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Schur–Agler and Herglotz–Agler classes of functions: positive-kernel decompositions and transfer-function realizations

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For separable Hilbert spaces $\mathcal{U}$, $\mathcal{Y}$, we define the operator-valued Schur class $\mathcal{S}(\mathcal{U},\mathcal{Y})$ (over the unit disk $\mathbb{D}$) to consist of all holomorphic functions S on $\mathbb{D}$ with values in the closed unit ball of the space $\mathcal{L}(\mathcal{U},\mathcal{Y})$ of bounded linear operators from $\mathcal{U}$ to $\mathcal{Y}$, i.e., subject to $\|S(\zeta)\| \le 1$ for all $\zeta \in \mathbb{D}$. The following result linking the theories of holomorphic functions, linear operators, and input/state/output linear systems is now well known (see e.g. [1] for a full discussion where multivariable extensions are also treated).

Theorem 1: Given a function $S\colon \mathbb{D} \to \mathcal{L}(\mathcal{U},\mathcal{Y})$, the following are equivalent:

(1) $S \in \mathcal{S}(\mathcal{U},\mathcal{Y})$.

(2) The de Branges–Rovnyak kernel

$$K_S(\omega,\zeta) = \frac{I - S(\omega)^* S(\zeta)}{1 - \overline{\omega}\zeta}$$

is a positive kernel on $\mathbb{D}$, i.e., there is an auxiliary Hilbert space $\mathcal{X}$ and a holomorphic function $H\colon \mathbb{D} \to \mathcal{L}(\mathcal{U},\mathcal{X})$ which gives rise to a *Kolmogorov decomposition* for K_S:

$$K_S(\omega,\zeta) = H(\omega)^* H(\zeta).$$

(3) S has a unitary transfer-function realization, i.e., there is an auxiliary Hilbert state space $\mathcal{X}$ and a unitary colligation matrix

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

so that

$$S(\zeta) = D + \zeta C(I - \zeta A)^{-1} B \text{ for } \zeta \in \mathbb{D}.$$

(3′) Condition (3) above holds where the colligation matrix $\mathbf{U}$ is taken to be any of (i) coisometric, (ii) isometric, or (iii) contractive.

It is natural to seek extensions of the Schur class to the multivariable setting where the disk $\mathbb{D}$ is replaced by the polydisk

$$\mathbb{D}^d = \{\zeta = (\zeta_1,\dots,\zeta_d) \in \mathbb{C}^d \colon |\zeta_k| < 1 \text{ for } k = 1,\dots,d\}.$$

Define the d-variable Schur class $\mathcal{S}_d(\mathcal{U},\mathcal{Y})$ to consist of holomorphic functions $S\colon \mathbb{D}^d \to \mathcal{L}(\mathcal{U},\mathcal{Y})$ subject to $\|S(\zeta)\| \le 1$ for all $\zeta \in \mathbb{D}^d$. It was the profound observation of Agler [2] that, unless $d \le 2$, a characterization of $\mathcal{S}_d(\mathcal{U},\mathcal{Y})$ of the same form as Theorem 1 is not possible. Instead, we define what is now called the Schur–Agler class, denoted as $\mathcal{SA}(\mathbb{D}^d, \mathcal{L}(\mathcal{U},\mathcal{Y}))$, to consist of holomorphic functions $S\colon \mathbb{D}^d \to \mathcal{L}(\mathcal{U},\mathcal{Y})$ such that $\|S(T_1,\dots,T_d)\| \le 1$ whenever $T = (T_1,\dots,T_d)$ is a commutative d-tuple of strict contraction operators on a fixed separable infinite-dimensional Hilbert space $\mathcal{K}$. Here

$$S(T_1,\dots,T_d) = \sum_{n \in \mathbb{Z}_+^d} S_n \otimes T^n$$

(convergence in the strong operator topology) where $S(\zeta) = \sum_{n \in \mathbb{Z}_+^d} S_n \zeta^n$ is the multivariable Taylor expansion for S centered at the origin $0 \in \mathbb{D}^d$ and where we use standard multivariable notation: $\zeta^n = \zeta_1^{n_1} \cdots \zeta_d^{n_d}$, $T^n = T_1^{n_1} \cdots T_d^{n_d}$ if $n = (n_1,\dots,n_d) \in \mathbb{Z}_+^d$. The following result due to Agler [2] (see also [3], [4]) has had a profound impact on the subject over the years.

Theorem 2: Given a function $S\colon \mathbb{D}^d \to \mathcal{L}(\mathcal{U},\mathcal{Y})$, the following are equivalent:

(1) $S \in \mathcal{SA}(\mathbb{D}^d, \mathcal{L}(\mathcal{U},\mathcal{Y}))$.

(2) S has an *Agler decomposition* in the sense that there exist $\mathcal{L}(\mathcal{U})$-valued positive kernels $K_1,\dots,K_d$ on $\mathbb{D}^d$ such that

$$I - S(\omega)^* S(\zeta) = \sum_{k=1}^{d} (1 - \overline{\omega}_k \zeta_k) K_k(\omega,\zeta).$$

(3) S has a unitary Givone–Roesser d-dimensional transfer-function realization, i.e., there is an auxiliary Hilbert state space $\mathcal{X}$ with a d-fold orthogonal direct-sum decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_d$ together with a unitary colligation matrix

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

so that

$$S(\zeta) = D + C(I - P(\zeta)A)^{-1} P(\zeta) B \text{ for } \zeta \in \mathbb{D}^d,$$

where we have set

$$P(\zeta) = \zeta_1 P_1 + \cdots + \zeta_d P_d$$

with P_k the orthogonal projection of $\mathcal{X}$ onto $\mathcal{X}_k$ for each $k = 1,\dots,d$.

(3′) Condition (3) above holds where the colligation matrix $\mathbf{U}$ is taken to be any of (i) coisometric, (ii) isometric, or (iii) contractive.

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We obtain parallel results for the following linear-fractional transformed versions of the Schur–Agler class:

1) The Herglotz–Agler class over the unit polydisk $\mathbb{D}^d$, denoted $\mathcal{HA}(\mathbb{D}^d, \mathcal{L}(\mathcal{U}))$ defined as the class of all holomorphic functions $F\colon \mathbb{D}^d \to \mathcal{L}(\mathcal{U})$ such that $F(T)$ has positive real part
$$F(T) + F(T)^* \geq 0$$
for all commutative d-tuples $T = (T_1, \dots, T_d)$ of strict contractions on a Hilbert space $\mathcal{K}$.
2) The Schur–Agler class over the right polyhalfplane
$$\Pi^d = \{z = (z_1, \dots, z_d) \in \mathbb{C}^d \colon z_k + \overline{z}_k > 0 \quad \text{for } k = 1, \dots, d\},$$
denoted by $\mathcal{SA}(\Pi^d, \mathcal{L}(\mathcal{U}, \mathcal{Y}))$, consisting of all holomorphic $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued functions s on Π^d such that $\|s(A)\| \leq 1$ for all strictly accretive commutative d-tuples $A = (A_1, \dots, A_d)$ of operators on $\mathcal{K}$ (i.e., such that $A_k + A_k^* \geq cI$ for some constant $c > 0$, $k = 1, \dots, d$).
3) The Herglotz–Agler class over Π^d, denoted by $\mathcal{HA}(\Pi^d, \mathcal{L}(\mathcal{U}))$, consisting of all holomorphic functions f on Π^d such that $f(A) + f(A)^* \geq 0$ for all strictly accretive commutative d-tuples A of operators on $\mathcal{K}$.

For the single-variable case such realization results have been explored in a systematic way in [5] and [6]. For the multivariable setting, apart from the now classical Schur–Agler class over the polydisk $\mathcal{SA}(\mathbb{D}^d, \mathcal{L}(\mathcal{U}, \mathcal{Y}))$, the only results along these lines which we are aware of are those in the recent paper of Agler–McCarthy–Young [7] and of Agler–Tully-Doyle–Young [8], [9] (in the setting of the Nevanlinna–Agler class of functions where the right polyhalfplane and the right operator halfplane in the definition of the Herglotz–Agler class are replaced by the upper polyhalfplane and the upper operator halfplane).

The approach in [5] (in the single-variable setting) is to use a linear-fractional-transformation (LFT) change of variables (on the domain and/or the range side) to reduce the desired result to the corresponding result for the Schur class over the unit disk. This is also the main tool in [7], [8], [9]: use an LFT Cayley-transform change of variables to reduce results for the Nevanlinna–Agler class to the corresponding known results for the Schur–Agler class. However the procedure is rather intricate due to the added subtleties involved in handling points at infinity in the multivariable case.

In contrast, the approach in [6] is to apply a projective version of the lurking isometry argument (roughly, a lurking-isotropic-subspace argument in a Kreĭn-space setting) to arrive at the desired realization result via a direct but unified Kreĭn-space geometric argument. One of the main contributions of the present work is to extend this approach to the multivariable setting. The main difficulty is to guarantee that a naturally defined isotropic subspace is actually a graph space with respect to a system of coordinates not coming from a fundamental decomposition of the ambient Kreĭn space. We show how this difficulty can be overcome for the case of the $\mathbb{D}^d$-Herglotz–Agler class and Π^d-Schur–Agler class. For the Π^d-Herglotz–Agler class, we are able to overcome the difficulty only in a special case (associated with the imposition of a growth condition at infinity), thereby recovering parallel results from [7]. For the most general Π^d-Herglotz–Agler function f, we follow the LFT change-of-variable approach of [8] combined with the more general realization formalism (Schur complement of an operator pencil) suggested by the work of Bessmertnyĭ (see [10], [11], [12], [13], [14]) to arrive at a realization formula for the most general Π^d-Herglotz–Agler function. We note that the original Bessmertnyĭ class involved additional symmetries leading to strong rigidity results. It was conjectured in [15] that an appropriate weakening of the metric conditions for the Bessmertnyĭ operator pencil should lead to a representation for the most general Π^d-Herglotz–Agler function. Here we show that this conjecture is correct once one identifies the appropriate modification: one must allow the nonhomogeneous skew-adjoint term in the nonhomogeneous Bessmertnyĭ operator pencil to be unbounded (more precisely, a certain flip Π-impedance-conservative system node in the sense of [5]).

There has been a lot of work on transfer-function realization for the single-variable Schur and Herglotz classes over the right half plane. The most influential for our point of view toward multivariable generalizations is the work of Arov-Nudelman [16] and of Staffans and collaborators (see [17], [5], [6], [18], [19] as well as the treatise [20] and the references there). There is also a complementary approach to such realization theory (upper halfplane rather than right halfplane version) with emphasis on the theory of selfadjoint extensions of densely defined symmetric operators on a Hilbert space (see [21], [22], [23] as well as the recent book [24] and the references there).

Our results are presented in full detail in [25]. We also mention that parallel results for the four classes under discussion ($\mathbb{D}^d$-Schur–Agler, $\mathbb{D}^d$-Herglotz–Agler, Π^d-Schur–Agler and Π^d-Herglotz–Agler) for the rational matrix-valued (Cayley) inner case, where the emphasis is on obtaining realizations with finite-dimensional state space, are obtained in our companion paper [26].

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