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Control Lyapunov Function Construction using Symplectic Structure for Partial Differential Equations

Gou Nishida[1] and Bernhard Maschke[2]

Abstract— **This paper presents a systematic way of constructing control Lyapunov functions for a class of partial differential equations with distributed controls in terms of a geometric viewpoint, a symplectic structure. The equations are assumed to include up to second order derivatives with respect to a time coordinate and higher order derivatives with respect to spatial coordinates.**

I. INTRODUCTION

A. Background

Lyapunov functions play a central role in stability analysis and stabilization problems in control theory [1]. However, a general way of finding Lyapunov functions has not been elaborated. In this paper, we derive a necessary condition of constructing control Lyapunov functions for a class of partial differential equations from symplectic structures. We restrict our scope to systems including up to second order derivatives with respect to the time coordinate and higher order derivatives with respect to spatial coordinates for simplification of notations. Our method is based on the geometry of differential forms; therefore, we assume an integrable solution for a given system although analytical solutions in closed-form are not required. We summarize the novelties of our methods in comparison with other methods [2], [3], [4], [5], [6] that studied an inverse problem for reconstructing some potentials as follows:

- Systems of partial differential equations can be treated.
- (Instantaneous) Hamiltonian is considered as a Lyapunov function; however, the original systems are needed to be Hamiltonian systems.
- Topology of system domain can be considered by using the geometry of differential forms.

This paper discusses the above first two matters. On the last topic, we study only the trivial case that the systems are defined on a manifold that is homeomorphic to a Euclidian space. Moreover, we consider only distributed controls here.

The control Lyapunov functions considered in this paper are calculated from the Lagrangians of the systems. However, a Lagrangian cannot be regarded as a Lyapunov function in general. Thus, we transform Lagrangians into generalized energies that can be regarded as Hamiltonians. Hamiltonians can be regarded as the total energies of the systems in terms of the instantaneous multi-symplectic structure in classical field theory [7], [8], [9]. A generalized energy can be regarded as a Lyapunov function if it is positive definite. If it is not positive definite, e.g., saddle around an equilibrium point, we shape the energy in terms of passivity-based controls [4], [10].

The multi-symplectic structure is defined on a jet bundle that is an extended state space for treating higher order variables. In this formalism, the systems are written by means of the differential forms, and we can systematically describe the inverse problem of the variational calculus. The inverse problem means that if a differential operator matrix defining a given system is self-adjoint, which is equivalent to the exactness of differential forms [11], there exist Lagrangians for the system [12], [13]. In the solvable cases, we can derive generalized energies from the Lagrangians through the Legendre transformation.

Although the above concept is simple, the remained problem is that a given system is not always self-adjoint. To deal with this problem, we decompose the system into a self-adjoint subsystem and a non-self-adjoint subsystem by using the geometric property of the differential forms. For the decomposed system, we can derive the Lagrangian for the self-adjoint subsystem by using a homotopy operator [12]. Homotopy operators act as integration for differential forms on manifolds [2], [3], [6], [5], [14]. Then, the system is asymptotic stable if the non-self-adjoint subsystem is dissipative. Hence, we only have to consider a compensation input for satisfying the dissipativity to stabilize the systems.

II. MATHEMATICAL PRELIMINARY

A. Control Lyapunov function

A (global) control Lyapunov function for an undistributed system $\Delta = 0$ relative to the equilibrium state x^0 is a continuous function $V\colon \mathcal{X} \to \mathbb{R}$ such that the following properties hold: (i) V is proper at x^0, i.e., the set $\{x \in \mathcal{X} \mid V(x) \leq L\}$ is compact for each $L > 0$. (ii) V is positive definite on $\mathcal{X}$, i.e., $V(x^0) = 0$ and $V(x) > 0$ for each $x \neq x^0$ in $\mathcal{X}$. (iii) There is a time T for each $x \neq x^0$ in $\mathcal{X}$, and a control u is admissible for x such that, for the path ξ corresponding to u and x, $V(\xi(t)) \leq V(x)$ for all $t \in [0, T)$ and $V(\xi(T)) < V(x)$. If $\mathcal{X}$ is some neighborhood $\mathcal{O}$ of x^0, V is called a local control-Lyapunov function.

B. Lyapunov functional

A Lyapunov functional [15] for a global attractor $\mathcal{A}$ is a functional $\mathscr{V}\colon \mathcal{A} \to \mathbb{R}$ such that (i) $\mathscr{V}$ is continuous on

[1]Gou Nishida is with Department of Applied Analysis and Complex Dynamical Systems, Graduate School of Informatics, Kyoto University, 36-1, Yoshida-Honmachi, Sakyo-ku, Kyoto, Kyoto 606-8501, Japan (g.nishida@ieee.org)

[2]Bernhard Maschke is with Laboratoire d'Automatique et de Génie des Procédés, UMR CNRS 5007, Université Claude Bernard Lyon 1, Université de Lyon, 43 Bd du 11 Novembre 1918, F-69622 Villeurbanne Cedex, FRANCE (maschke@lagep.cpe.fr)

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$\mathcal{A}$, (ii) $\mathscr{V}$ is nonincreasing along trajectories ($\mathscr{V}(u(t))$ is nonincreasing as a function of t), (iii) if $\mathscr{V}(u(t)) = \mathscr{V}(u_0)$ for some $t > 0$ then u_0 is a fixed point, where $\mathcal{A}$ is the maximal compact invariant set with respect to the time evolution of a system and the minimal set that attracts all bounded set.

C. Representation of partial differential equations

Let X be an n-dimensional manifold that is homeomorphic to $\mathbb{R}^m \times \mathbb{R}$ as a spatial-time domain for partial differential equations, where we assumed that X can be split into the spatial domain $\mathbb{R}^m$ and the time axis $\mathbb{R}$, and $n = m + 1$. We denote the local coordinates of X by $x = (x^0, x^1, \cdots, x^m)$, $x^0 = t$ is the time coordinate, and $x^s = (x^1, \cdots, x^m)$ is the set of spatial coordinates.

By regarding functions $u(x)$ of x as independent variables, we consider a trivial bundle $M = X \times U$, where the typical fiber U is homeomorphic to $\mathbb{R}^l$, we denote the local coordinates of U by $u = (u^1, \cdots, u^l)$, and thus those of M by $(x^i; u^a)$ for $1 \leq a \leq l$.

Moreover, by considering k-th order derivative functions of u with respect to x for $0 \leq k \leq r$ as independent variables, we defined a k-th order jet bundle $J^rM = X \times U \times J^1U \times \cdots \times J^rU$, where we denote the local coordinates of J^rM by $(x^i; u^a_I)$ for $0 \leq i \leq m$ and $1 \leq a \leq l$, and I is the multi-index which is a simplified subscript for describing all combinations of derivatives. The order of I is written by $|I|$ such as $0 \leq |I| \leq r$ if u_I includes up to r-th order derivatives. Hence, u_I for $k = |I|$ means all k-th order derivative functions with respect to the repeated combination $I = \{i_1, \cdots, i_k\}$ selected from x. For example, in the case of $n = 2$, $l = 2$, $r = 2$, $x = (t, y)$, and $u = (v, w)$, we denote the local coordinates of J^2M by $(x; u_I) = (t, y; v, w, v_t, v_y, w_t, w_y, v_{tt}, v_{ty}, v_{yy}, w_{tt}, w_{ty}, w_{yy})$.

Let $\Delta(x, u_I) = 0$ be a system of r-th order partial differential equations defined on a jet bundle $J^{2r}M$ restricted to a closed submanifold $\mathcal{Z} = \mathcal{Y} \times \mathcal{T} \subset X$, where $\mathcal{Y}$ means the state space at a point in a time interval $\mathcal{T}$, $\Delta = 0$ means $\Delta_a = 0$ for any $1 \leq a \leq l$, and $\Delta = (\Delta_1, \cdots, \Delta_l) \in (C^\infty(J^rM))^l$.

D. Variational problems described by differential forms

Let us introduce differential forms on the simplified variational bi-complex without using contact forms [12], [13] over jet bundles. Such differential forms can describe variational problems in a unified way.

Definition 2.1: We define the following horizontal j-form on J^rM:

$$\eta = \sum_{i_1 \cdots i_j} c^{i_1 \cdots i_j}\, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_j}, \tag{1}$$

where the sum with respect to $i_1 \cdots i_j$ consists of $\alpha = \binom{m}{j}$ combinations, $c^{i_1 \cdots i_j} \in (C^\infty(J^rM))^\alpha$, and $\Omega^j_h(J^rM)$ is the space of horizontal j-forms.

Definition 2.2: We define the following vertical k-form on J^rM:

$$\omega = \sum_{\substack{a(1)\cdots a(k)\\ I(1)\cdots I(k)}} c^{I(1)\cdots I(k)}_{a(1)\cdots a(k)}\, du^{a(1)}_{I(1)} \wedge du^{a(2)}_{I(2)} \wedge \cdots \wedge du^{a(k)}_{I(k)}, \tag{2}$$

where the pair of indexes $\{a(j), I(j) \mid 1 \leq j \leq k\}$ consists of combinations of $\{a, I\}$ that are differ from each other, the sum with respect to $\{a(1) \cdots a(k), I(1) \cdots I(k)\}$ consists of $\beta = \binom{l \cdot \sharp I}{k}$ combinations, $c^{I(1)\cdots I(k)}_{a(1)\cdots a(k)} \in (C^\infty(J^rM))^\beta$, and $\Omega^k_v(J^rM)$ is the space of vertical k-forms.

Definition 2.3: For any $\eta \in \Omega^j_h(J^rM)$ and any $\omega \in \Omega^k_v(J^rM)$, we define a (j,k)-form on J^rM by $\omega \wedge \eta \in \Omega^{j,k}(J^rM)$, where $\Omega^{j,k}(J^rM)$ is the space of (j,k)-forms.

Definition 2.4: Let Z be a j-dimensional submanifold of the base space X. We define the following functional (j,k)-form on J^rM:

$$\psi = \int_Z \omega \wedge \eta \in \mathscr{F}^{j,k}(J^rM), \tag{3}$$

where $j = \dim Z$, $\mathscr{F}^{j,k}(J^rM)$ is the space of functional (j,k)-forms on J^rM, and the wedge $\wedge$ might be omitted.

Definition 2.5: For any $\omega \in \Omega^k_v(J^rM)$, we define the vertical derivative operator

$$\begin{aligned} d_v \colon \Omega^k_v(J^rM) &\to \Omega^{k+1}_v(J^rM);\\ \omega &\mapsto \sum_{a,I} \frac{\partial c^{I(1)\cdots I(k)}_{a(1)\cdots a(k)}}{\partial u^a_I} du^a_I \wedge du^{a(1)}_{I(1)} \wedge \cdots \wedge du^{a(k)}_{I(k)}, \end{aligned} \tag{4}$$

where the sum with respect to $\{a, I\}$ consists of combinations that are not included in $\{a(j), I(j) \mid 1 \leq j \leq k\}$.

Definition 2.6: For any function or any differential from on J^rM, we define total derivative operator with respect to x^i

$$D_i = \frac{\partial}{\partial x^i} + \sum_{a=1}^{l} \sum_{|I|=0}^{\infty} u^a_{Ii} \frac{\partial}{\partial u^a_I}, \tag{5}$$

where $u_{Ii} = \partial u_I / \partial x^i$, D_i is locally equivalent to partial differentials, and it satisfies linearity and the Leibniz rule, D_i and d_v commute, and $D_i\, du_I = d(D_i u_I) = du_{Ii}$.

A Lagrange density $\mathcal{L}\, dx \in \Omega^{m,0}(J^rM)$, the coefficient of a Lagrange density $\mathcal{L} \in \Omega^0_v(J^rM)$, and a Lagrange density functional $\mathscr{L} = \int \mathcal{L}\, dx \in \mathscr{F}^{m,0}(J^rM)$ are collectively called a Lagrangian hereafter.

Definition 2.7: We define the variational derivative operator

$$\begin{aligned} \delta \colon \mathscr{F}^{m,k}(J^rM) &\to \mathscr{F}^{m,k+1}(J^rM);\\ \int_Z \xi\, dx &\mapsto \int_Z d_v \xi\, dx. \end{aligned} \tag{6}$$

Definition 2.8: Let us consider a Lagrangian $\mathscr{L} = \int_Z \mathcal{L}(x, u_I)\, dx \in \mathscr{F}^{m,0}(J^rM)$. Then, $d_v\mathcal{L} = \Delta\, du \equiv 0$ determined by $\delta\mathscr{L} = 0$ is an Euler-Lagrange equation $\Delta = 0$, where $\Delta = (\Delta_1, \cdots, \Delta_l) \in (C^\infty(J^rM))^l$, and we have used integration by parts in the transformation of $d_v\mathcal{L} = \Delta\, du$.

E. Self-adjointness and Euler-Lagrange equations

If a given system $\Delta = 0$ is an Euler-Lagrange equation derived from a variational problem of a Lagrangian $\mathcal{L} \in \Omega^{n,0}(J^r M)$, if and only if, *the Fréchet derivative* $\mathcal{D}_\Delta$ is self adjoint: $\mathcal{D}^*_\Delta = \mathcal{D}_\Delta$ [12, pp. 329 and 364], where we have defined the operator $\mathcal{D}_\Delta$ and its adjoint operator $\mathcal{D}^*_\Delta$ for a test function $f = (f_1, \cdots, f_l) \in (C^\infty(J^{2r}M))^l$ as the following $(l \times l)$-matrix operator with (a, b)-components for $1 \le a \le l$ and $1 \le b \le l$:

$$(\mathcal{D}_\Delta)_{ab}(f_b) = \sum_{|I|=0}^{r} \frac{\partial \Delta_a}{\partial u_I^b} D_I f_b, \tag{7}$$

$$(\mathcal{D}^*_\Delta)_{ab}(f_b) = \sum_{|I|=0}^{r} (-1)^{|I|} D_I \left(\frac{\partial \Delta_b}{\partial u_I^a} f_b \right). \tag{8}$$

Note that, in the case of $D_\emptyset = 1$ and $\mathcal{D}^*_\Delta$, we must use the Leibniz rule in the calculations regarding $D_I(\,\cdot\,)$.

III. MAIN RESULTS

This section shows the local Lyapunov function construction using the homotopy formula for a class of systems defined on trivial manifolds, i.e., homeomorphic to Euclidian spaces. Let $\Delta(x, u_I) = 0$ be a system of r-th order partial differential equations defined on $J^{2r}M$ restricted to a compact submanifold $\mathcal{Z} \subset X$ with the compact boundary $\partial\mathcal{Z}$. Then, the space of smooth functions on $\mathcal{Z}$ is dense the Hilbert space $H^r(\mathcal{Z})$. All functions are assumed to be smooth without remark hereafter, and we don't mention the stability analysis for each particular system in $H^r(\mathcal{Z})$ as in the usual way of discussions in the analytical methods for PDE. Moreover, we assume that the equations include up to second order derivatives with respect to the time coordinate and higher order derivatives with respect to spatial coordinates, and the system has a unique equilibrium set (possibly periodic orbits). The last assumption is equivalent to the ignore of the condition (iii) in Section II-B.

A. Decomposition of partial differential equations

For a given system of partial differential equations, we have the following decomposition theorem.

Theorem 3.1 ([14]): Let us consider a system $\Delta = \Delta_1\, du^1 + \cdots + \Delta_l\, du^l = 0$ defined on a vertically star-shaped domain over $\mathcal{Z}$ in $J^{2r}M$. Then the system can be described as follows:

$$\Delta = \Delta^E + \Delta^D, \tag{9}$$

where $\Delta^E = (\Delta^E_1, \cdots, \Delta^E_l) \in (C^\infty(J^{2r}M))^l$, $\Delta^D = (\Delta^D_1, \cdots, \Delta^D_l) \in (C^\infty(J^{2r}M))^l$, $\mathcal{D}^*_{\Delta^E} = \mathcal{D}_{\Delta^E}$, and $\mathcal{D}^*_{\Delta^D} \neq \mathcal{D}_{\Delta^D}$.

Moreover, the Lagrangian of the subsystem Δ^E can be derived from the homotopy operator $h_v \colon \Omega^k_v(J^{2r}M) \to \Omega^{k-1}_v(J^{2r}M)$ with respect to $\omega \in \Omega^k_v(J^{2r}M)$ for $k \ge 1$:

$$\begin{aligned}\mathcal{L} &= h_v(\Delta^E\, du) \\ &= \int_0^1 \sum_{a=1}^{l} u^a \cdot \Delta^E_a(x, \lambda u_I)\, d\lambda \in \Omega^0_\sim(J^r M),\end{aligned} \tag{10}$$

where we have defined the quotient space $\Omega^k_\sim(J^r M)$ such that $\xi + \operatorname{Div}\zeta \in \Omega^k_v(J^r M)$ is identified with $\xi \in \Omega^k_v(J^r M)$ for any $\zeta \in (\Omega^k_v(J^r M))^n$, and the total divergence $\operatorname{Div}\zeta = \sum_{j=0}^m D_j \zeta_j \in \Omega^k_v(J^r M)$.

B. Power balance equation of partial differential equations

From a Lagrangian $\mathcal{L}$ of a subsystem $\Delta^E = 0$, we can derive the following generalized energy $\mathcal{E}$.

Definition 3.1 ([8]): Let us consider a system of partial differential equations at most second order with respect to the time coordinate, and the following instantaneous Legendre transformation of the multi-symplectic formalism:

$$\begin{cases} p_a^{Kt} = \dfrac{\partial \mathcal{L}'}{\partial u^a_{Kt}} - D_s p_a^{Kts} & \text{for } 1 \le |Kt| \le r, \\ p_a^{Kts} = 0 \quad \text{if } |Kt| = r, \end{cases} \tag{11}$$

where $D_s f = \sum_{j=1}^m D_j f_j$ is the total divergence with respect to spatial variables for any function $f = (f_1, \cdots, f_m)$. Then, the generalized energy on $J^{2r}M|_\mathcal{Z}$ is given by

$$\begin{aligned}\mathscr{E} &= \int_\mathcal{Z} \mathcal{E}(x, u_I)\, dx^1 \wedge \cdots \wedge dx^m \\ &= \int_\mathcal{Z} \left\{ p_a^{Kt} u^a_{Kt} - \mathcal{L}(x, u_I) \right\} dx^1 \wedge \cdots \wedge dx^m,\end{aligned} \tag{12}$$

where $0 \le |K| \le r - 1$, $0 \le |I| \le r$, and p_a^{Kt} has been regarded as a function of (x, u_I) by substituting (11) into it.

From the Hamiltonian representation of the system (9), we can derive the following relation of the energy variation of the system.

Proposition 3.2 ([9]): At an instantaneous time, the power balance equation of (9) is given by

$$\frac{\partial \mathscr{E}}{\partial t} = \int_\mathcal{Z} \left(e^a_p f^a_p + \sum_{k=0}^{r-1} e^a_{qk} f^a_{qk} + e^a_d f^a_d \right) dx^{(s)}, \tag{13}$$

where $dx^{(s)} = (-1)^{s-1} dx^1 \wedge \cdots \wedge dx^{s-1} \wedge dx^{s+1} \wedge \cdots \wedge dx^m$, $1 \le a \le l$, $0 \le |K| \le r - 1$ means the multi-index with respect to spatial variables, $|K_k| = k$, and we have defined

$$\begin{cases} (f^a_p, e^a_p) = \left((-1)^{|Kt|} D_{Kt} \dfrac{\partial \mathcal{L}}{\partial u^a_{Kt}},\ u^a_t \right), \\ (f^a_{qk}, e^a_{qk}) = \left(D_t u^a_{\bar{K}_k},\ \dfrac{1}{w(\bar{K}_k)} \dfrac{\partial \mathcal{L}}{\partial u^a_{\bar{K}_k}} \right), \\ (f^a_d, e^a_d) = \left(-u^a_t,\ -\Delta^D_a \right). \end{cases} \tag{14}$$

by using $\mathcal{L} \in \Omega^0_v(J^{2r}M)$ such that $d_v\mathcal{L} = \Delta^E\, du$.

Lemma 3.3: If the energy flow $\Delta^D_a u^a_t$ is dissipative, the generalized energy (12) of the system decreases.

Proof: In the case of $e^a_d f^a_d = 0$, the power $\int_\mathcal{Z} (e^a_p f^a_p + \sum_{k=0}^{r-1} e^a_{qk} f^a_{qk})\, dx^{(s)}$ is zero in (13), because the system is conservative. Therefore, if the energy flows out through $e^a_d f^a_d = \Delta^D_a u^a_t$, $\partial\mathscr{E}/\partial t < 0$. ■

Remark 3.1: The Legendre transformation (11) (or Lagrangian) is called almost regular if it is submersion to the image P^{r-1}, where P^{r-1} is a closed embedded subbundle of multi-symplectic manifold Z^{r-1}. Then, vector fields on

P^{r-1} can be pull-backed to $J^{2r}M$ by the Legendre transformation [7]. However, for the aim of controls, such a one-to-one correspondence between Lagrangian and Hamiltonian systems is not required. We only have to obtain at least one Hamiltonian as a Lyapunov function from a given system.

C. System representation

We apply the concept described in the previous section to the following particular system.

Theorem 3.4: Consider the following system of partial differential equations:

$$\Delta_a = u_t^a - f^a(x, u_I) - g^a(x, u_I) = 0, \tag{15}$$

where $g^a(x, u_I) = g_s^a(x, u_I) + g_d^a(x, u_I)$ is the control input such that $g_s^a(x, u_I)$ is designed for shaping the energy of the system and $g_d^a(x, u_I)$ is used for assigning damping effects to the system. Now, we assume that the system (15) is intagrable, i.e., there is a distribution on $J^{2r}M$. We define Δ_a^E and Δ_a^D for $g^a(x, u_I) = 0$. If the generalized energy $\tilde{\mathcal{E}}$ of $\tilde{\Delta}_a^E = \Delta_a^E + g_s^a(x, u_I)$ is positive definite, globally proper, smooth, and $\{\Delta_a^D - g_d^a(x, u_I)\}u_t^a$ is dissipative, then (15) is asymptotic stable.

Proof: In this case, we can regard the generalized energy function (12) as a Lyapunov function. From *Lemma 3.3*, we can obtain the dissipativity of the system. ∎

IV. EXAMPLES

In this section, we describe the control Lyapunov function construction method by using the following two examples in a constructive way.

A. Linear partial differential equation

Let us consider the following system:

$$\begin{cases} \Delta_1 = -u_{tt} + w_x - g^1 = 0, \\ \Delta_2 = -v_{tt} - 2u + v_x - g^2 = 0, \\ \Delta_3 = -w_{tt} + u - g^3 = 0, \end{cases} \tag{16}$$

where $u = u(x,t)$, $v = v(x,t)$ and $w = w(x,t)$ are the state variables, g^1, g^2 and g^3 are the distributed controls, and we have defined $n = 2$, $l = 3$ and $r = 1$. We assume that Δ_1, Δ_2 and Δ_3 are the stationary conditions with respect to the variations of u, v and w, respectively, in the case of $g^a = \Delta_a^D = 0$, i.e., these terms are regarded as external forces to equations of motion.

We first consider the case of the autonomous system with $g^1 = g^2 = g^3 = 0$. Then, the system (16) is not self-adjoint; however, the following Lagrangian for the exact subsystem of (16) can be calculated by (10):

$$\mathcal{L} = \frac{1}{2}\left(u_t^2 + v_t^2 + w_t^2 + uw + vv_x + uw_x\right) - uv. \tag{17}$$

Thus, we can decompose (16) into the following two subsystems:

$$\begin{cases} \Delta_1^E = -u_{tt} - v + \dfrac{w}{2} + \dfrac{w_x}{2}, & \Delta_1^D = v - \dfrac{w}{2} + \dfrac{w_x}{2}, \\ \Delta_2^E = -v_{tt} - u, & \Delta_2^D = -u + v_x, \\ \Delta_3^E = -w_{tt} + \dfrac{u}{2} - \dfrac{u_x}{2}, & \Delta_3^D = \dfrac{u}{2} + \dfrac{u_x}{2}. \end{cases} \tag{18}$$

The generalized energy can be calculated by (12) as follows:

$$\begin{aligned} \mathcal{E} &= u_t p_1^t + v_t p_2^t + w_t p_3^t - \mathcal{L} \\ &= u_t \frac{\partial \mathcal{L}}{\partial u_t} + v_t \frac{\partial \mathcal{L}}{\partial v_t} + w_t \frac{\partial \mathcal{L}}{\partial w_t} - \mathcal{L} \\ &= \frac{1}{2}\left(u_t^2 + v_t^2 + w_t^2 - vv_x - uw - uw_x\right) + uv, \end{aligned} \tag{19}$$

where we have introduced the dual variable $p_a^t = \partial\mathcal{L}/\partial u_t^a$ of u_t^a, i.e., an instantaneous multi-momentum in the multi-symplectic manifold. The energy $\mathcal{E}$ includes all state variables of the system; however, it is not positive definite. Note that there is no need to take into account spatial derivatives in the momenta in (19), because the energy corresponds to an instantaneous Hamiltonian defined in the instantaneous multi-symplectic formalism [8]. An instantaneous Hamiltonian means the conserved quantity with respect to the time evolution.

Next, let us find controls that can transform (19) to be positive definite. That is, we shape (19) as follows:

$$\mathcal{E} = \tilde{\mathcal{E}} + \mathcal{E}_s, \tag{20}$$

$$\begin{aligned} \tilde{\mathcal{E}} &= \frac{1}{2}\left(u_t^2 + v_t^2 + w_t^2\right) \\ &\quad + \frac{1}{4}\left\{2(u+v)^2 + (v-v_x)^2 + (u-w)^2 + (u-w_x)^2\right\}, \end{aligned} \tag{21}$$

$$\mathcal{E}_s = -\frac{1}{4}\left(4u^2 + 3v^2 - v_x^2 - w^2 - w_x^2\right). \tag{22}$$

If we can eliminate $\mathcal{E}_s$ from $\mathcal{E}$ by some control input, then the generalized energy of the system becomes $\tilde{\mathcal{E}} > 0$. The change $\mathcal{E}_s$ affects the Lagrangian as $\tilde{\mathcal{L}} = \mathcal{L} - \mathcal{E}_s$. The stationary condition of the variational problem of $\tilde{\mathcal{L}}$ is

$$\begin{cases} \tilde{\Delta}_1^E = -u_{tt} - 2u - v + \dfrac{1}{2}(w + w_x) = 0, \\ \tilde{\Delta}_2^E = -v_{tt} + \dfrac{1}{2}(v_{xx} - 3v) - u = 0, \\ \tilde{\Delta}_3^E = -w_{tt} + \dfrac{1}{2}(u - w - u_x + w_{xx}) = 0. \end{cases} \tag{23}$$

The system (23) can be realized by setting

$$\begin{cases} g_s^1 = 2u + v + \dfrac{1}{2}(w_x - w), \\ g_s^2 = \dfrac{1}{2}(3v - v_{xx}) - u + v_x, \\ g_s^3 = \dfrac{1}{2}(u + w + u_x - w_{xx}) \end{cases} \tag{24}$$

in (16) as

$$\Delta_a = \tilde{\Delta}_a^E + \Delta_a^D - g_d^a = 0 \tag{25}$$

for $1 \le a \le 3$, where $g^a = g_s^a + g_d^a$ and g_d^a will be designed for stabilizing the systems in the next step.

Finally, let us consider the stabilization of the conservative system (25) by means of the input g_d^a for $1 \le a \le 3$. The time derivative of $\tilde{\mathcal{E}}$ of the system (25) is

$$\begin{aligned}\frac{d\tilde{\mathcal{E}}}{dt} &= \frac{\partial\tilde{\mathcal{E}}}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial\tilde{\mathcal{E}}}{\partial u_t}\frac{\partial u_t}{\partial t} + \frac{\partial\tilde{\mathcal{E}}}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial\tilde{\mathcal{E}}}{\partial v_t}\frac{\partial v_t}{\partial t} \\ &\quad + \frac{\partial\tilde{\mathcal{E}}}{\partial w}\frac{\partial w}{\partial t} + \frac{\partial\tilde{\mathcal{E}}}{\partial w_t}\frac{\partial w_t}{\partial t} \\ &= \frac{1}{2}u_t(-2g^1 + 4u + 2v - w + w_x) \\ &\quad + \frac{1}{2}v_t(-2g^2 - 2u + 3v + v_x) \\ &\quad + \frac{1}{2}w_t(-2g^3 + u + w). \qquad (26)\end{aligned}$$

Note that we don't explicitly use the expressions in (24) in the following, i.e., we will directly calculate the whole g^a through the above relation. Thus, $d\tilde{\mathcal{E}}/dt < 0$ if

$$-2g^1 + 4u + 2v - w + w_x = -u_t, \qquad (27)$$

$$-2g^2 - 2u + 3v + v_x = -v_t, \qquad (28)$$

$$-2g^3 + u + w = -w_t. \qquad (29)$$

Therefore, if we can realize the inputs,

$$g^1 = 2u + \frac{u_t}{2} + v - \frac{w}{2} + \frac{w_x}{2}, \qquad (30)$$

$$g^2 = -u + \frac{3v}{2} + \frac{v_t}{2} + \frac{v_x}{2}, \qquad (31)$$

$$g^3 = \frac{u}{2} + \frac{w}{2} + \frac{w_t}{2}, \qquad (32)$$

the system (16) is controlled to be asymptotic stable.

Figure 1 illustrates the time responses of the state variables u, v and w with respect to the initial conditions, $u(0,x) = 0.1\sin(0.1\pi x)$, $v(0,x) = 0$ and $w(0,x) = 0$ under the boundary conditions, $u(t,-L) = u(t,L)$, $v(t,-L) = v(t,L)$ and $w(t,-L) = w(t,L)$, where $L = 10$. Figure 2 illustrates the time responses of the controlled system with (32) in the same setting. We can see that the unstable original system is controlled to be asymptotic stable.

B. Nonlinear partial differential equation

Let us consider the following system:

$$\begin{aligned}\Delta_1 &= v_{tt} - v_{xx} - 0.1v_t v \\ &\quad - (1 - v^2)(1 + 2v) - g^1 = 0, \qquad (33)\end{aligned}$$

where x is the spatial variable, $v = v(x,t)$ is the distributed state, $g^1 = g^1(x,t)$ is the distributed control, we defined $n = 2$, $l = 1$ and $r = 1$, and we assumed that Δ_1 is the stationary conditions with respect to the variations of v for $g^1 = \Delta_1^D = 0$.

Let $g^1 = 0$. The system (33) is not self-adjoint. Thus, in the same way of the previous example, the following Lagrangian for Δ_1^E can be calculated:

$$\mathcal{L} = v + v^2 - \frac{v^3}{3} - \frac{v^4}{2} + \frac{v^2 v_t}{20} + \frac{v_t^2}{2} - \frac{v_x^2}{2}, \qquad (34)$$

where (33) has been written as

$$\begin{cases}\Delta_1^E = v_{tt} - v_{xx} - (1 - v^2)(1 + 2v), \\ \Delta_1^D = -0.1v_t v.\end{cases} \qquad (35)$$

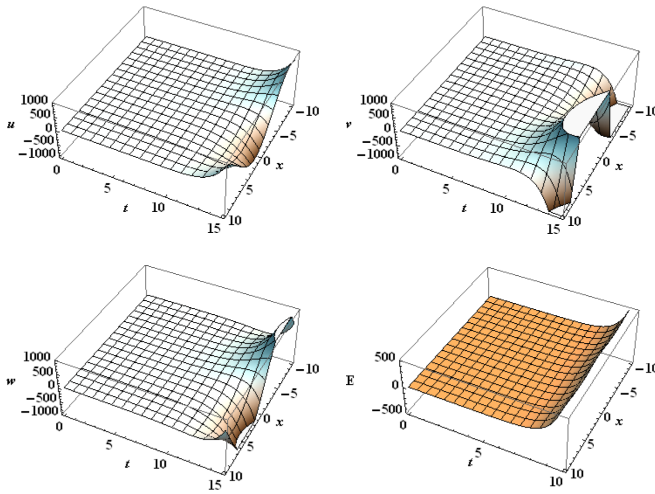

Fig. 1. Initial-value time responses (upper left: u, upper right: v, lower left: w, lower right: $\mathcal{E}$)

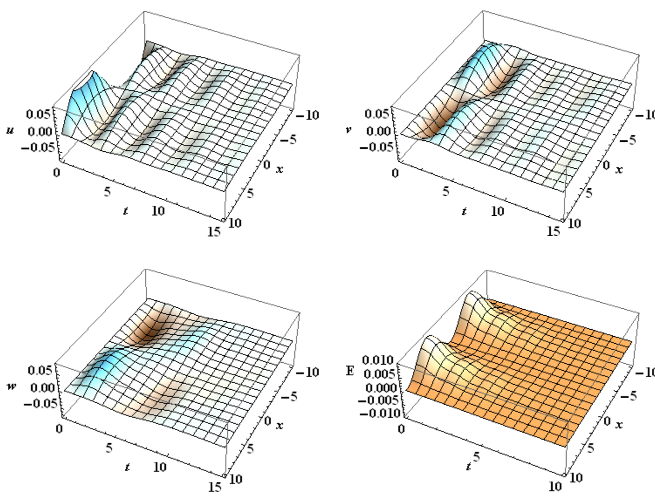

Fig. 2. Controlled time responses (upper left: u, upper right: v, lower left: w, lower right: $\tilde{\mathcal{E}}$)

The generalized energy can be calculated by (12) as follows:

$$\begin{aligned}\mathcal{E} &= v_t p_1^t - \mathcal{L} = v_t\frac{\partial\mathcal{L}}{\partial v_t} - \mathcal{L} \\ &= -v - v^2 + \frac{v^3}{3} + \frac{v^4}{2} + \frac{v_t^2}{2} + \frac{v_x^2}{2}, \qquad (36)\end{aligned}$$

where $p_1^t = \partial\mathcal{L}/\partial v_t$. We shape (36) to be positive definite. The function $-v - v^2 + v^3/3 + v^4/2 + C$ is positive definite with respect to $v = 1$, where C is a constant such that $C \geq 7/6$; therefore, we don't shape this term. Consider $\mathcal{E} = \tilde{\mathcal{E}} + \mathcal{E}_s$, where $\mathcal{E}_s = C$. Such $\mathcal{E}_s$ does not affect the variational problem. The solution of the problem of $\tilde{\mathcal{L}} = \mathcal{L}$ is

$$\tilde{\Delta}_1^E = \Delta_1^E = 0. \qquad (37)$$

Thus, we set $g_s^1 = 0$ in (33), and we will design g_d^1 for stabilizing the systems

$$\Delta_1 = \Delta_1^E + \Delta_1^D - g_d^1 = 0. \tag{38}$$

Here, we calculate the time derivative of $\tilde{\mathcal{E}} = \mathcal{E}$,

$$\begin{aligned}\frac{d\tilde{\mathcal{E}}}{dt} &= \frac{\partial \tilde{\mathcal{E}}}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial \tilde{\mathcal{E}}}{\partial v_t}\frac{\partial v_t}{\partial t} \\ &= \{-v_{xx} - 0.1 v_t v - (1-v^2)(1+2v)\} v_t + v_t v_{tt} \\ &= v_t(v_{xx} + 0.1 v_t v - g^1). \end{aligned} \tag{39}$$

Hence, $d\tilde{\mathcal{E}}/dt < 0$ if $v_{xx} + 0.1 v_t v - g^1 = -v_t$. If we can set

$$g^1 = v_{xx} + 0.1 v_t v + v_t, \tag{40}$$

the system (33) is asymptotic stable at $v = 1$. Although the energy flow of $\Delta_1^D v_t = -0.1 v v_t^2$ is non dissipative, the controller (40) can be reduced as $\bar{g}^1 = v_{xx} + v_t$ if the magnitude of the energy flow $|(v_{xx} + v_t)v_t|$ is sufficiently larger than $|0.1 v v_t^2|$.

Figure 3 shows the time responses of u with respect to the initial conditions, $u(0, x) = \exp(-x^2)$ and $u_t(0, x) = 0$ under the boundary condition $u(t, -L) = u(t, L)$, where $L = 20$. Figure 4 shows the time responses of the stabilized system with $\bar{g}^1$ in the same setting. We can see that the unstable original system is controlled to be asymptotic stable.

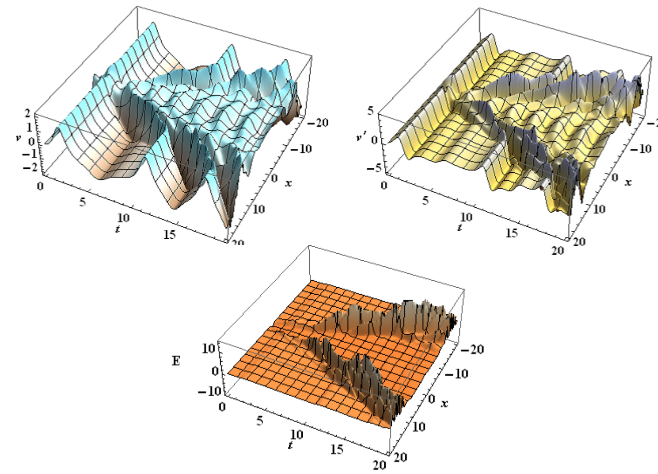

Fig. 3. Initial-value time responses (left: v, right: v_t, bottom: $\mathcal{E}$)

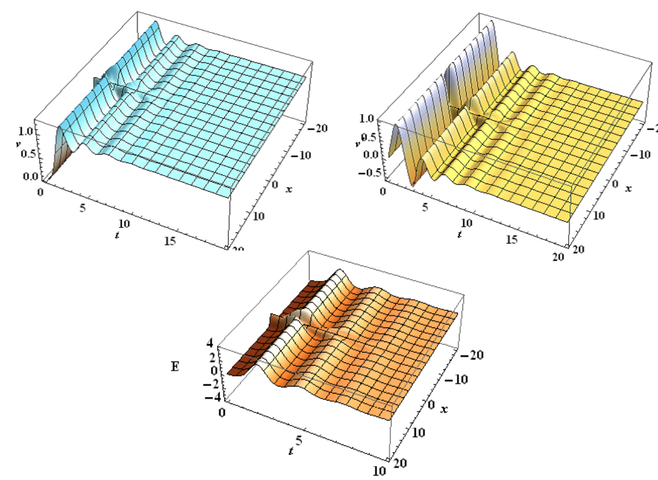

Fig. 4. Controlled time responses (left: v, right: v_t, bottom: $\tilde{\mathcal{E}}$)

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REFERENCES

[1] E.D. Sontag, *Mathmetical Control Theory*, 2nd ed., Springer, 1998.

[2] A. Banaszuk and J. Hauser, "Approximate feedback linearization: A homotopy operator approach", *SIAM J. Control and Optimization*, Vol.34, No.5, pp. 1533-1554, 1996.

[3] W.M. Sluis, A. Banaszuk, J. Hauser and R.M. Murray, "A Homotopy Algorithm for Approximating Geometric Distributions by Integrable Systems", *Systems & Control Letters*, No. 27, pp. 285-291, 1996.

[4] B. Maschke, R. Ortega and A.J. Van der Schaft, "Energy-based Lyapunov functions for forced Hamiltonian systems with dissipation", *IEEE Trans. Automatic Control*, Vol. 45, No. 8, pp. 1498–1502, 2000.

[5] G. Nishida and M. Yamakita, "Disturbance Structure Decomposition for Distributed-Parameter Port-Hamiltonian Systems", in Proc. *the 43rd IEEE Conf. Decision and Control*, Bahamas, pp.2082–2087, Dec., 2004.

[6] M. Guay and N. Hudon, "Stabilization of Nonlinear Systems via Potential-based Realization", in Proc. *IFAC Symp. on Nonlinear Control Systems*, Toulouse, pp. 122–127, 2013.

[7] M.J. Gotay, "A Multisymplectic Framework for Classical Field Theory and the Calculus of Variations: I. Covariant Hamiltonian Formalism", *Mechanics, Analysis and Geometry: 200 Years after Lagrange*, M. Francaviglia (ed.), Elsevier Science Pub. B.V., pp. 203–235, 1991.

[8] M.J. Gotay, "A Multisymplectic Framework for Classical Field Theory and the Calculus of Variations II: Space + time decomposition", *Differential Geometry and its Applications*, Vol. 1, pp. 375–390, 1991.

[9] G. Nishida and B. Maschke, "Implicit Representation for Passivity-Based Boundary Controls", in Proc. *the 4th IFAC Workshop on Lagrangian and Hamiltonian Methods for Non-Linear Control*, Bertinoro, pp. 200-207, 2012.

[10] A.J. van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control*, 2nd revised and enlarged edition, Springer Communications and Control Engineering series, Springer-Verlag, London, 2000.

[11] S. Morita, *Geometry of Differential Forms*, American Mathematical Society, 2001.

[12] P.J. Olver, *Applications of Lie Groups to Differential Equations: Second Edition*, Springer-Verlag, New York, 1993.

[13] I.M. Anderson, "Introduction of the variational bicomplex", *in Contemporary Mathematics No. 132: Mathematical Aspects of Classical Field Theory*, A.M.S., 1992, pp. 51–73.

[14] G. Nishida, M. Yamakita and Z. Luo, "Virtual Lagrangian Construction Method for Infinite Dimensional Systems with Homotopy Operators", *F. Bullo et al. (Eds.): Lagrangian and Hamiltonian methods for nonlinear control 2006*, LNCIS 366, pp. 75–86, Springer, 2007.

[15] J. C. Robinson, *Infinite-Dimensional Dynamical Systems*, Cambridge Univ. Press., 2001.