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Robust Synchronization of Directed Networks with Coprime Factor Perturbed Agent Dynamics

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Abstract— **This paper deals with robust synchronization of directed and undirected multi-agent networks with uncertain agent dynamics. Given a network with identical nominal dynamics, we allow uncertainty in the form of coprime factor perturbations of the transfer matrix of the agent dynamics. These perturbations are assumed to be stable and have $\mathcal{H}_\infty$-norm that is bounded by an a priori given desired tolerance. We derive state space equations for dynamic observer based protocols that achieve robust synchronization in the presence of such uncertainty. We show that this robust synchronization of the network by the dynamic protocol is equivalent to robust stabilization of a single linear system by all controllers from a given set of feedback controllers. The synchronizing protocols are expressed in terms of real symmetric solutions to algebraic Riccati equations related to the nominal agent dynamics, and contain a weighting factor depending on the eigenvalues of the graph Laplacian. We obtain an achievable interval, i.e. an interval such that for each value of the tolerance contained in this interval there exists a robustly synchronizing protocol. For undirected networks, the supremum of this interval is proportional to the square root of the quotient of the smallest and the largest eigenvalue of the graph Laplacian.**

I. INTRODUCTION

Recently, extensive effort has been invested in the theory of distributed control of networked systems. Specifically, control of networked multi-agent systems has gathered a lot of attention. A networked multi-agent system is a dynamical system that consists of a group of input-output systems and an interaction topology that dictates which exchange of information is permitted between these systems and their neighbours. These input-output systems are called the *agents* of the network. The possible interactions between the agents are modeled by a graph, called the *network graph*, which specifies the neighbors of each agent. On this graph, the nodes represent the agents, and the edges represent the interaction topology. Depending on the context, the network graph can either be direct or undirected. An important object in the theory of networked multi-agent systems is the *Laplacian matrix* of the network graph. Many of the properties of a networked system can be expressed in terms of the eigenvalues of the Laplacian, see [1], [2].

Once the interaction exchange between neighboring agents is specified, the network dynamics is completely determined by the agents dynamics and the interactions between agents and their neighbors. The form that this information exchange takes is often called a communication *protocol*. In the networked system, a protocol takes the role of a feedback controller acting only on locally available information. The feedback processor for a specific agent uses only information from that agent and its neighbors. One of the important problems in the theory of networked multi-agent systems is the design of protocols that achieve a desired overall network behavior.

Since many related problem formulations from different application areas involving interconnected dynamical systems can be cast in the framework of networked multi-agent systems, networked multi-agent systems have received a great deal of interest from several fields of scientific research. A well-known problem in the theory of networked systems is the *consensus problem*, see [3], [4], [5], [6], [7] and [8]. The consensus problem was examined more recently in [9] and [10]. In the problem of consensus, the goal is to reach a state of agreement on certain quantities of interest which depend on the states of each agent. This is to be achieved by means of local information exchange only. A communication protocol that achieves this goal is said to achieve consensus within the network.

In [11], results on consensus of linear multi-agent systems have been extended to accommodate the presence of uncertainty in the agent dynamics. While the agents in the network have identical *nominal* dynamics, the actual dynamics of each agent is *uncertain* in the sense that the transfer matrix of each agent is a perturbation of the common nominal dynamics. In [11], *additively perturbed* agent dynamics were considered, and conditions for the existence of dynamic protocols that achieve robust synchronization and methods to obtain such protocols were established.

In this paper, we consider directed and undirected networked multi-agent systems with *coprime factor perturbed* agent dynamics. We provide protocols that achieve robust synchronization for these kinds of perturbations, depending on the nominal agent dynamics and the spectrum of the Laplacian matrix of the network graph. We show that using these dynamic observer based protocols, for undirected graphs one can obtain an uncertainty tolerance that is proportional to the square root of the quotient of the smallest and largest eigenvalue of the graph Laplacian.

The outline of this paper is as follows. In section II, we introduce some notation, and formulate a version of the bounded real lemma that is instrumental in proving the main result. In section III we provide some basic graph theory and state some important properties of the Laplacian matrix. Next in section IV, the theory of synchronization of unperturbed linear multi-agent systems is briefly examined. In section V we provide a formulation of the problem of

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robust synchronization of coprime factor perturbed multi-agent systems. Finally in section VI we formulate the main theorem of this paper, which gives an interval of achievable values of the uncertainty tolerance. We then provide robustly synchronization protocols that achieve robust synchronization for any given tolerance that lies within this interval. These protocols are expressed in terms of the stabilizing solutions to two Riccati equations and eigenvalues of the graph Laplacian.

II. PRELIMINARIES

In this paper, we denote the set of all proper and stable real rational matrices by $\mathcal{RH}_\infty$. If $G \in \mathcal{RH}_\infty$, then $||G||_\infty$ denotes its $\mathcal{H}_\infty$-norm, $||G||_\infty = \sup_{\mathrm{Re}(\lambda)\geq 0} ||G(\lambda)||$, where $\mathrm{Re}(\lambda)$ denotes the real part of the complex number λ. A square matrix $H \in \mathbb{R}^{n\times n}$ is called Hurwitz if all its eigenvalues have strictly negative real part.

Let $\mathbb{R}$ denote the field of real numbers, $\mathbb{R}^n$ the n-dimensional Euclidean space and $\mathbb{R}^{n\times n}$ the space of $n \times n$ real matrices. Denote the field of complex numbers by $\mathbb{C}$. Let I_p denote the identity matrix of dimension p and I the identity matrix of appropriate dimension. The tensor or *Kronecker product* of matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is denoted by $A \otimes B$.

This paper will use ideas and results from $\mathcal{H}_\infty$-control. The $\mathcal{H}_\infty$-control problem dates back to work by G. Zames in [12] and the first solution in a state space setting was provided in [13]. A result that is instrumental in $\mathcal{H}_\infty$-control is the *bounded real lemma*. In this paper we will use a version of the bounded real lemma adapted to our purposes. The proof is omitted.

Lemma 1: Consider the system $\dot{x} = Ax + Bu$, $y = Cx + Du$ with transfer matrix $G(s) = C(sI-A)^{-1}B+D$. Assume $D^TD = I$ and A is Hurwitz. Let $\tau > 1$. The $\mathcal{H}_\infty$-norm $||G||_\infty$ of the transfer matrix from u to y satisfies $||G||_\infty < \tau$ if there exists $\epsilon > 0$ and a real symmetric solution P to the Riccati inequality

$$\begin{aligned} A^TP + PA + C^TC + \frac{1}{\tau^2-1}(PB + C^TD)(B^TP + D^TC) \\ \leq -\epsilon(PB + C^TD)(B^TP + D^TC). \quad (1) \end{aligned}$$

III. NETWORKS

In this paper, we consider networks whose interaction topologies are represented by directed or undirected graphs. A graph consists of a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, p\}$ is the set of *nodes*, and where $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of *edges*. Given two nodes $i, j \in \mathcal{V}$ with $i \neq j$, then an edge from i to j is represented by the pair $(i, j) \in \mathcal{E}$. A graph with the property that $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$ is called undirected. The *neighboring set* N_i of vertex i is defined as $N_i := \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$. For a graph $\mathcal{G}$, its adjacency matrix A is given by $A = (a_{ij})$, with $a_{ii} = 0$ and $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. The *Laplacian matrix* of $\mathcal{G}$ is defined as $L = (l_{ij})$, where we have $l_{ii} = \sum_{j\neq i} a_{ij}$, $l_{ij} = -a_{ij}$, $i \neq j$. In the case that $\mathcal{G}$ is undirected, the Laplacian L is a positive semi-definite real symmetric matrix and all its eigenvalues are nonnegative real. If $\mathcal{G}$ is directed, then L is not necessarily symmetric, and the eigenvalues of L are not guaranteed to be real. In this case, still, all eigenvalues of L have nonnegative real part. Since all row-sums of L are zero, i.e. $\sum_j l_{ij} = 0\ \forall i$, zero is an eigenvalue of L with eigenvector $\mathbf{1} := (1, \ldots, 1)^T$. Consequently, L has at most rank $p - 1$.

It is a well-known fact that the Laplacian matrix of an undirected graph has rank $p - 1$ if and only if the graph is connected, that is, if for every distinct pair of nodes i and j, there exists a path from i to j. Under this condition, the zero eigenvalue of L has multiplicity one. The $p - 1$ nonzero eigenvalues of L can be ordered increasingly as $0 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_p$. A directed graph contains a spanning tree if there exists a node i such that there exists a directed path from node i to every other node j. A directed graph contains a spanning tree if and only if its Laplacian has rank $p-1$. The eigenvalues of L are in general not real and its set of nonzero eigenvalues is denoted in arbitrary order by $\{\lambda_2, \lambda_3, \ldots, \lambda_p\}$. Since the Laplacian matrix of an undirected graph is symmetric, it can be *diagonalized* by an orthogonal transformation U that brings it to the form $\Lambda := U^TLU = \mathrm{diag}(0, \lambda_2, \ldots, \lambda_p)$, which is denoted by Λ. The Laplacian matrix of a directed graph can be brought to *upper triangular form* by a complex unitary transformation U: $U^*LU = \Lambda_u$, where Λ_u is a complex upper diagonal matrix with $0, \lambda_2, \ldots, \lambda_p$ on the diagonal.

IV. MULTI-AGENT SYSTEMS

In this section, the problem of synchronization of multi-agent systems is formulated. Since the main interest of this paper is that of robust synchronization of *perturbed* multi-agent systems, we will only briefly discuss the unperturbed case. In this paper we consider multi-agent systems with p agents, where the communication topology of the system is represented by a directed or undirected graph $\mathcal{G}$ with Laplacian matrix denoted by L. For each agent i of the network, the nominal agent dynamics is given by one and the same finite-dimensional linear time-invariant system

$$\dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i. \quad (2)$$

For each i, the state $x_i \in \mathbb{R}^n$, and the input signal u_i and output signal y_i take values in $\mathbb{R}^m$ and $\mathbb{R}^q$, respectively. It is a standing assumption in this paper that (A, B) is stabilizable and (C, A) is detectable. Hence, there exist F and G such that both $A + BF$ and $A - GC$ are Hurwitz.

Following [10], [11], these agents are then interconnected using an observer-based *dynamic* protocol of the form

$$\begin{aligned} &\dot{w}_i = Aw_i + B\sum_{j\in\mathcal{N}_i}(u_i - u_j) + G(\sum_{j\in\mathcal{N}_i}(y_i - y_j) - Cw_i), \\ &u_i = Fw_i. \end{aligned} \quad (3)$$

for $i = 1, 2, \ldots, p$. The structure of this protocol is as follows. Each controller is able to observe the disagreement output signal $\sum_{j\in\mathcal{N}_i}(y_i - y_j)$ and the relative input $\sum_{j\in\mathcal{N}_i}(u_i - u_j)$ of its corresponding agent. The differential equation in (3) acts as an *observer* for the relative state $\sum_{j\in\mathcal{N}_i}(x_i - x_j)$ of agent i. The protocol state w_i is an estimate of this quantity. It is easily verified that the error $e_i := w_i - \sum_{j\in\mathcal{N}_i}(x_i - x_j)$ has error dynamics $\dot{e}_i = (A - GC)e_i$, which is asymptotically stable if $A - GC$ is Hurwitz. This estimate is then fed back to the agent by means of a static feedback.

By interconnecting the agents using the above protocol, we obtain the closed-loop dynamics of the entire network. Denote $\mathbf{x} := \mathrm{col}(x_1, x_2, \ldots, x_p)$, $\mathbf{u} := \mathrm{col}(u_1, u_2, \ldots, u_p)$, $\mathbf{y} := \mathrm{col}(y_1, y_2, \ldots, y_p)$, and $\mathbf{w} = \mathrm{col}(w_1, w_2, \ldots, w_p)$. The network dynamics is now

$$\begin{pmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{w}}\end{pmatrix} = \begin{pmatrix} I\otimes A & I\otimes BF\\ L\otimes GC & (I\otimes(A-GC)) + (L\otimes BF)\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{w}\end{pmatrix}. \tag{4}$$

Next, we state the prevalent definition of synchronization of such a network.

Definition 1: The network with agent dynamics (2) is said to be synchronized by protocol (3) if for all $i, j = 1, 2, \ldots, p$ we have that $x_i(t) - x_j(t) \to 0$ and $w_i(t) - w_j(t) \to 0$ as $t \to \infty$.

In [11], it is shown that for networks that are undirected and connected, or directed and contain a spanning tree, synchronization of the network with agent dynamics (2) by protocol (3) is equivalent to the stabilization of a single linear system by all controllers from a set of $p - 1$ related controllers. We will use a similar idea in the next section.

V. ROBUST SYNCHRONIZATION

While the nominal agent dynamics is still given by the unperturbed dynamics (2), we now allow uncertainty in the form of coprime factor perturbations of the nominal agent dynamics. The agents have identical nominal transfer matrices given by $G(s) = C(sI - A)^{-1}B$. It is well known that there exists a coprime factorization of G of the form $G = M^{-1}N$ with $M, N \in \mathcal{RH}_\infty$ such that $NN^* + MM^* = I$, where $N^*(s) := N^T(-s)$, see e.g. [14]. Such a factorization is called a normalized coprime factorization over $\mathcal{RH}_\infty$. The perturbed transfer function for each agent is now assumed to be given by

$$G_{(\Delta_M\ \Delta_N)} := (M + \Delta_M)^{-1}(N + \Delta_N),$$

where $\Delta_M, \Delta_N \in \mathcal{RH}_\infty$. In this paper, we consider all such perturbations with $||(\Delta_M\ \Delta_N)||_\infty \le \gamma$, where $\gamma > 0$ is a desired uncertainty tolerance. It is well-known, see e.g. [14], that the coprime factor perturbed dynamics of agent i can be represented by the feedback interconnection of the auxiliary plant

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i + QC^Td_i, \quad y_i = Cx_i + d_i,\\ z_i &= \begin{pmatrix}C\\0\end{pmatrix}x_i + \begin{pmatrix}0\\I\end{pmatrix}u_i + \begin{pmatrix}I\\0\end{pmatrix}d_i. \end{aligned} \tag{5}$$

with the feedback gain

$$d_i = (-\Delta_M\ \Delta_N)z_i. \tag{6}$$

The matrix Q appearing in the above dynamics is the unique real symmetric solution to the Riccati equation

$$AQ + QA^T - QC^TCQ + BB^T = 0, \tag{7}$$

such that $A - QC^TC$ is Hurwitz. The matrix Q is called the stabilizing solution of (7). For the sake of simplicity, we will sometimes denote $\Delta := (-\Delta_M\ \Delta_N)$ and write the feedback loop in the form $d_i = \Delta z_i$. Next, we give a definition of the robust synchronization problem.

Definition 2: Given a desired tolerance $\gamma > 0$, the protocol (3) is said to *robustly synchronize* the network (4) if for all $\Delta_M, \Delta_N \in \mathcal{RH}_\infty$ with $||(\Delta_M\ \Delta_N)||_\infty \le \gamma$ we have that for all $i, j = 1, 2, \ldots, p$

$$x_i(t) - x_j(t) \to 0,\ w_i(t) - w_j(t) \to 0$$

as $t \to \infty$. The tolerance γ is called the *synchronization radius* of the network.

For the robust synchronization problem, we consider a modified version of protocol (3). A *weighting factor* on the Laplacian matrix L of the network graph is used, denoted by N.

$$\begin{aligned} \dot{w}_i &= Aw_i + B\sum_{j\in\mathcal{N}_i}\frac{1}{N}(u_i - u_j)\\ &\quad + G(\sum_{j\in\mathcal{N}_i}\frac{1}{N}(y_i - y_j) - Cw_i),\\ u_i &= Fw_i. \end{aligned} \tag{8}$$

The positive real valued parameter N is introduced as an extra design parameter, for which a value has to be determined next to the gain matrices F and G.

To start off, we now first examine conditions under which, for a given uncertainty tolerance $\gamma > 0$, there exists a robustly synchronizing protocol. After interconnecting the agents with protocol (8), the overall network dynamics can be conveniently represented by again denoting the aggregate agent and controller state vectors by $\mathbf{x} = \mathrm{col}(x_1, x_2, \ldots, x_p)$ and $\mathbf{w} = \mathrm{col}(w_1, w_2, \ldots, w_p)$, and the aggregate output and input vectors $\mathbf{y} = \mathrm{col}(y_1, y_2, \ldots, y_p)$ and $\mathbf{u} = \mathrm{col}(u_1, u_2, \ldots, u_p)$, respectively. The aggregate output and input vectors of the feedback-loop are denoted $\mathbf{d} = \mathrm{col}(d_1, d_2, \ldots, d_p)$ and $\mathbf{z} = \mathrm{col}(z_1, z_2, \ldots, z_p)$, respectively. Then by interconnecting the perturbed network (5), (6) with the protocol (8) we obtain that the dynamics of the overall perturbed network is given by

$$\begin{aligned} \begin{pmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{w}}\end{pmatrix} &= \begin{pmatrix} I\otimes A & I\otimes BF\\ \frac{1}{N}L\otimes GC & I\otimes(A-GC) + (\frac{1}{N}L\otimes BF)\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{w}\end{pmatrix}\\ &\quad + \begin{pmatrix} I\otimes QC^T\\ \frac{1}{N}L\otimes G\end{pmatrix}\mathbf{d},\\ \mathbf{z} &= (I\otimes\begin{pmatrix}C\\0\end{pmatrix}\ \ I\otimes\begin{pmatrix}0\\F\end{pmatrix})\begin{pmatrix}\mathbf{x}\\ \mathbf{w}\end{pmatrix} + (I\otimes\begin{pmatrix}I\\0\end{pmatrix})\mathbf{d}. \end{aligned}$$

and

$$\mathbf{d} = (I\otimes\Delta)\mathbf{z}.$$

Recall that if the network graph is undirected, then there exists a orthogonal transformation U that diagonalizes L.

Note that the first column of U is given by the normalized vector of ones, $\frac{1}{\sqrt{p}}\mathbf{1}$. As before, let $\Lambda = \mathrm{diag}(0, \lambda_2, \ldots, \lambda_p)$. By applying the state transformation

$$\begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{w}} \end{pmatrix} := \begin{pmatrix} U^T \otimes I & 0 \\ 0 & U^T \otimes I \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix}, \;\; \tilde{\mathbf{z}} = (U^T \otimes I)\mathbf{z},$$

and $\tilde{\mathbf{d}} = (U^T \otimes I)\mathbf{d}$, we obtain

$$\begin{aligned} \begin{pmatrix} \dot{\tilde{\mathbf{x}}} \\ \dot{\tilde{\mathbf{w}}} \end{pmatrix} &= \begin{pmatrix} I \otimes A & I \otimes BF \\ \frac{1}{N}\Lambda \otimes GC & I \otimes (A - GC) + (\frac{1}{N}\Lambda \otimes BF) \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{w}} \end{pmatrix} \\ &\quad + \begin{pmatrix} I \otimes QC^T \\ \frac{1}{N}\Lambda \otimes G \end{pmatrix} \tilde{\mathbf{d}}, \\ \tilde{\mathbf{z}} &= (I \otimes \begin{pmatrix} C \\ 0 \end{pmatrix} \;\; I \otimes \begin{pmatrix} 0 \\ F \end{pmatrix}) \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{w}} \end{pmatrix} + (I \otimes \begin{pmatrix} I \\ 0 \end{pmatrix}) \tilde{\mathbf{d}}, \\ \tilde{\mathbf{d}} &= (U^T \otimes I)(I \otimes \Delta)(U \otimes I)\tilde{\mathbf{z}} = (I \otimes \Delta)\tilde{\mathbf{z}}. \end{aligned} \tag{9}$$

Analogously as Theorem 4.2 in [11], the following theorem reduces the problem of robust synchronization to a problem of simultaneous robust stabilization, and gives *necessary and sufficient* conditions on the weighting parameter N and gains F and G such that protocol (8) robustly synchronizes the network. We first consider the case that the network topology is given by an undirected graph.

Theorem 1: Consider network with nominal agent dynamics given by (2). Assume that the network graph is a connected, undirected graph. Let $\gamma > 0$. Then the following statements are equivalent:

1) The dynamic protocol (8) synchronizes the network with perturbed agents

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i + QC^T d_i, \quad y_i = Cx_i + d_i, \\ z_i &= \begin{pmatrix} C \\ 0 \end{pmatrix} x_i + \begin{pmatrix} 0 \\ I \end{pmatrix} u_i + \begin{pmatrix} I \\ 0 \end{pmatrix} d_i, \\ d_i &= (-\Delta_M \; \Delta_N) z_i, \end{aligned} \tag{10}$$

for all $\Delta_M, \Delta_N \in \mathcal{R}H_\infty$ with $||(\Delta_M \; \Delta_N)||_\infty \leq \gamma$.

2) The perturbed system

$$\begin{aligned} \dot{x} &= Ax + Bu + QC^T d, \quad y = Cx + d, \\ z &= \begin{pmatrix} C \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} u + \begin{pmatrix} I \\ 0 \end{pmatrix} d, \\ d &= (-\Delta_M \; \Delta_N) z, \end{aligned} \tag{11}$$

is internally stabilized for all $\Delta_M, \Delta_N \in \mathcal{R}H_\infty$ with $||(\Delta_M \; \Delta_N)||_\infty \leq \gamma$ by all $p-1$ controllers

$$\dot{w} = Aw + Bu + G(y - Cw), \;\; u = \frac{1}{N}\lambda_i F w, \tag{12}$$

where $i = 2, \ldots, p$ and λ_i is the ith eigenvalue of the Laplacian L.

Proof: In this proof, we again use the shorthand notation Δ for $(-\Delta_M \; \Delta_N)$. Let H be any $p-1 \times p$ matrix such that $\ker H = \mathrm{im}\,\mathbf{1}$. Then $HU = (0 \; U_2)$, where U_2 has full rank. It is easily seen that $x_i(t) - x_j(t) \to 0$ as $t \to \infty$ for all i, j if and only if $(H \otimes I)\mathbf{x} \to 0$. Similarly, $w_i(t) - w_j(t) \to 0$ for all i, j if and only if $(H \otimes I)\mathbf{w} \to 0$. This is equivalent with $\tilde{x}_i(t) \to 0$ and $\tilde{w}_i(t) \to 0$ for $i = 2, 3, \ldots, p$.

(only if) First, we show that if dynamic protocol (8) robustly synchronizes the network, then the interconnection of the plant (11) with controller (12) is robustly stabilized. Assume that the network with perturbed agent dynamics (5) is synchronized by protocol (8) for all perturbations Δ with $||\Delta||_\infty \leq \gamma$. Take an arbitrary $\Delta \in \mathcal{R}H_\infty$ with $||\Delta||_\infty \leq \gamma$. We perturb each agent in the network (10) with Δ. Interconnecting (11) and (12) yields

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} &= \begin{pmatrix} A & \frac{1}{N}\lambda_i BF \\ GC & A - GC + \frac{1}{N}\lambda_i BF \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} \\ &\quad + \begin{pmatrix} QC^T \\ G \end{pmatrix} d, \\ z &= \begin{pmatrix} C & 0 \\ 0 & \frac{1}{N}\lambda_i F \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} d, \quad d = \Delta z. \end{aligned} \tag{13}$$

Since the network is robustly synchronized by the protocol, we have that $\tilde{x}_i \to 0$, $\tilde{w}_i \to 0$ as $t \to \infty$ for $i = 2, \ldots, p$ in (9). This implies that for $i = 2, \ldots, p$ the following systems are internally stable:

$$\begin{aligned} \begin{pmatrix} \dot{\tilde{x}}_i \\ \dot{\tilde{w}}_i \end{pmatrix} &= \begin{pmatrix} A & BF \\ \frac{1}{N}\lambda_i GC & A - GL + \frac{1}{N}\lambda_i BF \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ \tilde{w}_i \end{pmatrix} \\ &\quad + \begin{pmatrix} QC^T \\ \frac{1}{N}\lambda_i G \end{pmatrix} \tilde{d}_i, \\ \tilde{z}_i &= \begin{pmatrix} C & 0 \\ 0 & F \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ \tilde{w}_i \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} \tilde{d}_i, \quad \tilde{d}_i = \Delta \tilde{z}_i. \end{aligned}$$

By the simple state transformation $\tilde{w}_i = \frac{1}{N}\lambda_i \bar{w}_i$, we see that this system is equivalent with (13). Therefore, the system (13) is internally stable for $i = 2, 3, \ldots, p$.

(if) Next, assume that the $p-1$ controllers (12) stabilize system (11) for all $\Delta \in \mathcal{RH}_\infty$ with $||\Delta||_\infty \leq \gamma$. From the small gain theorem, it then follows that for $i = 2, 3, \ldots, p$ the closed-loop systems (13) are internally stable and the transfer matrices G_i from d to z satisfy $||G_i||_\infty < \frac{1}{\gamma}$. We show that the perturbed network is synchronized by protocol (8) for all perturbations Δ that satisfy $||\Delta||_\infty \leq \gamma$. This is done by showing that for $i = 2, 3, \ldots, p$ we have $\tilde{x}_i(t) \to 0$ and $\tilde{w}_i(t) \to 0$ as $t \to \infty$, where $\tilde{x}_i$ and $\tilde{w}_i$ satisfy (9). Denote $\bar{\mathbf{x}} = \mathrm{col}(\tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_p)$, $\bar{\mathbf{w}} = \mathrm{col}(\tilde{w}_2, \tilde{w}_3, \ldots, \tilde{w}_p)$, $\bar{\mathbf{z}} = \mathrm{col}(\tilde{z}_2, \tilde{z}_3, \ldots, \tilde{z}_p)$, and $\bar{\mathbf{d}} = \mathrm{col}(\tilde{d}_2, \tilde{d}_3, \ldots, \tilde{d}_p)$. Let $\Lambda_1 = \mathrm{diag}(\lambda_2, \lambda_3, \ldots, \lambda_p)$. From (9) we obtain

$$\begin{aligned} \begin{pmatrix} \dot{\bar{\mathbf{x}}} \\ \dot{\bar{\mathbf{w}}} \end{pmatrix} &= \begin{pmatrix} I_{p-1} \otimes A & I_{p-1} \otimes BF \\ \frac{1}{N}\Lambda_1 \otimes GC & I_{p-1} \otimes (A - GC) + (\frac{1}{N}\Lambda_1 \otimes BF) \end{pmatrix} \begin{pmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{w}} \end{pmatrix} \\ &\quad + \begin{pmatrix} I_{p-1} \otimes QC^T \\ \frac{1}{N}\Lambda_1 \otimes G \end{pmatrix} \bar{\mathbf{d}}, \\ \bar{\mathbf{z}} &= (I_{p-1} \otimes \begin{pmatrix} C \\ 0 \end{pmatrix} \;\; I_{p-1} \otimes \begin{pmatrix} 0 \\ F \end{pmatrix}) \begin{pmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{w}} \end{pmatrix} \\ &\quad + (I_{p-1} \otimes \begin{pmatrix} I \\ 0 \end{pmatrix}) \bar{\mathbf{d}}, \\ \bar{\mathbf{d}} &= (I_{p-1} \otimes \Delta)\bar{\mathbf{z}}. \end{aligned} \tag{14}$$

The transfer matrix of this system from $\bar{\mathbf{d}}$ to $\bar{\mathbf{z}}$ is a block diagonal matrix with $G_2, G_3, \ldots, G_p$ on the diagonal. We obtain that for G the transfer matrix of the network from $\bar{\mathbf{d}}$

to $\bar{\mathbf{z}}$ it holds that $||G||_\infty < \frac{1}{\gamma}$. By observing that the $\mathcal{H}_\infty$-norm of $I_{p-1} \otimes \Delta$ is less then or equal to γ and applying the small gain theorem, it follows that the system (14) is internally stable and $\bar{\mathbf{x}}(t) \to 0$ and $\bar{\mathbf{w}}(t) \to 0$ as $t \to \infty$ for all Δ with $||\Delta||_\infty \leq \gamma$. So indeed, we have $\tilde{x}_i \to 0$ and $\tilde{w}_i \to 0$ in (9) for $i = 2, 3, \ldots, p$. ∎

Next, we turn to the case that the network graph is directed. In fact, the results on the undirected case are comparable to those in the case that the network is directed and contains a spanning tree. In this case, robust synchronization of the plant (11) by the $p-1$ controllers (12) implies robust synchronization of (9). As noted before, there exists an complex unitary transformation U that brings L to upper diagonal form $U^*LU = \Lambda_u$, with Λ_u complex and $0, \lambda_2, \ldots, \lambda_p$ on the diagonal. Then in Theorem 1, statement (2) implies statement (1). This can be proven by letting the unitary transformation U take over the role of the orthogonal transformation and in the proof of Theorem 1 replacing Λ with Λ_u.

The main conclusion that can be drawn from Theorem 1 is that both for the undirected as the directed case, to synthesize a protocol that achieves robust synchronization of the network for a desired uncertainty tolerance $\gamma > 0$, it suffices to find a positive real number N, and gains F and G such that the single system (11) is robustly internally stabilized by all $p-1$ controllers (12) while achieving uncertainty tolerance γ. By applying the small gain theorem, this is equivalent with requiring that all of the controllers (12) solve the $\mathcal{H}_\infty$-control problem for the system (11) in the sense that the closed-loop system is internally stable and the transfer matrix G_i from d to z satisfies $||G_i|| < \frac{1}{\gamma}$.

VI. ROBUSTLY SYNCHRONIZING PROTOCOLS

In this section, we examine which values $\gamma > 0$ of the uncertainty tolerance can be achieved, given the nominal agent dynamics and the Laplacian matrix of the network graph of either a directed network containing a spanning tree or a connected undirected network. We characterize an achievable interval, such that for every γ within this interval, there exist a dynamic protocol of the form (8) that robustly synchronizes the network.

In addition tot the Riccati equation (7), we consider the Riccati equation

$$A^TP + PA - PBB^TP + C^TC = 0. \tag{15}$$

Next to the stabilizing solution Q of (7), let P be the unique real symmetric solution of (15) such that $A - BB^TP$ is Hurwitz. In the remainder of this paper, we denote the eigenvalue of the Laplacian with the maximal modulus by λ_M and the eigenvalue with minimal real part by λ_m, i.e.

$$\mathrm{Re}(\lambda_m) = \min_{i=2,\ldots,p} \mathrm{Re}(\lambda_i),$$
$$|\lambda_M| = \max_{i=2,\ldots,p} |\lambda_i|.$$

We now formulate the main theorem of this paper.

Theorem 2: Consider the network with perturbed agent dynamics (10). Assume the network is directed and contains a spanning tree. Define

$$\gamma^* = \frac{\mathrm{Re}(\lambda_m)}{|\lambda_M|}\frac{1}{\sqrt{1+\rho(PQ)}}. \tag{16}$$

Then, for all γ in the open interval $(0,\ \gamma^*)$, there exists a dynamic protocol of the form (8) that robustly synchronizes the network with uncertainty tolerance γ.

In the remainder of this section, we show how to construct such a protocol, i.e. how choose parameters N, F, and G. We prove that for $\gamma \in (0,\ \gamma^*)$, and for N, F and G provided in this section, the $p-1$ controllers (12) robustly stabilize the system (11) for all Δ with $||\Delta||_\infty \leq \gamma$. First, we provide a lemma that will be instrumental in the proof. Recall that Q denotes the stabilizing solution to (7) and P denotes the stabilizing solution of (15). Then the following holds:

Lemma 2: Let $\tau > 0$ be such that $\tau^2 < \frac{1}{1+\rho(PQ)}$. Then $(\frac{1}{\tau^2} - 1)I - PQ$ is nonsingular. Temporarily denote

$$\tilde{P} = ((\frac{1}{\tau^2} - 1)I - PQ)^{-1}P. \tag{17}$$

Then $\tilde{P}$ is a real symmetric solution of the algebraic Riccati equation

$$\begin{aligned} A^T\tilde{P} + \tilde{P}A + \tau^2C^TC - \frac{1}{\tau^2}\tilde{P}BB^T\tilde{P} \\ + \frac{1}{1-\tau^2}(\tilde{P}Q + \tau^2I)C^TC(Q\tilde{P} + \tau^2I) = 0. \end{aligned} \tag{18}$$

Proof: The proof follows from combining equations (7) and (15). ∎

We will use this lemma in the proof of the following theorem.

Theorem 3: Consider the network with with nominal agent dynamics (2). Assume that the network graph is directed and contains a spanning tree. The perturbed dynamics of agent i is then given by

$$\dot{x}_i = Ax_i + Bu_i + QC^Td_i, \quad y_i = Cx_i + d_i,$$
$$z_i = \begin{pmatrix} C \\ 0 \end{pmatrix} x_i + \begin{pmatrix} 0 \\ I \end{pmatrix} u_i + \begin{pmatrix} I \\ 0 \end{pmatrix} d_i.$$

with $d_i = (-\Delta_M\ \Delta_N)z_i$. Let γ be any uncertainty tolerance such that $\gamma \in (0,\ \gamma^*)$. Choose N any real number such that $N > \frac{|\lambda_M|^2}{\mathrm{Re}(\lambda_m)}$ and

$$\gamma^2 < \frac{\mathrm{Re}(\lambda_m)}{N}\frac{1}{1+\rho(PQ)}. \tag{19}$$

Finally, let $\eta > 0$ such that $\eta < \frac{\mathrm{Re}(\lambda_m)}{N}$ and

$$\frac{\gamma^2}{\eta} < \frac{1}{1+\rho(PQ)}. \tag{20}$$

Then $(\frac{\eta}{\gamma^2} - 1)I - PQ$ is nonsingular. Define

$$\begin{aligned} \tilde{P} &:= ((\frac{\eta}{\gamma^2} - 1)I - PQ)^{-1}P, \\ F &:= -\frac{1}{\gamma^2}B^T\tilde{P}, \\ G &:= QC^T. \end{aligned} \tag{21}$$

Then the dynamic protocol (3) with N, F and G as chosen above robustly synchronizes the network for all perturbations $\Delta_M, \Delta_N \in \mathcal{RH}_\infty$ with $||(\Delta_M\ \Delta_N)||_\infty \leq \gamma$.

Proof: Again denote $(-\Delta_M\ \Delta_N)$ by Δ. By Theorem 1, it suffices to prove that the $p-1$ controllers (12) with N, F and G chosen as above, internally stabilize the system

$$\dot{x} = Ax + Bu + QC^T d, \quad y = Cx + d,$$
$$z = \begin{pmatrix} C \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} u + \begin{pmatrix} I \\ 0 \end{pmatrix} d,$$

while accomplishing $||G_i||_\infty < \frac{1}{\gamma}$, where G_i is the transfer matrix from d to z $(i = 2, 3, \ldots, p)$ in the closed-loop system given by (13). To show that this is indeed the case, we apply the following state transformation:

$$\begin{pmatrix} \tilde{x} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}.$$

Which results in the dynamics below:

$$\begin{aligned} \begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{w}} \end{pmatrix} &= \begin{pmatrix} A + \mu_i BF & -\mu_i BF \\ 0 & A - GC \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{w} \end{pmatrix} + \begin{pmatrix} QC^T \\ QC^T - G \end{pmatrix} d, \\ z &= \begin{pmatrix} C & 0 \\ \mu_i F & -\mu_i F \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{w} \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} d, \end{aligned} \tag{22}$$

where $\mu_i := \frac{\lambda_i}{N}$ for $i = 2, 3, \ldots, p$. To proceed, we will apply Lemma 1 to the system (22). We will show that there exists a real symmetric solution to the Riccati inequality associated with the system. We consider the closed-loop system (22). For ease of notation, we label the system matrices as

$$\tilde{A}_i = \begin{pmatrix} A + \mu_i BF & -\mu_i BF \\ 0 & A - GC \end{pmatrix}, \qquad \tilde{B}_i = \begin{pmatrix} QC^T \\ QC^T - G \end{pmatrix},$$
$$\tilde{C}_i = \begin{pmatrix} C & 0 \\ \mu_i F & -\mu_i F \end{pmatrix}, \qquad \tilde{D} = \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

Take the controller and observer gains F and G as defined in (21). Now we will apply Lemma 1 to show that $||G_i||_\infty < \frac{1}{\gamma}$, where G_i is the transfer function from d to z $(i = 2, 3, \ldots, p)$ in (22). We will show that there exists a suitable choice for Z_i such that Z_i is a solution to the Riccati inequality (1) with $\tau = \frac{1}{\gamma}$, and ϵ sufficiently small:

$$\begin{aligned} \tilde{A}_i^* Z_i + Z\tilde{A}_i + \tilde{C}_i^* \tilde{C}_i + \frac{\gamma^2}{1-\gamma^2}(Z_i \tilde{B}_i + \tilde{C}_i^* \tilde{D})(\tilde{B}_i^* Z_i + \tilde{D}^* \tilde{C}_i) \\ \leq -\epsilon (Z_i \tilde{B}_i + \tilde{C}_i^* \tilde{D})(\tilde{B}_i^* Z_i + \tilde{D}^* \tilde{C}_i) \end{aligned} \tag{23}$$

Lemma 1 requires that the matrices $\tilde{A}_i$ $(i = 2, 3, \ldots, p)$ are Hurwitz. The set of eigenvalues of $\tilde{A}_i$ is the union of those of $A + \mu_i BF$ and $A - GC$. We immediately see that $A - GC = A - QC^T C$ is Hurwitz. Showing that $A + \mu_i BF$ is Hurwitz for $i = 2, 3, \ldots p$ can be done in a similar way. Next, for $i = 2, 3, \ldots, p$, let Y_i be the unique positive semi-definite solution to the Lyapunov equation

$$\begin{aligned} Y_i(A - QC^T C) + (A - QC^T C)^T Y_i \\ + \left(\frac{|\mu_i|^2}{\gamma^4} + \frac{a_i}{\gamma^4} \right) \tilde{P} BB^T \tilde{P} = 0, \end{aligned} \tag{24}$$

where a_i is any real number such that

$$a_i \geq \frac{|\mu_i|^2(|\mu_i|^2 - \mathrm{Re}(\mu_i)^2)}{\mathrm{Re}(\mu_i)(\mathrm{Re}(\mu_i) - \eta)} \geq 0.$$

Since $A - QC^T C$ is Hurwitz, we can find a solution Y_i for any such a_i. Next, take

$$Z_i := \begin{pmatrix} \frac{k_i}{\gamma^2} \tilde{P} & 0 \\ 0 & Y_i \end{pmatrix}, \tag{25}$$

where $k_i = \frac{|\mu_i|^2}{\mathrm{Re}(\mu_i)}$. It can then be shown that for ϵ sufficiently small, Z_i indeed satisfies (23). After performing the addition in (23), it turns out that both sides of the expression are Hermitian 2×2 block matrices. We will now show that the inequality holds in two steps. First, we will consider the upper left corner of the left hand side of (23). This upper left corner is given by

$$\begin{aligned} \frac{k_i}{\gamma^2} \Bigg[\tilde{P}A + A^T \tilde{P} - \frac{\mathrm{Re}(\mu_i)}{\gamma^2} \tilde{P} BB^T \tilde{P} + \frac{\gamma_2}{k_i} C^T C \\ + \frac{k_i}{\gamma^2} \frac{\gamma^2}{1-\gamma^2} (\tilde{P}Q + \frac{\gamma^2}{k_i} I) C^T C (Q\tilde{P} + \frac{\gamma^2}{k_i} I) \Bigg]. \end{aligned} \tag{26}$$

Next, rewrite (26) as

$$\begin{aligned} \frac{k_i}{\gamma^2} \Bigg[& \tilde{P}A + A^T \tilde{P} + \frac{\gamma^2}{\eta} C^T C + (\frac{\gamma^2}{k_i} - \frac{\gamma^2}{\eta}) C^T C \\ & - \frac{\eta}{\gamma^2} \tilde{P} BB^T \tilde{P} + (\frac{\eta - \mathrm{Re}(\mu_i)}{\gamma^2}) \tilde{P} BB^T \tilde{P} \\ & + \frac{\eta}{\eta - \gamma^2} (\tilde{P}Q + \frac{\gamma^2}{\eta} I) C^T C (Q\tilde{P} + \frac{\gamma^2}{\eta} I) \\ & + \frac{k_i}{1-\gamma^2} (\tilde{P}Q + \frac{\gamma^2}{k_i} I) C^T C (Q\tilde{P} + \frac{\gamma^2}{k_i} I) \\ & - \frac{\eta}{\eta - \gamma^2} (\tilde{P}Q + \frac{\gamma^2}{\eta} I) C^T C (Q\tilde{P} + \frac{\gamma^2}{\eta} I) \Bigg] \end{aligned}$$

By applying Lemma 2 with $\tau = \frac{\gamma}{\sqrt{\eta}}$, it follows that this is equal to

$$\begin{aligned} \frac{k_i}{\gamma^2} \Bigg[& \gamma^2 (\frac{1}{k_i} - \frac{1}{\eta}) C^T C \\ & + \frac{1}{\gamma^2} (\eta - \mathrm{Re}(\mu_i)) \tilde{P} BB^T \tilde{P} \\ & + (\frac{k_i}{1-\gamma^2} \frac{\gamma^4}{k_i^2} - \frac{\eta}{\eta - \gamma^2} \frac{\gamma^4}{\eta^2}) C^T C \\ & + (\frac{k_i}{1-\gamma^2} \frac{\gamma^2}{k_i} - \frac{\eta}{\eta - \gamma^2} \frac{\gamma^2}{\eta}) (\tilde{P}QC^T C + C^T CQ\tilde{P}) \\ & + (\frac{k_i}{1-\gamma^2} - \frac{\eta}{\eta - \gamma^2}) \tilde{P}QC^T CQ\tilde{P} \Bigg]. \end{aligned}$$

This expression can be greatly simplified by using the following definitions:

$$\begin{aligned} \alpha_i &:= \frac{k_i}{1-\gamma^2} \frac{\gamma^4}{k_i^2} - \frac{\eta}{\eta - \gamma^2} \frac{\gamma^4}{\eta^2} - \gamma^2 (\frac{1}{\eta} - \frac{1}{k_i}), \\ \beta_i &:= \frac{k_i}{1-\gamma^2} \frac{\gamma^2}{k_i} - \frac{\eta}{\eta - \gamma^2} \frac{\gamma^2}{\eta}, \\ \delta_i &:= \frac{k_i}{1-\gamma^2} - \frac{\eta}{\eta - \gamma^2} \end{aligned}$$

Observe that we can now express (26) in terms of the previous definitions as

$$\frac{k_i}{\gamma^2}\left[\frac{1}{\gamma^2}(\eta - \mathrm{Re}(\mu_i))\tilde{P}BB^T\tilde{P} + \begin{pmatrix} I & \tilde{P}Q \end{pmatrix}\begin{pmatrix} \alpha_i C^T C & \beta_i C^T C \\ \beta_i C^T C & \delta_i C^T C \end{pmatrix}\begin{pmatrix} I \\ Q\tilde{P} \end{pmatrix}\right] \quad (27)$$

Now, the top left corner of the right hand side of (23) is given by

$$-\epsilon\frac{k_i}{\gamma^2}\begin{pmatrix} I & \tilde{P}Q \end{pmatrix}\begin{pmatrix} \frac{\gamma^2}{k_i}C^T C & C^T C \\ C^T C & \frac{k_i}{\gamma^2}C^T C \end{pmatrix}\begin{pmatrix} I \\ Q\tilde{P} \end{pmatrix}$$

and all other corners are equal to zero. It can be shown that for ϵ sufficiently small, we have that

$$\frac{k_i}{\gamma^2}\begin{pmatrix} I & \tilde{P}Q \end{pmatrix}\begin{pmatrix} \alpha_i C^T C & \beta_i C^T C \\ \beta_i C^T C & \delta_i C^T C \end{pmatrix}\begin{pmatrix} I \\ Q\tilde{P} \end{pmatrix} \leq -\epsilon\frac{k_i}{\gamma^2}\begin{pmatrix} I & \tilde{P}Q \end{pmatrix}\begin{pmatrix} \frac{\gamma^2}{k_i}C^T C & C^T C \\ C^T C & \frac{k_i}{\gamma^2}C^T C \end{pmatrix}\begin{pmatrix} I \\ Q\tilde{P} \end{pmatrix}. \quad (28)$$

From (24), (27), and (28) we then have that the Riccati inequality (23) holds if

$$\begin{pmatrix} \frac{k_i}{\gamma^4}(\eta - \mathrm{Re}(\mu_i))\tilde{P}BB^T\tilde{P} & \frac{k_i\mu_i - |\mu_i|^2}{\gamma^4}\tilde{P}BB^T\tilde{P} \\ \frac{k_i\mu_i^* - |\mu_i|^2}{\gamma^4}\tilde{P}BB^T\tilde{P} & -\frac{a_1}{\gamma^4}\tilde{P}BB^T\tilde{P} \end{pmatrix} \leq 0. \quad (29)$$

Here, the top left corner of the matrix is what *remains* of (27) after we take inequality (28) into consideration. The other corners are equal to the corresponding corners of the left hand side of (23). It can be checked by straightforward computation that this inequality holds. Since (28) and (29) hold, it follows that (23) is satisfied by our choice of Z_i in (25) for ϵ sufficiently small. By Lemma 1 we obtain that for $i = 2, 3, \ldots, p$ the transfer matrix G_i of (22) satisfies $||G_i|| < \frac{1}{\gamma}$. Finally, since in Theorem 1 statement (2) implies statement (1), we obtain that the dynamic protocol robustly synchronizes the network. ■

The results of Theorem 2 and Theorem 3 hold in the case the network graph directed and contains a spanning tree, and consequently still hold in the special case it is undirected. However we can improve the supremum (16) of the achievable interval if we assume that the network graph is undirected. Recall that in this case, all eigenvalues of the Laplacian matrix L are real. We then have that $\mathrm{Re}(\lambda_i) = |\lambda_i| = \lambda_i$ for $i = 2, 3, \ldots, p$ and k_i in (25) reduces to $\mu_i = \frac{\lambda_i}{N}$. The achievable interval for undirected networks is given in the following corollary.

Corollary 1: Consider the network with perturbed agent dynamics (10). Assume the network graph is undirected and connected. Define

$$\gamma^* = \sqrt{\frac{\lambda_2}{\lambda_p}}\frac{1}{\sqrt{1+\rho(PQ)}}. \quad (30)$$

Then, for all $\gamma \in (0,\ \gamma^*)$ there exists a dynamic protocol of the form (8) that robustly synchronizes the network with uncertainty tolerance γ.

Proof: The proof is omitted here due to space limitations. ■

VII. CONCLUSIONS

In this paper, we have considered the problem of robust synchronization of directed and undirected multi-agent networks with uncertain agent dynamics. For a given network with identical nominal dynamics for each agent, uncertainty is introduced in the sense that the agent dynamics are coprime factor perturbations a common nominal dynamics. These perturbations are assumed to be stable and have $\mathcal{H}_\infty$-norm that is bounded by an a priori desired uncertainty tolerance. This paper characterizes an interval for the values of the uncertainty tolerance that can be achieved and provides a dynamic observer based protocol that achieves robust synchronization of the network for all perturbations within such a desired tolerance. It has been proven that robust synchronization of the network by the dynamic protocol is equivalent to robust stabilization of a single linear system by all feedback controllers from a given finite set of controllers. The protocols provided are expressed in terms of the stabilizing, real symmetric positive semi-definite solutions to algebraic Riccati equations associated with the nominal dynamics, and contain a weighting factor that depends on the Laplacian matrix of the network graph. For the class of protocols in this paper and for a directed network, the supremum of the achievable interval is proportional to the quotient of the smallest real part and the largest modulus of the eigenvalues of the graph Laplacian. For undirected graphs, these eigenvalues are real and the supremum of the achievable interval is proportional to the square root of the quotient of the smallest and largest eigenvalues of the Laplacian.

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