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Estimation of Cycle-Slipping for Phase Synchronization Systems

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***Abstract*— For multidimensional and infinite-dimensional control systems with periodic differentiable nonlinearities and denumerable sets of equilibria the problem of cycle-slipping is investigated. By means of Lyapunov periodic functions, Kalman-Yakubovich-Popov lemma, and Popov functionals new frequency-algebraic estimates for a number of slipped cycles are obtained.**

***Index Terms*— Multivariable systems, asymptotic properties, scalar and vector Lyapunov functions, Popov-type stability of feedback systems, frequency-response methods.**

I. INTRODUCTION

In this paper we examine asymptotic behavior of control systems based on the principle of *phase synchronization*, or *phase synchronization systems* (PSS), see [6] and references therein. These systems, sometimes referred to as synchronous or pendulum-like control systems, involve periodic nonlinearities and typically have infinite sequence of equilibria points. An important class of PSS is constituted by *phase-locked* systems, which are based on the seminal idea of phase-locked loop (PLL) and widely used in telecommunications and electronics [1], [7], [8].

Recent decades a vast literature examining asymptotic behavior and other dynamical properties of PSSs has been published, motivated by numerous applications of these systems in mechanical, electric, electronic and telecommunication engineering. Most of these papers address the problem of gradient-like behavior, aiming at obtaining conditions which guarantee convergence of the solutions to equilibria, which means that the generators of the system are synchronized for any initial state. For details and bibliography see e.g. [6], [9] and references therein.

But as a rule the synchronous regime of gradient-like system is preceded by cycle slipping, i.e. the increase of the absolute phase error. Its amplitude depends on the initial state of PSS and is an important characteristic of the transient process of the system.

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The phenomenon of cycle slipping was set forth in the book [14], for mathematical pendulum with viscous friction proportional to the square of angular velocity. For mathematical pendulum the number of full rotations around the point of suspension was called the number of slipped cycles.

The extension of this notion to PSSs is as follows. Suppose that a gradient-like phase synchronization system has a Δ -periodic input and let $\sigma(t)$ be its phase error. They say that the output function $\sigma(t)$ has slipped $k \in \mathbf{N} \bigcup \{0\}$ cycles if there exists such a moment $\widehat{t} \geq 0$ that

$$|\sigma(\widehat{t}) - \sigma(0)| = k\Delta, \qquad (1)$$

however for all $t \geq 0$ one has

$$|\sigma(t) - \sigma(0)| < (k+1)\Delta. \qquad (2)$$

So to give the adequate description for behavior of PSSs one must establish possibly close estimates for the number of slipped cycles. And since large number of slipped cycles is undesirable for PSSs the problem of its estimation is important.

In the paper [2] the problem of cycle slipping was considered for multidimensional PSSs. By periodic Lyapunov-like functions and the Kalman-Yakubovich-Popov (KYP) lemma some frequency-algebraic estimates were obtained. The results of [2] were formulated in terms of LMI-solvability in [16]. The estimates of [2] were extended to discrete-time and distributed parameter PSSs in the paper [12] and the monograph [5] respectively. Distributed parameter PSSs were investigated by the method of a priori integral indices with the help of Popov-like functionals of special type.

In this paper two various mathematical models of PSSs are examined. They are the systems of ODE and Volterra integro-differential equations with periodic differentiable nonlinear functions. The former model gives adequate mathematical description of PSSs without distributed components, for example of mathematical pendulum and of phase-locked loops (PLLs). The latter is good for PLLs and communications systems with time-delay as well as for self-synchronizing mechanical systems. We exploit borrowed from [10] generalized periodic Lyapunov-like functions for ODE and corresponding Popov-like functionals for Volterra equations. As a result new frequency-algebraic estimates for the number of slipped cycles are obtained.

II. PRELIMINARIES

In this paper we shall consider multidimensional and infinite-dimensional systems of indirect control with scalar output σ and a periodic scalar input $\varphi(\sigma)$. Their mathematical description is the system of ordinary differential

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equations and integro-differential Volterra equation respectively. The methods of investigation for multidimensional and infinite-dimensional system are different. But the method of construction the periodic Lyapunov-like function and the special Popov-like functional is the same. Consequently all the requirements on the function φ and all the constants and new functions introduced in order to formulate and to prove the main results are the same for the two types of systems.

This section is devoted to the preliminary description of function $\varphi(\sigma)$ and the auxiliary functions.

We suppose that $\varphi(\sigma)$ $(\sigma \in \mathbf{R})$ is a Δ-periodic function with two simple zeros on $[0, \Delta)$.

Let us for definiteness suppose that

$$\int_0^\Delta \varphi(\sigma) d\sigma \leq 0. \tag{3}$$

The function φ is continuously differentiable. So

$$\alpha_1 \leq \frac{d\varphi}{d\sigma} \leq \alpha_2 \quad (\sigma \in \mathbf{R}). \tag{4}$$

with $\alpha_1 \alpha_2 < 0$. Let us define the function

$$\Phi(\sigma) = \sqrt{(1 - \alpha_1^{-1}\varphi'(\sigma))(1 - \alpha_2^{-1}\varphi'(\sigma))}.$$

We shall need the functions

$$r_j(k, \vartheta, x) = \frac{\int_0^\Delta \varphi(\sigma) d\sigma + (-1)^j \frac{x}{\vartheta k}}{\int_0^\Delta |\varphi(\sigma)| d\sigma} \quad (j = 1, 2),$$

$$r_{0j}(k, \vartheta, x) = \frac{\int_0^\Delta \varphi(\sigma) d\sigma + (-1)^j \frac{x}{\vartheta k}}{\int_0^\Delta \Phi(\sigma)|\varphi(\sigma)| d\sigma} \quad (j = 1, 2),$$

$$r_{1j}(k, \vartheta, \varepsilon, \tau, x) = \frac{\int_0^\Delta \varphi(\sigma) d\sigma + (-1)^j \frac{x}{\vartheta k}}{\int_0^\Delta |\varphi(\sigma)| \sqrt{\varepsilon + \tau \Phi^2(\sigma)} d\sigma} \quad (j = 1, 2)$$

and the matrices $T_j(k, \vartheta, x) =$

$$= \left\| \begin{array}{ccc} \varepsilon & \dfrac{a\vartheta r_j(k,\vartheta,x)}{2} & 0 \\ \dfrac{a\vartheta r_j(k,\vartheta,x)}{2} & \delta & \dfrac{a_0\vartheta r_{0j}(k,\vartheta,x)}{2} \\ 0 & \dfrac{a_0\vartheta r_{0j}(k,\vartheta,x)}{2} & \tau, \end{array} \right\|$$

where $a_0 = 1 - a$, $a \in [0, 1]$.

Let us also introduce the functions

$$F_j(\sigma) = \varphi(\sigma) - r_j|\varphi(\sigma)| \quad (j = 1, 2),$$

$$\Psi_j(\sigma) = \varphi(\sigma) - r_{0j}|\varphi(\sigma)|\Phi(\sigma),$$

$$Y_j(\sigma) = \varphi(\sigma) - r_{1j}|\varphi(\sigma)|P(\varepsilon, \tau, \sigma),$$

where

$$P(\varepsilon, \tau, \sigma) = \sqrt{\varepsilon + \tau \Phi^2(\sigma)}.$$

All the results are formulated in terms of the transfer function of the linear part of the control system. We shall denote if by $K(p) \quad (p \in \mathbf{C})$.

The main requirement of the theorems is that for all $\omega \geq 0$ the frequency-domain inequality

$$\begin{array}{l} Re\{\vartheta K(i\omega) - \tau(K(i\omega) + \alpha_1^{-1} i\omega)^*(K(i\omega) + \alpha_2^{-1} i\omega)\} \\ -\varepsilon |K(i\omega)|^2 - \delta \geq 0 \quad (i^2 = -1) \end{array} \tag{5}$$

is valid. Here ε, ϑ, τ and δ are parameters and symbol (*) is used for a complex conjugate number.

III. MULTIDIMENSIONAL PHASE SYSTEMS

Consider an autonomous control system

$$\left. \begin{array}{l} \dfrac{dz}{dt} = Az + b\varphi(\sigma) \quad (z \in \mathbf{R}^m, \sigma \in \mathbf{R}), \\ \dfrac{d\sigma}{dt} = c^* z + \rho\varphi(\sigma). \end{array} \right\} \tag{6}$$

Here A — $m \times m$ - real matrix, b and c are real m - vectors, ρ is a number, and symbol (*) is used for Hermitian conjugation.

We suppose that A is a Hurwitz matrix, the pairs (A, b) and (A, c) are controllable and observable respectively.

First we shall present a number of auxiliary Lyapunov-type assertions.

Lemma 1. *Suppose there exist such numbers $k \in \mathbf{N}$, $a \in [0, 1]$, $\vartheta \neq 0$, positive ε, δ, τ and such continuously differentiable functions $\sigma(t)$ and $W(t)$ that the following conditions are fulfilled:*

1)$W(t) \geq 0$ for $t \geq 0$;

2)

$$\begin{array}{l} \frac{dW(t)}{dt} + \vartheta\varphi(\sigma(t))\frac{d\sigma(t)}{dt} + \varepsilon\left(\frac{d\sigma(t)}{dt}\right)^2 + \delta\varphi^2(\sigma(t)) \\ +\tau\Phi^2(\sigma(t))\left(\frac{d\sigma(t)}{dt}\right)^2 \leq 0, \forall t \geq 0; \end{array} \tag{7}$$

3) matrices $T_j(k, \vartheta, W(0)) \quad (j = 1, 2)$ are positive definite.

Then

$$|\sigma(t) - \sigma(0)| < k\Delta, \quad t \geq 0. \tag{8}$$

Lemma 1 is proved in [9] with the help of Lyapunov functions

$$V_j(t) = W(t) + \vartheta(a \int_{\sigma(0)}^{\sigma(t)} F_j(\sigma)\, d\sigma + a_0 \int_{\sigma(0)}^{\sigma(t)} \Psi_j(\sigma)\, d\sigma).$$

We shall prove here a modification of Lemma 1.

Lemma 2. *Suppose there exist such numbers $k \in \mathbf{N}$, $\vartheta \neq 0$, ε, δ, $\tau > 0$ and such functions $\sigma(t) \in C^1(0, \infty)$ and $W(t) \in C^1(0, \infty)$ that the conditions 1) and 2) of Lemma 1 are fulfilled and*

$$4\delta > \vartheta^2(r_{1j}(k, \vartheta, \varepsilon, \tau, W(0)))^2 \quad (j = 1, 2). \tag{9}$$

Then the assertion (8) is true.

Proof: Let ε_0 be so small that

$$4\delta > \vartheta^2(r_{1j}(k,\vartheta,\varepsilon,\tau,W(0)+\varepsilon_0)^2 \quad (j=1,2). \tag{10}$$

Consider the functions $Y_j(\sigma)$ with

$$r_{1j} = r_{1j}(k,\vartheta,\varepsilon,\tau,W(0)+\varepsilon_0),$$

and two Lyapunov-like functions

$$V_j(t) = W(t) + \vartheta \int_{\sigma(0)}^{\sigma(t)} Y_j(\sigma)d\sigma \quad (j=1,2).$$

Their derivatives are as follows

$$\frac{dV_j(t)}{dt} = \frac{dW(t)}{dt} + \vartheta Y_j(\sigma(t))\dot{\sigma}(t). \tag{11}$$

From condition 2) of Lemma 1 we conclude that

$$\begin{aligned}\frac{dV_j}{dt} \le -\varepsilon\dot{\sigma}^2 - \delta\varphi^2(\sigma) - \tau(\Phi(\sigma)\dot{\sigma})^2 - \\ -\vartheta r_{1j}P(\varepsilon,\tau,\sigma)\,|\varphi(\sigma)|\,\dot{\sigma} \quad (j=1,2),\end{aligned} \tag{12}$$

or

$$\begin{aligned}\frac{dV_j}{dt} \le -(P(\varepsilon,\tau,\sigma)\dot{\sigma})^2 - \delta|\varphi(\sigma)|^2 \\ -\vartheta r_{1j}|\varphi(\sigma)|P(\varepsilon,\tau,\sigma)\dot{\sigma}.\end{aligned} \tag{13}$$

It follows from (10) that

$$\frac{dV_j(t)}{dt} \le 0 \quad (j=1,2), \tag{14}$$

and consequently for all $t>0$

$$V_j(t) \le V_j(0) = W(0) \quad (j=1,2). \tag{15}$$

Suppose now that for a certain $\bar{t}$ we have

$$\sigma(\bar{t}) = \sigma(0) + k\Delta. \tag{16}$$

Then

$$V_1(\bar{t}) = W(\bar{t}) + \vartheta k \int_0^{\Delta} Y_1(\sigma)d\sigma = W(\bar{t}) + W(0) + \varepsilon_0. \tag{17}$$

So $V_1(\bar{t}) > W(0)$ which contradicts (15).

If we suppose that

$$\sigma(\bar{t}) = \sigma(0) - k\Delta \tag{18}$$

we can use $V_2(t)$ and establish that

$$\begin{aligned}V_2(\bar{t}) = W(\bar{t}) - \vartheta k \int_0^{\Delta} Y_2(\sigma)d\sigma = \\ = W(\bar{t}) + W(0) + \varepsilon_0 > W(0).\end{aligned} \tag{19}$$

Lemma 2 is proved. ∎

In order to use the two lemmas for estimation of solutions of system (6) we exploit certain approaches from monographs [3], [5].

Let us introduce the function $\xi(t) = \frac{d}{dt}\varphi(\sigma(t))$ and the $(m+1)$-vector-function

$$y(t) = \left\| \begin{array}{c} z(t) \\ \varphi(\sigma(t)) \end{array} \right\|.$$

We introduce also $(m+1)$ -vectors

$$L = \left\| \begin{array}{c} O \\ 1 \end{array} \right\|, \quad D = \left\| \begin{array}{c} c \\ \rho \end{array} \right\|$$

and the $(m+1)\times(m+1)$-matrix

$$Q = \left\| \begin{array}{cc} A & b \\ O & 0 \end{array} \right\|.$$

Functions $y(t)$ and $\xi(t)$ satisfy the system

$$\left.\begin{aligned}\frac{dy(t)}{dt} &= Qy(t) + L\xi(t), \\ \frac{d\sigma(t)}{dt} &= D^*y(t)\end{aligned}\right\}. \tag{20}$$

The controlability of (A,b) implies the controlability of (Q,L) [3].

Consider the quadratic form of $y \in \mathbf{R}^{m+1}$, $\xi \in \mathbf{R}$:

$$\begin{aligned}G(y,\xi) = 2y^*H(Qy+L\xi) + \varepsilon y^*DD^*y + \vartheta y^*LD^*y \\ -\tau(D^*y - \alpha_1^{-1}\xi)(\alpha_2^{-1}\xi - D^*y) + \delta y^*LL^*y.\end{aligned}$$

Here $H=H^*$ is a real $(m+1)\times(m+1)$-matrix and ε, ϑ, τ and δ are parameters.

Consider the function

$$K(p) = -\rho + c^*(A - pE_m)^{-1}b \qquad (p \in \mathbf{C}),$$

where E_m is a unit $m\times m$-matrix, which is the transfer function of linear part of (6) from the input φ to the output $(-\dot{\sigma}(t))$. It is demonstrated in [3] by means of the KYP lemma that if the frequency-domain inequality (5) is true for $\omega \ge 0$ then there exists such real matrix $H=H^*$ that

$$G(y,\xi) \le 0, \quad \forall y \in \mathbf{R}^{m+1}, \quad \xi \in \mathbf{R}. \tag{21}$$

Let $(z(t),\sigma(t))$ be a solution of (6) with the initial data $(z(0),\sigma(0))$. The corresponding solution $y(t)$ of (20) has the initial data

$$y(0) = \left\| \begin{array}{c} z(0) \\ \varphi(\sigma(0)) \end{array} \right\|.$$

Note that $y(t)$ is bounded (since A is Hurwitz matrix and $\varphi(\sigma)$ is a bounded function).

Theorem 1. *Suppose there exist such $k \in \mathbf{N}$, $a \in [0,1]$, positive ε, δ, τ and $\vartheta \ne 0$ that the following conditions are fulfilled:*

1) for all $\omega \ge 0$ the frequency-domain inequality (5) is true;

2) matrices $T_j\,(k,\vartheta,y^*(0)Hy(0)-I)$ $(j=1,2)$, where $I = inf_{t\in R_+} y^*(t)Hy(t)$, are positive definite for a certain matrix $H=H^*$, satisfying (21).

Then for the solution of (6) with initial data $(z(0),\sigma(0))$ the estimate

$$|\sigma(t) - \sigma(0)| < \Delta k \tag{22}$$

is true for all $t>0$.

Proof: The proof is based on Lemma 1, where $\{z(t),\sigma(t)\}$ is the solution of system (6) with initial data $(z(0),\sigma(0))$. Let

$$W(t) = y^*(t)Hy(t) - I$$

where $y(t)$ is the solution of (20) with $y(0) = \left\| \begin{array}{c} z(0) \\ \varphi(\sigma(0)) \end{array} \right\|$.
We have $W(t) \geq 0$ for $t \geq 0$ and

$$\frac{dW(t)}{dt} = 2y^*(t)H\left(Qy(t) + L\xi(t)\right). \tag{23}$$

Condition 1) of the theorem guarantees that (21) is true, whence as $D^*y = \dot{\sigma}$ and $L^*y = \varphi(\sigma(t)$ it follows that

$$\begin{array}{l} \frac{dW(t)}{dt} \leq -\vartheta\varphi(\sigma(t))\dot{\sigma}(t) - \varepsilon\dot{\sigma}^2(t) - \delta\varphi^2(\sigma(t)) - \\ -\tau\dot{\sigma}^2(t)\left(1 - \alpha_1^{-1}\varphi'(\sigma(t))\right)\left(1 - \alpha_2^{-1}\varphi'(\sigma(t))\right). \end{array} \tag{24}$$

So condition 2) of Lemma 1 is fulfilled. Condition 3) of Lemma 1 coincides with condition 2) of the theorem. It follows from Lemma 1 that estimate (22) holds for all $t \geq 0$. Theorem 1 is proved.

■

Theorem 2. *Suppose for certain parameters ε, δ, $\tau > 0$, $\vartheta \neq 0$ and $k \in \mathbf{N}$ the following conditions are fulfilled:*

1) for all $\omega \geq 0$ the frequency-domain inequality (5) is true;

2)

$$4\delta > \vartheta^2(r_{1j}(k, \vartheta, \varepsilon, \tau, y(0)^*Hy(0) - I))^2 \quad (j = 1, 2) \tag{25}$$

where I and H are defined in the text of Theorem 1.
Then for the solution of (6) with the initial data $(z(0), \sigma(0))$ the estimate (22) is true for all $t \geq 0$.

The proof of Theorem 2 is based on Lemma 2. It is just alike the proof of Theorem 1 with the same functions $\sigma(t)$ and $W(t)$.

IV. INFINITE-DIMENSIONAL PHASE SYSTEMS

Consider the integro-differential Volterra equation

$$\begin{array}{l} \dot{\sigma}(t) = \alpha(t) + \rho\varphi(\sigma(t-h)) - \\ -\int\limits_0^t \gamma(t-\tau)\varphi(\sigma(\tau))d\tau \quad (t > 0, h \geq 0), \end{array} \tag{26}$$

where $\alpha(t)$, $\gamma(t)$ are functions and h, ρ are numbers.

The initial condition for (26) has the form

$$\sigma|_{t\in[-h;0]} = \sigma^0(t) \in C([-h, 0]) \tag{27}$$

We suppose that $\alpha(t)$ is continuous and tends to 0 as t goes to infinity. We suppose also that for a certain positive c the following inclusions are true

$$e^{ct}\gamma(t), \alpha(t) \in L_2[0, +\infty).$$

The transfer function of the linear part of (26) is as follows

$$K(p) = -\rho e^{-hp} + \int\limits_0^{+\infty} e^{-pt}\gamma(t)dt \quad (p \in \mathbf{C}).$$

In this section we are going to use the method of a priori integral indices [11] with Popov-like functionals destined specially for systems with periodic nonlinearities. In order to formulate the theorem about frequency-algebraic evaluation of the cycle-slipping we need the estimates of certain Popov-like functionals.

Let $\sigma(t)$ be the solution of (26), (27). Let T be an arbitrary positive number.

Introduce the functions

$$\eta(t) = \varphi(\sigma(t)), \tag{28}$$

$$\mu(t) = \left\{ \begin{array}{lll} 0 & \text{for} & t < 0 \\ t & \text{for} & t \in [0, 1] \\ 1 & \text{for} & t > 1 \end{array} \right\}, \tag{29}$$

$$\begin{array}{c} \sigma_0(t) = \alpha(t) + (1 - \mu(t-h))\rho\zeta_T(t-h) - \\ -\int\limits_0^t (1 - \mu(\tau - h))\gamma(t-\tau)\zeta_T(\tau)d\tau. \end{array} \tag{30}$$

Let ϑ, δ, ε, τ be parameters. Consider the integrals

$$\begin{array}{c} I_{1T} = -\int\limits_0^{\ell} \{(1 - \mu(t))\dot{\sigma}(t)\eta(t) + \delta(1 - \mu^2)\eta^2(t) + \\ +\alpha_1^{-1}\alpha_2^{-1}\dot{\eta}^2(t) - \left(\dot{\widehat{\mu(t)\eta(t)}}\right)^2\alpha_1^{-1}\alpha_2^{-1}\tau + \\ +\tau\left(\alpha_1^{-1} + \alpha_2^{-1}\right)\dot{\sigma}(t)\left(\dot{\eta}(t) - \dot{\widehat{\mu(t)\eta(t)}}\right)\}dt, \end{array} \tag{31}$$

where $\ell = T$, if $T < 1$, and $\ell = 1$, if $T \geq 1$;

$$\begin{array}{c} I_{2T} = \int\limits_0^T \{-\vartheta\sigma_0(t)\mu(t)\eta(t) - 2(\varepsilon + \tau)\dot{\sigma}(t)\sigma_0(t) + \\ +(\tau + \varepsilon)\dot{\sigma}_0^2(t) + \tau\left(\alpha_1^{-1} + \alpha_2^{-1}\right)\sigma_0(t)\dot{\widehat{\mu(t)\eta(t)}}\}dt; \end{array} \tag{32}$$

Note that the function $\eta(t)$ is bounded for $t \geq 0$ as well as the functions $\dot{\sigma}(t)$ and $\dot{\eta}(t)$. So one can construct the estimates

$$I_{1T} \geq -q_1(\delta, \vartheta, \tau, \alpha_1, \alpha_2), \tag{33}$$

$$I_{2T} \geq -q_2(\vartheta, \varepsilon, \tau, \alpha_1, \alpha_2). \tag{34}$$

where positive q_1, q_2 depend on $max(\varphi(\sigma))$, $\int\limits_0^\infty \alpha^2(t)dt$, $\int\limits_0^\infty \gamma^2(t)dt$.

We shall also need the value

$$q_3 = \frac{\sqrt{\tau}\,W^2}{8(\varepsilon + \tau)\sqrt{\delta|\alpha_1|\alpha_2}} \cdot \max\varphi^2(\sigma) \tag{35}$$

where

$$W = \vartheta + \sqrt{\tau\delta|\alpha_1|\alpha_2}\left(\alpha_1^{-1} + \alpha_2^{-1}\right). \tag{36}$$

Let

$$Q(\vartheta, \delta, \varepsilon, \tau, \alpha_1, \alpha_2) = q_1 + q_2 + q_3. \tag{37}$$

Theorem 3. *Suppose there exist such positive ϑ, ε, δ, τ, $a \in [0, 1]$ and natural k that the following conditions are fulfilled*

1) for all $\omega \geq 0$ the frequency-domain inequality (5) is true;

2) the matrices $T_j(k, \vartheta, Q)$ $(j = 1, 2)$, where the value of Q is defined by (37), are positive definite.

Then for the solution of (26) with the initial condition (27) the estimate

$$|\sigma(0)-\sigma(t)|<k\Delta \qquad (38)$$

is true for all $t>0$.

Proof: Let $\sigma(t)$ be the solution of (26), (27). Let T be an arbitrary positive number.

Introduce the functions

$$\zeta_T(t)=\left\{\begin{array}{ll}\eta(t) & t\le T\\ \eta(T)e^{\lambda(T-t)} & t>T>1 \quad (\lambda>0).\end{array}\right\},$$

$$\eta_T(t)=\mu(t)\zeta_T(t),$$

where $\eta(t)$ and $\mu(t)$ are defined by (28) and (29),

$$\sigma_T(t)=\rho\eta_T(t-h)-\int_0^t\gamma(t-\tau)\eta_T(\tau)d\tau.$$

For $t\in[0,T]$ we have

$$\dot\sigma(t)=\sigma_0(t)+\sigma_T(t). \qquad (39)$$

with $\sigma_0(t)$ defined by (30). Consider a set of functionals

$$\rho_T=\int_0^{+\infty}\left\{\vartheta\sigma_T(t)\eta_T(t)+\delta\eta_T^2(t)+\varepsilon\sigma_T^2(t)+\right.$$
$$\left.+\tau(\sigma_T(t)-\alpha_1^{-1}\dot\eta_T(t))(\sigma_T(t)-\alpha_2^{-1}\dot\eta_T(t))\right\}dt$$

It is demonstrated in [4] by means of a priori indices method that if the frequency-domain inequality (5) is true, then for any $T>0$

$$\rho_T<0. \qquad (40)$$

We shall use the representation

$$\rho_T=I_T+I_{1T}+I_{2T}+I_{4T}, \qquad (41)$$

where

$$I_T=\int_0^T\left\{\vartheta\dot\sigma(t)\varphi(\sigma(t))+\varepsilon\dot\sigma^2(t)+\delta\varphi^2(\sigma(t))+\right.$$
$$\left.+\tau(\alpha_1^{-1}\dot\varphi(\sigma(t))-\dot\sigma(t))(\alpha_2^{-1}\dot\varphi(\sigma(t))-\dot\sigma(t))\right\}dt;$$

$$I_{4T}=\int_T^\infty\left\{\vartheta\sigma_T(t)\eta_T(t)+\delta\eta_T^2(t)+(\varepsilon+\tau)\sigma_T^2(t)-\right.$$
$$\left.-\left(\alpha_1^{-1}+\alpha_2^{-1}\right)\tau\sigma_T(t)\dot\eta_T(t)+\tau\alpha_1^{-1}\alpha_2^{-1}\dot\eta_T^2(t)\right\}dt.$$

The integrals I_{1T},I_{2T} are defined by (31) and (32).

From (40) and (41) we have

$$I_T\le -I_{1T}-I_{2T}-I_{4T}. \qquad (42)$$

In virtue of the form of $\eta_T(t)$ we have

$$I_{4T}=\int_T^\infty\left\{\vartheta\sigma_T(t)\eta(T)e^{\lambda(T-t)}+\delta\eta^2(T)e^{2\lambda(T-t)}+\right.$$
$$+(\varepsilon+\tau)\sigma_T^2(t)+\lambda\left(\alpha_1^{-1}+\alpha_2^{-1}\right)\tau\sigma_T(t)\eta(T)e^{\lambda(T-t)}+$$
$$\left.+\lambda^2\tau\alpha_1^{-1}\alpha_2^{-1}\eta^2(T)e^{2\lambda(T-t)}\right\}dt.$$

Let

$$\lambda=\sqrt{\frac{\delta|\alpha_1|\alpha_2}{\tau}}. \qquad (43)$$

Then

$$I_{4T}\ge-\int_T^\infty\frac{W^2}{4(\varepsilon+\tau)}\cdot\eta^2(T)e^{2\lambda(T-t)}dt$$

with W defined by (36).

Consequently

$$I_{4T}\ge-\frac{\sqrt{\tau}\,W^2}{8(\varepsilon+\tau)\sqrt{\delta|\alpha_1|\alpha_2}}\cdot\eta^2(T).$$

So

$$I_{4T}\ge -q_3(\delta,\varepsilon,\tau,\alpha_1,\alpha_2) \qquad (44)$$

where q_3 is defined by (35).

From (42), (33), (34), and (44) we obtain

$$I_T\le q_1+q_2+q_3\equiv Q(\vartheta,\delta,\varepsilon,\tau,\alpha_1,\alpha_2). \qquad (45)$$

Let $\varepsilon_0>0$ be so small that matrices $T_j(Q+\varepsilon_0)$ are positive definite.

Let us consider the functions $F_j(\sigma)$, $\Psi_j(\sigma)$ $(j=1,2)$, used in previous section, with

$$r_j=r_j(k,\vartheta,Q+\varepsilon_0),$$
$$r_{0j}=r_{0j}(k,\vartheta,Q+\varepsilon_0),$$

It is true that

$$\begin{array}{c}I_T=\vartheta a\int_{\sigma(0)}^{\sigma(T)}F_j(\sigma)d\sigma+\vartheta a_0\int_{\sigma(0)}^{\sigma(T)}\Psi_j(\sigma)+\\ +\int_0^T\left\{\vartheta\dot\sigma(t)\varphi(\sigma(t))+\varepsilon\dot\sigma^2(t)+\delta\varphi^2(\sigma(t))-\right.\\ -\vartheta aF_j(\sigma(t))\dot\sigma(t)-\vartheta a_0\Psi_j(\sigma(t))\dot\sigma(t)+\\ \left.+\tau\dot\sigma^2(t)\Phi(\sigma(t))\right\}dt \quad (j=1,2).\end{array} \qquad (46)$$

In virtue of condition 2) of the theorem, the third term in the right hand part of (46) is the integral of positive definite quadratic form. So

$$I_T\ge\vartheta\left(a\int_{\sigma(0)}^{\sigma(T)}F_j(\sigma)d\sigma+a_0\int_{\sigma(0)}^{\sigma(T)}\Psi_j(\sigma)d\sigma\right) \quad (j=1,2). \qquad (47)$$

Suppose that

$$\sigma(t_1)=\sigma(0)+k\Delta.$$

Then

$$\int_{\sigma(0)}^{\sigma(t_1)}F_1(\sigma)d\sigma=k\int_0^\Delta F_1(\sigma)d\sigma=\frac{1}{\vartheta}(Q+\varepsilon_0),$$

$$\int_{\sigma(0)}^{\sigma(t_1)}\Psi_1(\sigma)d\sigma=k\int_0^\Delta\Psi_1(\sigma)d\sigma=\frac{1}{\vartheta}(Q+\varepsilon_0).$$

Then

$$I_{t_1}\ge Q+\varepsilon_0>Q. \qquad (48)$$

which contradicts with (45). So our hypothesis is wrong. With the help of $F_2(\sigma)$ and $\Psi_2(\sigma)$ we prove that

$$\sigma(t) \neq \sigma(0) - k\Delta.$$

As a result for all $t > 0$

$$\sigma(0) - k\Delta < \sigma(t) < \sigma(0) + k\Delta.$$

Theorem 3 is proved. ■

V. CONCLUSION

The paper is devoted to the problem of cycle-slipping for multidimensional and infinite-dimensional control systems with periodic nonlinear input and denumerable set of equilibria (phase synchronization systems). The case of differentiable nonlinearities is considered. The problem is investigated with the help of Lyapunov direct method, the Kalman-Yakubovich-Popov lemma, and the method of a priory integral indices. In the paper new types of Lyapunov-like functions and Popov-like functionals are used. As a result a series of multiparametric frequency-algebraic estimates for the number of slipped cycles for the output of the system is established. New frequency-algebraic estimates for deviation of the output of the system from its initial state can be extended to phase synchronization systems with vector input, just in the same way as the estimates by Ershova and Leonov [2] have been extended in [15], [13].

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