfrak{c}^*)$ is said to be optimal or solution if the integral in (3) reaches on $(x^*, \mathfrak{c}^*)$ its least possible finite value over all elements from $\mathcal{P}$.

The extension concept expressed by the problem (3)–(6) naturally generalizes the approaches by R.V. Gamkrelidze from [1], by R.W. Rishel from [2], by R.T. Rockafellar from [4], and also the recently constructed extension from [9] arranged for the linear in v case.

IV. EXISTENCE THEOREM

The main goal of any extension undertaken in Optimal Control theory is to ensure the existence of solution to the extended problem. The following theorem demonstrates consistency of the extension (3)–(6) in the just mentioned sense.

Theorem 1: Suppose that the control problem (1) allows the impulsive extension of order ω. Let the set A or B be compact, $\mathcal{P} \neq \varnothing$, and assume that there exist constants κ, and $C > 0$ such that for any $(x, \mathfrak{c}) \in \mathcal{P}$, where $\mathfrak{c} = \{\mu, \nu_t, \nu_s^\tau\}$, at least one of the following conditions is satisfied:

a) $\|\mu\| \le \kappa \displaystyle\int_{t_0}^{t_1} \langle \bar{g}_0(x, v, t), d\mathfrak{c} \rangle + C$;
b) $\|\mu\| \le C$.

Then, the problem (3)–(6) has a solution.

The proof is based on the discontinuous time variable change. We propose only a sketch below. The main idea of the proof is to reduce the problem (3)–(6) to a conventional convex control problem by using the discontinuous time variable change π and the "cosmic"-like variable change Θ introduced in Section II.

Let us set

$$\Theta_\omega(v) = \frac{\omega(|v|) \cdot v}{|v|(1 + \omega(|v|))}, \ \ v \in \mathbb{R}^k,$$

and $\Theta_\omega(v) = \Pi(v)$ when $v \in S_\infty^{k-1}$.

Consider the conventional optimal control problem with generalized controls:

$$\begin{aligned} \text{Minimize} \quad & \int_0^{s_1} \langle f_0(x, u, \chi), \xi_s \rangle ds, \\ \text{subject to} \quad & \dot{x} = \langle f(x, u, \chi), \xi_s \rangle, \\ & \dot{\chi} = \alpha, \ \text{a.a.}\ s \in [0, s_1], \\ & x(0) \in A, \ x(s_1) \in B, \\ & \chi(0) = t_0, \ \chi(s_1) = t_1, \\ & \varphi(x, \chi) \le 0, \\ & \alpha(s) \in [0,1], \\ & \alpha(s) + \langle |u|, \xi_s \rangle = 1 \ \text{a.a.}\ s, \\ & \operatorname{supp}(\xi_s) \subseteq U \ \forall\, s \in [0, s_1]. \end{aligned} \tag{8}$$

Here, $f_0(x, u, \chi) = \bar{g}_0(x, \Theta_\omega^{-1}(u), \chi)$, $f(x, u, \chi) = \bar{g}(x, \Theta_\omega^{-1}(u), \chi)$, and $U = \Theta_\omega(\bar{V})$.

The time s_1 in (8) is not fixed, unlike in the problem (3)–(6) considered on the fixed time interval $[t_0, t_1]$. The controls in (8) are $\alpha(s)$, and ξ_s, where α is usual control (measurable function) and ξ_s is generalized control having support in U. Note that the set U is compact since it is the image of the compact $\bar{V}$ at the continuous map Θ_ω. Thereby the problem (8) is a conventional autonomous convex control problem with free time in the sense of [1].

In our work, we show how the two problems, (3)–(6) and (8), are equivalent. That is, for every process $(x, \mathfrak{c}) \in \mathcal{P}$ there exists an admissible process $(\tilde{x}, \chi, \alpha, \xi_s, s_1)$ of the problem (8), such that the values of the minimizing integrals are equal, and vice-versa. If so, if a solution exists to one of the problems, then it also exists in the other problem. However the existence of solution to the auxiliary problem (8) can be easily established via the property of weakly sequential compactness of generalized controls, thanks to the conditions a) or b). This implies the existence of solution to the original problem (3)–(6). $\square$

Note 1: Since any Borel measure can be weakly-* approximated by absolutely continuous measures, and in view of R.V. Gamkrelidze approximation lemma (see [1]),[1] by using the discontinuous time variable change, and the definition of the generalized impulsive control $\mathfrak{c}$, it is easy to derive that for any $(x, \mathfrak{c}) \in \mathcal{P}$ there exists a sequence of control processes (x_i, v_i) of the problem (1), not necessarily admissible in (1), such that

- $\displaystyle\int_{t_0}^{t_1} \omega(|v_i|)dt \to \|\mu\|$,
- $\displaystyle\int_{t_0}^{t_1} g_0(x_i, v_i, t)dt \to \int_{t_0}^{t_1} \langle \bar{g}_0(x, v, t), d\mathfrak{c} \rangle$,
- $x_i(t) \to x(t), \;\; \forall\, t \in (T \setminus \mathrm{Ds}(\mu)) \cup \{t_0\} \cup \{t_1\}$, and
- $\displaystyle\limsup_{i\to\infty} \max_{t\in T} \varphi^j(x_i(t), t) \le 0 \;\; \forall\, j = 1, .., l$.

This means that the conditions a), b) of the Theorem 1 will be clearly satisfied as soon as there exist positive constants ε, κ, and C, and a number i, such that

a′) $\displaystyle\int_{t_0}^{t_1} \omega(|v|)dt \le \kappa \int_{t_0}^{t_1} g_0(x, v, t)dt + C$, or

b′) A, and V are compact, and

$$\int_{t_0}^{t_1} \omega(|v|)dt \le \kappa \int_{t_0}^{t_1} g^i(x, v, t)dt + C,$$

where $v(t) \in V$, $\dot{x} = g(x, v, t)$ with $x(t_0) \in A$, $\displaystyle\max_{t\in T} \varphi^j(x(t), t) \le \varepsilon \; \forall\, j$.

Thanks to a′) and b′) and Theorem 1, it becomes possible to guarantee a priori whether the extension (3)–(6) is successful or not, by operating only in terms of the original problem (1) which is being extended.

Example 1. The so-called Dido problem:

$$\begin{aligned} &\text{Maximize} && \int_{-1}^{1} x(t)dt, \\ &\text{subject to} && x(-1) = x(1) = 0,\; x(t) \ge 0, \\ & && \int_{-1}^{1} \sqrt{1 + (\dot{x})^2}dt = l. \end{aligned} \tag{9}$$

Continuous solution fails to exist when the length of the arc l is sufficiently large. (Dido problem is a typical example of the so called isoperimetric problem. It is fairly common the solution to fail to exist for this class of problems.) By means of what has been said, the extension (3)–(6) is successful for problem (9) in the sense that there exists a solution to the extended problem. Let us confirm it. By setting $\omega(\xi) = \xi$, $v = \dot{x}$, $g^1(x, v) := v$, $g^2(x, v) = \sqrt{1 + v^2}$, we have

$$\int_{-1}^{1} |v|dt \le \int_{-1}^{1} \sqrt{1 + |v|^2}dt = \int_{-1}^{1} g^2(x, v)dt.$$

Then, the condition b′) guarantees the existence of solution. (Above we introduce an extra state variable y such that $\dot{y} = \sqrt{1 + |v|^2}$.)

Example 2. Minimization of the norm of a function in $\mathbb{L}_1$ over the elements of the unit sphere in $\mathbb{L}_2$

$$\begin{aligned} &\text{Minimize} && \int_{0}^{1} |f(t)|dt, \\ &\text{subject to} && \int_{0}^{1} |f(t)|^2 dt = 1. \end{aligned} \tag{10}$$

The solution does not exist for tis problem. Indeed the infimum, here, is zero, but it is not reached due to the integral constraint. However, with $\omega(\xi) = \xi^2$, $v = f$, and $g^1(x, v) := v^2$, condition b′), again, works and then, the solution exists in the extended sense.

In the realm of the current research, what will the extended solutions to (9) and (10) be?

For the Dido problem (9), as $l \le \pi$ the solution is classical: it is an arc of the circle, connecting the two points $t = -1$ and $t = 1$. As $l > \pi$ the solution is discontinuous. Let us describe it.

By definition, we have

$$\bar{g}(x, v) = \frac{v}{1 + |v|}, \;\; \bar{g}_0(x, v) = \frac{x}{1 + |v|},$$

and $\bar{g}(x, v) = \pm 1$, $\bar{g}_0(x, v) = 0$, when $v \in S^0_\infty$ (i.e., when $v = \pm\infty$). Then, the optimal generalized impulsive control $\mathfrak{c}$ is

$$\mu = r\delta_{-1} + r\delta_1 + \mu_a,\; \nu_t = \delta_{a(t)},\; \nu_s^{-1} = \delta_{+\infty},\; \nu_s^1 = \delta_{-\infty},$$

where $r = \dfrac{l - \pi}{2}$, μ_a is the absolutely continuous measure such that $\frac{d\mu}{dt} = |a(t)|$, and

$$a(t) = \frac{-t}{\sqrt{1 - t^2}}.$$

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[1] This lemma, in particular, says that any generalized control can be weakly approximated by conventional controls, that is in the sense of weak convergence of generalized controls.

The optimal trajectory $x(t)$ is given by

$$x(t) = \begin{cases} 0, & t = -1, \\ r + \sqrt{1-t^2}, & t \in (-1,1), \\ 0, & t = 1, \end{cases}$$

and $x_0(s) = rs$, and $x_1(s) = r - rs$.

Regarding the problem (10) the extended solution is trivial. In fact, there are infinitely many solutions. For example,

$$\mu = \delta_\tau,\ \nu_t = 0,\ \nu_s^\tau = \delta_\infty \text{ or } \nu_s^\tau = \delta_{-\infty},$$

where $\tau \in [0,1]$ is chosen arbitrary.

There are many other classical examples, such as the Euler problem or catenary, to which the extension can be found via this approach, also for case when the condition a$'$) begins to work. See the book [15], where a number of such examples is collected.

V. CONCLUSIONS

Some optimal control problems do not have solution in the class of classic controls. This suggests the need of a relaxation or extension of the control problem ensuring the existence of a solution in some enlarged class of controls. This work aimed at the development of an extension when the control dynamics is

$$\dot{x}(t) = g(x(t), v(t), t),$$

where the control function $v(t)$ takes its values in some closed set V, not necessarily bounded. In order to minimize a integral functional over arcs satisfying this equation we searched for a solution in an extended sense enabling to $x(t)$ to have jumps. To achieve this goal, we exploited the approach of R.V. Gamkrelidze based on the generalized controls. The proposed extension, however, relates to discontinuous arcs, what leads to the notion of generalized impulsive control. The fact that V is not compact requires a certain compactification to ensure weak compactness of generalized impulsive controls. In turn, this is crucial for the existence of solution. As a result, a general theorem for the existence of solution was found.

Besides that, a number of classical examples such like "The minimal surface problem", "Dido problem", "Catenary" and others find proper extensions in the framework of the approach proposed above.

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