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Input-to-State Stability for Discrete-Time Monotone Systems

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***Abstract*— It is well known that input-to-state stability admits an astonishing number of equivalent characterizations. Here it is shown that for monotone systems on $\mathbb{R}^n_+$ there are some additional characterizations that are useful for network stability analysis. These characterizations include system theoretic properties, algebraic properties, as well as the problem of finding simultaneous bounds on solutions to a collection of inequalities.**

I. Introduction

We consider $\mathbb{R}^n$ equipped with the standard partial order given by the relations

$$x \le y \iff x_i \le y_i \text{ for } i = 1, \dots, n,$$
$$x < y \iff x \le y \text{ and } x \ne y,$$
$$x \ll y \iff x_i < y_i \text{ for } i = 1, \dots, n.$$

We also need the negations of these notions, i.e., $x \not\le y$ if and only if there exists an index i such that $x_i > y_i$. The other negations are defined in a similar manner. The *nonnegative orthant* $\mathbb{R}^n_+$ is the set of all $x \in \mathbb{R}^n$ such that $x \ge 0$. Similarly, $\mathbb{R}^{n\times m}_+$ denotes the set of matrices with nonnegative entries.

Recall that a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is monotone if $x \le y$ implies $f(x) \le f(y)$. In the sequel let $g\colon \mathbb{R}^n_+ \times \mathbb{R}^m_+ \to \mathbb{R}^n_+$ be a continuous and monotone mapping with $g(0,0) = 0$.

In recent years, especially in the context of stability analysis of networks of dynamical systems by means of small-gain theory and comparison principles [1–3, 5, 6, 8–14, 16, 19–21, 23, 24, 26, 27], the following problems have appeared frequently.

1) For a given $w \in \mathbb{R}^m_+$ find a bound on the maximal solution $s \in \mathbb{R}^n_+$ of the vector inequality
$$s \le g(s, w). \tag{1}$$
2) Find unbounded paths $\sigma\colon \mathbb{R}_+ \to \mathbb{R}^n_+$ and $\rho\colon \mathbb{R}_+ \to \mathbb{R}^m_+$ with $\sigma(0) = 0$ and $\rho(0) = 0$ so that
$$g(\sigma(r), \rho(r)) \ll \sigma(r) \text{ for all } r > 0. \tag{2}$$
3) Verify that for all $s \in \mathbb{R}^n_+$, $s \ne 0$,
$$g(s, 0) \not\ge s, \tag{3}$$
or, equivalently, that for all $s \in \mathbb{R}^n_+$,
$$g(s, 0) \ge s \implies s = 0. \tag{4}$$
4) Verify that the discrete-time system[1]
$$s^+ \le g(s, w), \tag{5}$$
respectively,
$$s^+ = g(s, w) \tag{6}$$
is input-to-state stable from w to s.

The important observation made in this paper is that the listed conditions are, essentially, all equivalent in the sense that one problem can be solved if any of the others can be solved. The precise formulation of this statement will be given in the next section, after some technical concepts have been introduced.

Some of the relations 1)–4) may appear more familiar for functions g of a very special form. To this end let $A \in \mathbb{R}^{n\times n}_+$, $B \in \mathbb{R}^{n\times m}_+$, and let $g(s, w) := As + Bw$. Then it is clear that the stability of the time-discrete systems (5) and (6) is intimately related to the condition that the spectral radius $\rho(A)$ is less than unity. This in turn allows to compute $s = (I - A)^{-1}Bw$, the unique maximal solution to (1). Perhaps not so well known is that $\rho(A) < 1$ if and only if (3) holds, see, e.g., [22]. Finally, the paths σ and ρ are closely connected to an eigenvector of A via the celebrated Perron-Frobenius Theorem, cf. [22].

For functions g of a special form, some of the above conditions and their relations have been investigated in previous works, in particular [16, 22, 23, 26].

It should be stressed that the equivalences here are strongly based on the monotonicity of g and the resulting forward invariance of $\mathbb{R}^n_+$ with respect to (6). In this regard our results are not a mere extension of the impressive list of equivalent characterizations of input-to-state stability given in [7] and [15], as the equivalences in those works also hold without the monotonicity requirement.

II. Definitions and main results

A function $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, and satisfies $\gamma(0) = 0$. It is said to be of class $\mathcal{K}_\infty$ if in addition it is unbounded. A function $\lambda\colon \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{L}$ if it is non-increasing and satisfies $\lim_{s\to\infty} \lambda(s) = 0$. A function $\beta\colon \mathbb{R}^2_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if for every fixed $s > 0$, $\beta(\cdot, s)$ is of class $\mathcal{K}$ and $\beta(s, \cdot)$ is of class $\mathcal{L}$. By $|\cdot|$ we denote the max-norm and by $\phi(k, s, w(\cdot))$, $k \ge 0$, the flow generated by (6). Observe that by monotonicity of g we always have

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[1]The notation $s^+ = f(s)$ is short-hand for $s[k+1] = f(s[k])$.

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$\phi(k, s, w(\cdot)) \leq \phi(k, s, \sup_l w(l))$, so that in many cases we can dispense with constant inputs. For constant inputs w we will use the notation $g_w^k(s) := \phi(k, s, w)$. The vector $(1, \ldots, 1)^\top$ will be denoted by e.

System (6) is *input-to-state stable (ISS) from w to s*, c.f. [25], if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $s \in \mathbb{R}_+^n$, $w \in \mathbb{R}_+^m$,

$$|g_w^k(s)| \leq \beta(|s|, k) + \gamma(|w|). \tag{7}$$

A continuous function $V: \mathbb{R}_+^n \to \mathbb{R}_+$ is an *ISS Lyapunov function for system* (6) if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}$ such that for all s and w,

$$\alpha_1(|s|) \leq V(s) \leq \alpha_2(|s|) \quad \text{and} \tag{8}$$

$$V(s) \geq \gamma(|w|) \implies V(g(s, w)) - V(s) \leq -\alpha_3(V(s)). \tag{9}$$

If it exists then without loss of generality the function V can be assumed to be smooth, cf. [15].

System (6) has the *asymptotic gain (AG) property* (cf. [7]) if there exists a $\gamma \in \mathcal{K}$ such that for all $s \in \mathbb{R}_+^n$ and $w \in \mathbb{R}_+^m$,

$$\limsup_{k \to \infty} |g_w^k(s)| \leq \gamma(|w|). \tag{10}$$

We will call a continuous monotone function $\zeta: \mathbb{R}_+^n \to \mathbb{R}_+^m$ *proper* if there exists a function $\tilde{\alpha} \in \mathcal{K}_\infty$ such that for all $s \in \mathbb{R}_+^n$,

$$\tilde{\alpha}(|s|)e \leq \zeta(s). \tag{11}$$

Observe that a proper function $\zeta: \mathbb{R}_+^n \to \mathbb{R}_+^m$ is positive definite if and only if there exists an $\hat{\alpha} \in \mathcal{K}_\infty$ such that for all $s \in \mathbb{R}_+^n$,

$$|\zeta(s)| \leq \hat{\alpha}(|s|). \tag{12}$$

A continuous, monotone function $g: \mathbb{R}_+^n \times \mathbb{R}_+^m \to \mathbb{R}_+^n$ with $g(0, 0) = 0$ is called *eventually increasable* if for all $s \in \mathbb{R}_+^n$ there exists a $k \geq 1$ and $w \in \mathbb{R}_+^m$ such that

$$s \leq g_w^k(s). \tag{13}$$

Theorem 1: The following statements are equivalent:

ISS-LF: there exists a monotone ISS Lyapunov-function for (6);

GAS: there exists a proper and positive definite map $\zeta: \mathbb{R}_+^n \to \mathbb{R}_+^m$ so that the origin is globally asymptotically stable with respect to

$$s^+ = f(s) := g(s, \zeta(s)); \tag{14}$$

ISS: system (6) is input-to-state stable;

AG: system (6) has the asymptotic gain property. △

Theorem 2: The following statements are equivalent:

UOC: (Uniform order condition) there exists a proper and positive definite map $\zeta: \mathbb{R}_+^m \to \mathbb{R}_+^n$ such that

$$g(s, w) \not\geq s \text{ for all } s \not\leq \zeta(w); \tag{15}$$

NP: (Neumann property) there exists a proper and positive definite $\zeta: \mathbb{R}_+^m \to \mathbb{R}_+^n$ such that for all $s \in \mathbb{R}_+^n$, $w \in \mathbb{R}_+^m$,

$$s \leq g(s, w) \implies s \leq \zeta(w). \qquad \triangle$$

Theorem 3: The properties listed in Theorem 1 imply those in Theorem 2. If g is eventually increasable, then the reverse implication holds as well. △

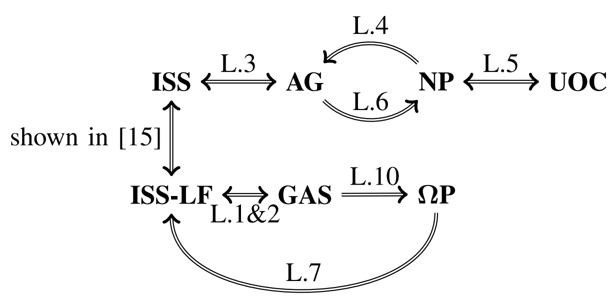
Theorem 4: The following property,

ΩP: there exist proper and positive definite $\sigma: \mathbb{R}_+ \to \mathbb{R}_+^n$ and $\rho: \mathbb{R}_+ \to \mathbb{R}_+^m$ such that

$$\text{for all } r > 0, \quad g\big(\sigma(r), \rho(r)\big) \ll \sigma(r), \tag{16}$$

implies every property listed in Theorems 1 and 2. Conversely, if g is eventually increasable, then every the properties listed in Theorems 1 and 2 implies ΩP. △

The proof of the Theorems will be given in the appendix, divided into multiple lemmas. In particular, the following implications will be shown:



Observe that only in Lemmas 4 and 10 the map g is assumed to be eventually increasable.

Example 1: Without the condition that g is eventually increasable the implication NP to ΩP may not hold, as the following example demonstrates. Let $\gamma: \mathbb{R}_+ \to \mathbb{R}_+$, $f: \mathbb{R}_+^2 \to \mathbb{R}_+^2$, and $g: \mathbb{R}_+^2 \times \mathbb{R}_+^m \to \mathbb{R}_+^2$ be given by

$$\gamma(r) := \frac{r^2}{1 + r} \text{ and } g(s, w) := f(s) := \begin{pmatrix} \gamma(s_1) + s_2 \\ \gamma(s_2) \end{pmatrix}.$$

By considering separately the cases $s_1 = 0$, s_2 arbitrary and $s_2 = 0$, s_1 arbitrary, we see that $f(s) \not\geq s$ for all $s > 0$. Note that g is not eventually increasable, since f does not depend on w.

The origin is GAS for $s^+ = f(s)$, since the second component of any trajectory decreases strictly to zero, so that eventually $f^k(s)$ is dominated by a point in the region $\Omega = \{s \in \mathbb{R}_+^2 : f(s) \ll s\}$ (it is sufficient to see that eventually $f_2^k(s) < 1$); monotonicity then implies $f^k(s) \to 0$ and also stability of the origin, cf. Corollary 1 in the appendix. Hence system (6) is ISS and, moreover, by Theorem 3 it satisfies NP.

However, an unbounded path σ as in (16) cannot exist, because $f(s) \ll s$ implies $s_1 \leq 1$, see Figure 1. ▽

Remark 1: The *generalized small-gain condition*, i.e., for all $s \in \mathbb{R}^n_+$, $s > 0$,

$$\Gamma_\mu(s) \ngeq s$$

where $\Gamma = (\gamma_{ij}) \in (\mathcal{K} \cup \{0\})^{n\times n}$, $\mu\colon \mathbb{R}^n \to \mathbb{R}^n$ a *monotone aggregation function*, and $\Gamma_\mu(s)_i = \mu\big(\gamma_{i1}(s_1), \ldots, \gamma_{in}(s_n)\big)$, cf. [4, 17, 22], which is frequently seen in the context of network small-gain theorems, is of course a special case of UOC ($s \nleq \zeta(0) \iff s > 0$). △

Remark 2: Property NP can equivalently be stated as follows: There exists a $\gamma \in \mathcal{K}_\infty$ such that for all $s \in \mathbb{R}^n_+$,

$$w \in \mathbb{R}^m_+, \quad s \le g(s,w) \implies |s| \le \gamma(|w|). \qquad (17)$$

To see that (△) implies (17) we apply norms to (△) in order to get

$$|s| \le |\zeta(w)| \le |\zeta(|w|e)| \le \max_{1\le i\le m} \zeta_i(|w|e) =: \gamma(|w|),$$

and from there it is plain to see that $\gamma \in \mathcal{K}_\infty$ because ζ is proper and positive definite. For the other direction define $\zeta(w) := \gamma(|w|)e$. Then $|s| \le \gamma(|w|)$ implies $s \le |s|e \le \gamma(|w|)e = \zeta(w)$. △

Remark 3: A Lyapunov function for a monotone system of the form

$$s^+ = f(s), \quad s \in \mathbb{R}^n_+,$$

can always be assumed to be itself a monotone function $V\colon \mathbb{R}^n_+ \to \mathbb{R}_+$. This follows from converse Lyapunov results like the converse ISS Lyapunov result [15] or the converse Lyapunov result [18] for autonomous systems. Indeed, constructions like [18] utilize Sontag's Lemma on $\mathcal{KL}$ functions to define a Lyapunov function $V\colon \mathbb{R}^n_+ \to \mathbb{R}_+$ via

$$V(s) := \sup_{k\ge 0} \alpha(|f^k(s)|)e^k$$

with a locally Lipschitz $\alpha \in \mathcal{K}_\infty$ satisfying $\alpha(|f^k(s)|) \le \hat\alpha(|s|)e^{-2k}$ for some $\hat\alpha \in \mathcal{K}_\infty$. From this definition it is immediate that V must be monotone and continuous (even locally Lipschitz) in s. In the literature on converse Lyapunov theorems the candidate function V usually undergoes additional smoothing steps to obtain a continuously differentiable Lyapunov function, which we do not need here. △

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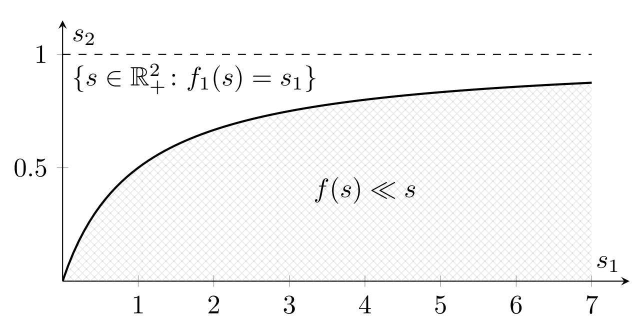

Fig. 1. The region $\Omega = \{s \in \mathbb{R}^2_+ : f(s) \ll s\}$ in Example 1.

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APPENDIX

Some of the properties in Theorems 1 and 2 use negations of order relations. It is thus sometimes easier to work with the negations of these properties in proofs, as we do in Lemmas 4 and 6. To this end we summarize the needed statements here.

$\neg$AG: For all $\gamma \in \mathcal{K}_\infty$ there exist $s \in \mathbb{R}^n_+, w \in \mathbb{R}^m_+$ such that
$$\limsup_{k\to\infty} |g^k_w(s)| > \gamma(|w|)\,. \tag{18}$$

$\neg$NP: For all proper and positive definite $\zeta : \mathbb{R}^m_+ \to \mathbb{R}^n_+$ there exist $s \in \mathbb{R}^n_+, w \in \mathbb{R}^m_+$ such that
$$s \le g(s,w) \tag{19}$$
and
$$s \not\le \zeta(w)\,. \tag{20}$$

Lemma 1: ISS-LF $\Longrightarrow$ GAS. △

Proof. We start with an ISS Lyapunov function $V\colon \mathbb{R}^n_+ \to \mathbb{R}_+$ which satisfies
$$V(s) \ge \gamma(|w|) \Longrightarrow V(g(s,w)) - V(s) \le -\alpha(V(s)) \tag{21}$$
for some $\gamma \in \mathcal{K}$ and $\alpha \in \mathcal{K}_\infty$. Define a proper and positive definite map $\zeta\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by $\zeta(s) := \gamma^{-1}(V(s))e$ and consider (14). The choice of ζ ensures that the decay condition in (21) is always satisfied, because
$$\begin{aligned} V(s) &= \gamma(\gamma^{-1}(V(s))) = \gamma(|\gamma^{-1}(V(s))e|) \\ &= \gamma(|\zeta(s)|) = \gamma(|w|). \end{aligned}$$
Hence, for (14) we have $V(f(s)) < V(s)$ whenever $s > 0$. We conclude by standard Lyapunov arguments that (14) is GAS and the proof is complete. □

Lemma 2: GAS $\Longrightarrow$ ISS-LF. △

Proof. Because (14) is globally asymptotically stable there exists a continuous, proper, radially unbounded, and monotone Lyapunov function $V\colon \mathbb{R}^n_+ \to \mathbb{R}_+$, cf. Remark 3, such that for some $\overline{\alpha}, \underline{\alpha} \in \mathcal{K}_\infty$ we have
$$\underline{\alpha}(|s|) \le V(s) \le \overline{\alpha}(|s|)\,. \tag{22}$$
The map ζ is proper and positive definite, hence there exists $\tilde\alpha \in \mathcal{K}_\infty$ such that (11) holds. Define $\gamma := \overline{\alpha} \circ \tilde\alpha^{-1} \in \mathcal{K}_\infty$. Now consider the case that $V(s) \ge \gamma(|w|)$. This implies
$$\overline{\alpha}(|s|) \ge \gamma(|w|) = \overline{\alpha}(\tilde\alpha^{-1}(|w|))$$
and thus
$$\tilde\alpha(|s|)e \ge |w|e \ge w.$$
Application of (11) yields
$$\zeta(s) \ge \tilde\alpha(|s|)e \ge w\,. \tag{23}$$
By monotonicity of V and using (23) we get
$$V(g(s,w)) \le V(g(s,\zeta(s))) < V(s),$$
proving that $V(s) \ge \gamma(|w|)$ implies $V(g(s,w)) < V(s)$. This shows the existence of an ISS-Lyapunov function for system (6). □

Lemma 3: ISS $\Longleftrightarrow$ AG. △

Proof. That input-to-state stability implies the asymptotic gain property is obvious. For the other direction consider the AG property (10). Fix $\varepsilon > 0$. Then for any $s_0 \in \mathbb{R}^n_+$ we can find a $T = T(s_0,\varepsilon) \in \mathbb{N}$ such that
$$\sup_{k\ge T} |g^k_w(s_0)| \le \gamma(|w|) + \varepsilon.$$
By monotonicity of the flow, respectively, g we obtain that for all $s \le s_0$,
$$\sup_{k\ge T} |g^k_w(s)| \le \sup_{k\ge T} |g^k_w(s_0)| \le \gamma(|w|) + \varepsilon.$$
One can verify that this property thus coincides with the (in general stronger) *uniform* asymptotic gain (UAG) property (uniform in the sense that the supremum is attained uniformly for initial conditions in compact sets and all inputs), cf. [7], where it is shown that UAG implies ISS. □

Lemma 4: Let g be eventually increasable. Then NP $\Longrightarrow$ AG. △

Proof. We will show $\neg$AG implies $\neg$NP. To this end let $\zeta : \mathbb{R}^m_+ \to \mathbb{R}^n_+$ be proper and positive definite. Choose $\gamma \in \mathcal{K}_\infty$ such that
$$\gamma(r)e \ge \zeta(re) \tag{24}$$
for all $r \in \mathbb{R}_+$. By $\neg$AG, see (18), there exist $s^* \in \mathbb{R}^n_+$ and $w^* \in \mathbb{R}^m_+$ so that
$$\limsup_{k\to\infty} |g^k_{w^*}(s^*)| > \gamma(|w^*|)\,. \tag{25}$$
Define $\overline{s} := \limsup_{k\to\infty} g^k_{w^*}(s^*)$.

First we assume that $\overline{s}$ is finite. Hence by (25) and using monotonicity of g and the norm we have

$$|\overline{s}| = |\limsup_{k\to\infty} g^k_{w^*}(s^*)| \geq \limsup_{k\to\infty} |g^k_{w^*}(s^*)| > \gamma(|w^*|). \tag{26}$$

Similarly, we deduce

$$\begin{aligned} g(\overline{s}, w^*) &= g(\limsup_{k\to\infty} g^k_{w^*}(s^*), w^*) \\ &\geq \limsup_{k\to\infty} g^{k+1}_{w^*}(s^*) = \overline{s}, \end{aligned} \tag{27}$$

which is (19), respectively, the first part of ¬NP. Because of the max-norm, (26), and (24) there exists an index i with

$$\overline{s}_i = |\overline{s}| > \gamma(|w^*|) \geq \zeta_i(|w^*|e) \geq \zeta_i(w^*), \tag{28}$$

which is equivalent to (20). Equation (27) together with (28) is ¬NP.

Now assume that at least one of the components of $\overline{s}$ is infinite. Thus

$$\limsup_{k\to\infty} g^k_{w^*}(s) \not< \infty\,. \tag{29}$$

Because g is eventually increasable, there exist $k \geq 1$ and $\overline{w} \geq w^*$ such that

$$s \leq g^k_{\overline{w}}(s)\,.$$

Applying $g_{\overline{w}}$ on both sides repetitively yields, by monotonicity of $g_{\overline{w}}$,

$$g^n_{\overline{w}}(s) \leq g^{n+k}_{\overline{w}}(s) \tag{30}$$

for all $n \in \mathbb{N}$. Observe that by monotonicity (29) still holds if w^* is replaced by $\overline{w}$ and hence for all monotone, proper, and positive definite $\zeta : \mathbb{R}^m_+ \to \mathbb{R}^n_+$ there exists a $K \in \mathbb{N}$ such that

$$s^\# := \sup_{K\leq l\leq K+k-1} g^l_{\overline{w}}(s) \not\leq \zeta(\overline{w}),$$

establishing (19). Using monotonicity of g gives

$$\begin{aligned} g(s^\#, \overline{w}) &= g\Big(\sup_{K\leq l\leq K+k-1} g^l_{\overline{w}}(s), \overline{w}\Big) \\ &\geq \sup_{K\leq l\leq K+k-1} g^{l+1}_{\overline{w}}(s) \\ &= \sup\Big\{g^{K+1}_{\overline{w}}(s), \dots, g^{K+k-1}_{\overline{w}}(s), g^{K+k}_{\overline{w}}(s)\Big\} \\ &\geq \sup\Big\{g^{K+1}_{\overline{w}}(s), \dots, g^{K+k-1}_{\overline{w}}(s), g^{K}_{\overline{w}}(s)\Big\} \\ &= s^\#, \end{aligned}$$

where in the last inequality we have used (30) for $n = K$. This establishes (20) and thus completes the proof. □

Lemma 5: NP $\Longleftrightarrow$ UOC. △

Proof. First note that UOC can be equivalently rephrased as:

> There exists a proper, monotone, and positive definite $\zeta : \mathbb{R}^m_+ \to \mathbb{R}^n_+$ such that for all $s \in \mathbb{R}^n_+$, $w \in \mathbb{R}^m_+$,
>
> $$s \not\leq \zeta(s) \implies g(s, w) \not\geq s.$$

This implication can be easily be rewritten to obtain NP, and thus the two properties are the same. □

Lemma 6: AG $\Longrightarrow$ NP. △

Proof. We will show that ¬NP implies ¬AG. To this end let $\gamma \in \mathcal{K}_\infty$ and define $\zeta\colon \mathbb{R}^m_+ \to \mathbb{R}^n_+$ by $\zeta(w) := \gamma(|w|)e$. It is easy to see that ζ is proper and positive definite. By ¬NP for this choice of γ and ζ there exist $s \in \mathbb{R}^n_+, w \in \mathbb{R}^m_+$ satisfying (19) and (20). With the monotonicity of g it follows from (19) that

$$s \leq g(s, w) \leq \dots \leq g^k_w(s)$$

for all $k \in \mathbb{N}$. And thus by taking norms

$$|s| \leq \limsup_{k\to\infty} |g^k_w(s)|. \tag{31}$$

Due to (20) there exists an index i such that $\zeta_i(w) < s_i$. Combining the latter with (31) results in

$$\gamma(|w|) = \zeta_i(w) < s_i \leq |s| \leq \limsup_{k\to\infty} |g^k_w(s)|$$

where the first equation is by definition of ζ. This establishes ¬AG, cf. (18). □

Lemma 7: ΩP $\Longrightarrow$ ISS-LF. △

Proof. Let $V(s) := \max_i \sigma_i^{-1}(s_i)$ and $\gamma(r) := \max_i \rho_i^{-1}(r)$, a class $\mathcal{K}_\infty$ function.

Clearly, V is monotone and satisfies

$$V(s) \leq \max_i \sigma_i^{-1}(|s|) =: \alpha_2(|s|)$$

and, since $\sigma\big(V(s)\big) \geq s$, we also have

$$\big(\max_i \sigma_i\big)\big(V(s)\big) = \big|\sigma\big(V(s)\big)\big| \geq |s|,$$

which implies $V(s) \geq \alpha_1(|s|)$ for $\alpha_1 = \big(\max_i \sigma_i\big)^{-1} \in \mathcal{K}_\infty$. These inequalities establish (8).

Now consider $s \in \mathbb{R}^n_+$, $w \in \mathbb{R}^m_+$, with $s > 0$ and assume that $V(s) \geq \gamma(|w|)$. It follows that $V(s) \geq \max_i \rho_i^{-1}(|w|) \geq \max_i \rho_i^{-1}(w_i)$, or equivalently, that $\rho\big(V(s)\big) \geq w$. Consequently, due to (16), we have

$$\begin{aligned} V(s) &= V\big(\sigma(V(s))\big) \\ &> V\Big(g\big(\sigma(V(s), \rho(V(s)))\big)\Big) \geq V\big(g(s, w)\big). \end{aligned}$$

This proves that V is an ISS Lyapunov function for system (6) with gain γ. □ Before showing that ΩP can be inferred from the other properties listed in Theorems 1 and 2, we turn to a few auxiliary results. By a (parametrized) *path* in $\mathbb{R}^n_+$ we mean a continuous function from a possibly unbounded interval into $\mathbb{R}^n_+$. Interchangeably, we sometimes refer to the image of such a function as a path.

Lemma 8: Let $f\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be continuous, monotone, satisfy $f(0) = 0$, and let the origin be globally attractive with respect to

$$s^+ = f(s). \tag{32}$$

Then the following assertions hold:

1) For all $s > 0$, $f(s) \not\geq s$ or, equivalently, $f(s) \geq s$ implies $s = 0$.

2) The set

$$\Omega = \Omega(f) := \{s \in \mathbb{R}^n_+ : f(s) \ll s\}$$

is radially unbounded in the sense that for all $r > 0$ there is an $s \in \Omega$ such that $|s| = r$. The set Ω enjoys the following properties:

a) If $s \in \Omega$ and $\lambda \in [0, 1)$ then $\lambda f(s)+(1-\lambda)s \in \Omega$.
b) If $u \in \Omega$ and $s \in \mathbb{R}^n_+$ is such that $f(u) \ll s \ll u$ then for all $\lambda \in [0, 1]$, $\lambda s + (1 - \lambda)u \in \Omega$.
c) Ω is open in $\mathbb{R}^n_+$.

3) The set

$$\Psi = \Psi(f) := \{s \in \mathbb{R}^n_+ : f(s) \leq s\}$$

is the topological closure of Ω in $\mathbb{R}^n_+$. The set Ψ is forward invariant under (32). Every point in Ψ is connected by a path in Ψ to the origin. The set

$$\begin{aligned}\Psi_\infty &:= \bigcap_{k\geq 0} f^{-1}[\Psi] \\ &= \{s \in \Psi\colon \text{ for every } k \geq 0 \text{ there is a} \\ &\qquad u \in \Psi \text{ such that } f^k(u) = s\}\end{aligned}$$

is a radially unbounded subset of Ψ in the sense that for all $r > 0$ there is an $s \in \Psi_\infty$ such that $|s| = r$. In particular, there exists a path $\overline{\sigma}\colon \mathbb{R}_+ \to \mathbb{R}^n_+$ such that

$$\overline{\sigma}(0) = 0, \qquad \overline{\sigma}(r) \in \Psi_\infty, \qquad \text{for all } r \geq 0, \quad (33)$$

for every i the function $\overline{\sigma}_i$ is non-decreasing, and for at least one i it is unbounded. △

Proof.

1) Assume the opposite, i.e., that there is an $s > 0$ so that $f(s) \geq s$. By monotonicity we then have $f^{k+1}(s) \geq f^k(s) \geq \ldots \geq s > 0$ for all $k \geq 0$, in contradiction to $f^k(s) \to 0$ as $k \to \infty$ since the origin is assumed to be attractive. Hence, no such $s > 0$ can exist, proving that indeed $f(s) \ngeq s$ for all $s > 0$.
2) Given that $f(s) \ngeq s$ whenever $s > 0$, the first claim is proven in [22, Theorem 3.3]. Properties (a)–(c) are a consequence of the defining inequality for Ω and monotonicity. Their verification is left to the reader.
3) Clearly, Ψ is the closure of Ω. As a consequence, Ψ is forward invariant under (32) and it enjoys the same interpolation properties (a)–(c) as Ω.
Every point in $s \in \Psi$ defines a trajectory of points tending to the origin. By application of (a), the linear interpolation of these points together with the origin yields a path connecting the point s to the origin.
Now fix $r > 0$ and consider the set Ψ_∞ intersected with the set

$$S_r := \{s \in \mathbb{R}^n_+ : \|s\|_1 = r\},$$

the sphere with respect to the 1-norm restricted to $\mathbb{R}^n_+$. The set S_r is compact, and so is $A_K := S_r \cap \bigcap_{k=0}^{K} f^{-k}[\Psi]$ for every $K \geq 0$. Due to the path-connectedness of Ψ, the sets A_K are all non-empty and form a descending sequence, i.e., $A_{K+1} \subseteq A_K$ for all $K \geq 0$. By a version of Cantor's Intersection Theorem the infinite intersection $\Psi_\infty \cap S_r = \bigcap_{K=0}^{\infty} A_K$ is non-empty. Trivially, we have $0 \in \Psi_\infty$. Since $r > 0$ was arbitrary, the first statement about Ψ_∞ is proven.
It is left to construct the path $\overline{\sigma}$. Fix an $r > 0$ from the previous step and take an $s^0 \in \Psi_\infty \cap S_r$. By the definition of Ψ_∞, there exists an entire trajectory of (32), $\phi\colon \mathbb{Z} \to \Psi_\infty$ with $\phi(0) = s^0$, $|\phi(k)| \to \infty$ for $k \to -\infty$, and $\phi(k) \to 0$ for $k \to \infty$. We write $s^k := \phi(k)$, $k \in \mathbb{Z}$.
Linear interpolation of these points combined with the origin yields a path $\widetilde{\sigma}\colon \mathbb{R} \to \Psi_\infty$ with $\widetilde{\sigma}(0) = s^0$ and the same limit behavior as ϕ, per

$$\widetilde{\sigma}(r) := (1 - \lambda)s^k + \lambda s^{k+1}$$

where $k(r) := \lfloor r \rfloor$, $\lambda(r) := r - \lfloor r \rfloor$, and $\lfloor r \rfloor$ denotes the largest integer less or equal to r. Then we obtain $\overline{\sigma}$ from $\widetilde{\sigma}$ via a time reversal like

$$\overline{\sigma}(r) := \begin{cases} 0 & \text{for } r = 0, \\ \widetilde{\sigma}(-\log r) & \text{for } r > 0, \end{cases}$$

so that we have $\overline{\sigma}(0) = 0$ and $|\overline{\sigma}(r)| \to \infty$ as $r \to \infty$. The monotonicity properties for $\widetilde{\sigma}$ and $\overline{\sigma}$ follow from the monotonicity of the trajectory ϕ. This completes the proof. □

The last item of the lemma has a noteworthy consequence:

Corollary 1: If the origin is attractive for $s^+ = f(s)$, where $f\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is continuous and monotone, then it is also stable. △

Proof. Just note that small points in the neighborhood of the origin can be dominated by a point $\omega \in \Omega$ in Ω. The ordering of solutions principle then dictates that no trajectory can escape the bound ω. □

Lemma 9: Consider f as in Lemma 8 and assume that in addition f is proper. Then Ψ_∞ is jointly unbounded in the sense that for every $u \in \mathbb{R}^n_+$ there is an $s \in \Psi_\infty$ satisfying $s \geq u$. The path $\overline{\sigma}$ can be chosen to be unbounded in every component. In addition, there exists a path $\sigma\colon \mathbb{R}_+ \to \mathbb{R}^n_+$ such that $\sigma(0) = 0$, $\sigma(r) \in \Omega$ for all $r > 0$, and for all i the functions σ_i are of class $\mathcal{K}_\infty$. In particular, the set Ω is jointly unbounded, too. △

Proof. Since f is proper, there exists an $\alpha \in \mathcal{K}_\infty$ such that $\alpha(|s|)e \leq f(s)$ for all $s > 0$. Consider the path $\overline{\sigma}\colon \mathbb{R}_+ \to \Psi_\infty$ given by Lemma 8. The function $r \mapsto |\overline{\sigma}(r)|$ is unbounded, so with increasing $r > 0$ also $\overline{\sigma}(r) \geq f(\overline{\sigma}(r)) \geq \alpha(|\overline{\sigma}(r)|)e$ increases in every component beyond any finite bound. This establishes that every individual component of $\overline{\sigma}$ is unbounded and hence that Ψ is jointly unbounded. By a perturbation argument, the path $\overline{\sigma}$ can be modified into a path σ with the same unboundedness properties, so that each component function is strictly increasing, i.e. of class $\mathcal{K}_\infty$. By the same perturbation argument, we obtain that

$f\big(\sigma(r)\big) \ll \sigma(r)$ for $r > 0$. This establishes the desired properties for Ω. □

Lemma 10: Let g be eventually increasable. Then GAS implies ΩP. △

Proof. Let $f(s) = g\big(s, \zeta(s)\big)$ and $\zeta \colon \mathbb{R}^n_+ \to \mathbb{R}^m_+$ be given by (14) and observe that f is proper. According to Lemma 9, there is a path $\sigma \colon \mathbb{R}_+ \to \mathbb{R}^n_+$ such that $f\big(\sigma(r)\big) \ll \sigma(r)$ for all $r > 0$. Define $\rho := \zeta \circ \sigma$, which, by construction, is proper and positive definite, as is σ. We obtain that for all $r > 0$,

$$g\big(\sigma(r), \rho(r)\big) = f(\sigma(r)) \ll \sigma(r).$$

This establishes ΩP. □

Finally, we aggregate the lemmas to prove the theorems.

A. Proof of Theorem 1

By Lemma 1, the existence of an ISS Lyapunov function implies GAS. By Lemma 2 the converse also holds. It was shown in [15] that for discrete-time systems input-to-state stability and the existence of an ISS Lyapunov function are equivalent. In fact, the Lyapunov function only needs to be continuous to imply ISS, and if there exists a continuous Lyapunov function, there also exists a smooth one. The equivalence of ISS and AG is shown in Lemma 3. □

B. Proof of Theorem 2

Refer to Lemma 5. □

C. Proof of Theorem 3

It suffices to consider only the two properties AG and NP. That AG implies NP is shown in Lemma 6. Under the additional assumption that g is eventually increasable, the converse implication is shown in Lemma 4. □

D. Proof of Theorem 4

Again it suffices to compare ΩP with any of the other properties listed in Theorems 1 and 2.

Lemma 7 establishes that ΩP is sufficient for ISS-LF. Lemma 10 shows that GAS implies ΩP when g is eventually increasable. □