21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

A note on formation keeping control with coarse information*

Matin Jafarian

Abstract— **This paper studies a chattering-free formation keeping control using very coarse information.**

I. INTRODUCTION

Formation keeping control is a cooperative control problem which aims at achieving a desired collective behavior, mainly forming a desired geometrical shape, for the agents of a multi-agent system. To achieve the task, the agents of the network should exchange information. One usual assumption in the literature is the exchange of full information among the agents. However, due to real world constraints, this assumption may not hold.

One way to approach the problem of communication constraints is to consider quantized exchanged information. Quantized control for the continuous-time dynamic agents implies that the agents update their control law whenever measurements cross the thresholds of their quantizer. There has been a great deal of interest in the literature in line with using quantized and binary information for coordination control. [3] adopted binary control laws in the framework of discontinuous dynamical systems and studied reaching consensus in finite-time. [2] has rigorously cast the problem in the framework of nonsmooth control systems. It has also introduced a new class of hybrid quantizers to deal with possible chattering phenomena. Line-deployment of kinematic agents using binary information was presented in [4]. [5] used binary controllers to achieve an exact formation as well as the reference velocity recovery for a network of second-order dynamic agents. The results prove the exact convergence to the desired formation despite using very coarse exchanged information. However, the control action starts fast switchings between -1 and $+1$ as soon as the system converges to the desired formation.

Motivated by [5], in this paper we study the fast switching behavior (chattering) resulting from the discontinuous controller. We show that some slight changes in the design of the controller can highly reduce the very fast switchings in [5]. However, the exact convergence will change to a practical convergence. This result can also be interesting for applications which aim at achieving a formation in a neighborhood.

This paper is organized as follows. Section II presents the analysis of controlling a double integrator (point mass) by a discontinuous spring. Section III analyzes the stability and convergence of a network of double-integrators using the results of Section II. Section IV concludes the paper.

II. DOUBLE INTEGRATOR CONTROLLED BY A DISCONTINUOUS SPRING

In this section, we analyze the stability and convergence of a double integrator controlled by a discontinuous spring modeled by

$$\begin{array}{rcl} \dot{x} & = & v \\ \dot{v} & = & -kv - u, \end{array} \tag{1}$$

where $x \in \mathbb{R}, v \in \mathbb{R}$ and $u \in \mathbb{R}$ are the position, the velocity and the control input respectively. We consider two different control laws as follows

$$\begin{array}{rcl} u_1 & = & \text{sign}\, x \\ u_2 & = & \text{sign}_\varepsilon\, x, \end{array} \tag{2}$$

where $u_1 : \mathbb{R} \to \{-1, +1\}$ and $u_2 : \mathbb{R} \to \{-1, 0, +1\}$ are defined as

$$\text{sign}\, x = \begin{cases} +1 & x \geq 0 \\ -1 & x < 0 \end{cases} \qquad \text{sign}_\varepsilon\, x = \begin{cases} +1 & x > \varepsilon \\ -1 & x < -\varepsilon \\ 0 & |x| \leq \varepsilon. \end{cases}$$

The right-hand side of (1) is discontinuous due to the discontinuity of sign and sign_ε functions at $x = 0$ and $|x| = \varepsilon$ respectively. Here, we adopt a Krasovskii notion of solution to analyze our discontinuous system [1]. We first present a brief review on Krasovsii notion of solution. Then, we study the properties of the equilibria of the system with u_1 and u_2. Define $X = (x, v)$ and let $F(X)$ be the set-valued map

$$F(X) = \begin{pmatrix} v \\ -kv \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathcal{K} u_i \tag{3}$$

where u_i is either equal to u_1 or u_2 and

$$\mathcal{K}\, \text{sign}\, x = \begin{cases} \{\text{sign}\, x\} & if \quad x \neq 0 \\ [-1, +1] & if \quad x = 0 \end{cases} \tag{4}$$

$$\mathcal{K}\, \text{sign}_\varepsilon\, x = \begin{cases} \{\text{sign}_\varepsilon\, x\} & if \quad |x| \neq \varepsilon \\ [-1, 0] & if \quad x = -\varepsilon \\ [0, +1] & if \quad x = \varepsilon. \end{cases} \tag{5}$$

We define $X(t) = (x(t), v(t))$ a Krasovskii solution to (1) on the interval $[0,\ t_1]$ if it is an absolutely continuous function which satisfies the differential inclusion

$$\dot{X}(t) \in F(X(t)) \tag{6}$$

for almost every $t \in [0, t_1]$, with F defined as in (3). Local existence of Krasovskii solutions to the differential inclusion above is guaranteed [1].

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*This work is partially supported by the Dutch Organization for Scientific Research (NWO) under the auspices of the project *QUantized Information Control for formation Keeping* (QUICK).

M. Jafarian is with ITM, Faculty of Mathematics and Natural Sciences, University of Groningen, the Netherlands, Tel: +31 50 363 3080. `{m.jafarian}@rug.nl`

A. *Equilibria*

Now, we study the properties of the equilibria of (1) with two different control laws u_1 and u_2. We recall that x_0 is a (Krasovskii) equilibrium if the function $x(t) \equiv x_0$ is a (Krasovskii) solution, that is if $0 \in F(X)$ (see [2]).

With $u = u_1$, the origin, $(0, 0)$, is the equilibrium of the system. With $u = u_2$, the set of (Krasovskii) equilibria of (1) is

$$\mathcal{E} = \{(x, v) | v = 0, -\varepsilon \le x \le \varepsilon\}.$$

In what follows, we present the stability analysis for the set of equibliria of the system. The analysis is based on an application of nonsmooth La Salle's invariance principle in [3].

Proposition 1 Any Krasovskii solution to (1) exists for all $t \ge 0$ and converges to the set of equilibria corresponding to the applied controller.

Proof: Take $V_1(x, v) = |x| + \frac{1}{2}v^2$, where $V_1(x, v)$ is a locally Lipschitz function. We evaluate the set-valued derivative $\dot{\overline{V_1}}(x, v)$ along (1). Define

$$\begin{aligned}\dot{\overline{V_1}}(x, v) = \{a \in \mathbb{R} :\ & \exists w \in F(x, v) \text{ s.t.} \\ & a = \langle w, p\rangle, \text{for all } p \in \partial V_1(x, v)\},\end{aligned}$$

where $F(x, v)$ is the set-valued map in (3). By definition of F in (3), for any $w \in F(x, v)$ there exists $w^x \in \mathcal{K}u_1$ such that

$$w = \begin{pmatrix} v \\ -kv \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} w^x.$$

Calculating the Clarke generalized gradient [1], $\partial V_1(x, v)$, we have

$$\partial V_1(x, v) = \{p : p = \begin{pmatrix} p^x \\ v \end{pmatrix} \text{ s.t. } p^x \in \mathcal{K}u_1\}.$$

Suppose that $\dot{\overline{V_1}}(x, v) \neq \emptyset$ and take $a \in \dot{\overline{V_1}}(x, v)$. Then by definition there exists $w \in F(x, v)$ such that $a = \langle w, p\rangle$ for all $p \in \partial V_1(x, v)$. Choose $p \in \partial V_1(x, v)$ such that $p^x = w^x$. Thus

$$a = \langle \begin{pmatrix} v \\ -kv \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} w^x, \begin{pmatrix} w^x \\ v \end{pmatrix} \rangle = \langle -kv, v\rangle.$$

Hence, for any $\dot{\overline{V_1}}(x, v) \neq \emptyset$, $\dot{\overline{V_1}}(x, v) = \{-kv^2\} \subseteq (-\infty, 0]$. Therefore, by La Salle's invariance principle the solutions of the system converge to the largest weakly invariant set where $v = 0$. From (1), any point $(x, 0)$ on this invariant set must satisfy $w^x = 0$. Multiplying by x, $xw^x = 0$ must be satisfied. Since $w^x \in \mathcal{K}\text{sign } x$, to have $xw^x = 0$, x must be zero. Therefore, Krasovskii solution to (1) will asymptotically converge to the origin.

Now, take $u = u_2$ and $V_2(x, v) = \frac{1}{2}v^2 + |x|_\varepsilon$, where $|x|_\varepsilon$ is defined as below

$$|x|_\varepsilon = \begin{cases} |x| & if \quad |x| > \varepsilon \\ \varepsilon & if \quad |x| \le \varepsilon. \end{cases}$$

Note that $V_2(x, 0) = \varepsilon$ if $x \in \mathcal{E}$, where $\mathcal{E}$ is the set of system's equilibria, otherwise $V_2(x, v) > \varepsilon$. Similar to the previous case, $\dot{\overline{V_2}}(x, v) = \{-kv^2\} \subseteq (-\infty, 0]$. Therefore, the largest weakly invariant set in this case contains all points $(x, 0)$ which satisfy $xw^x = 0$ where $w^x \in \mathcal{K}\ \text{sign}_\varepsilon\ x$. Based on the definition of $\mathcal{K}\ \text{sign}_\varepsilon\ x$, $xw^x = 0$ implies $-\varepsilon \le x \le \varepsilon$, which ends the proof. □

Remark 1 With $u = u_1$, we can find counterexamples to show that the finite-time convergence of (1) can not be achieved. Taking $u = u_2$, we can calculate the time at which the position variable x reaches the region of convergence $-\varepsilon \le x \le \varepsilon$ (*i.e.* if $|x(0)| > \varepsilon$). However, the velocity, v, converges to zero asymptotically.

B. *Fast switching of the control action*

As discussed previously, all solutions to (1) will asymptotically converge to the system's set of equilibria. In practice, the control law u_1 starts to show a very fast switching behavior while the system is converging to the origin. This behavior does not occur in the case of the control law u_2. Figure 1 shows the evolution of the control laws u_1 and u_2 for (1) with initial conditions $(0, 0)$ and $(0.1, 0)$ respectively. The initial conditions are chosen on the discontinuous surfaces of u_1 and u_2. As shown, the control law shows chattering for u_1, but not for u_2. Note that the choice of initial conditions does not affect the result.

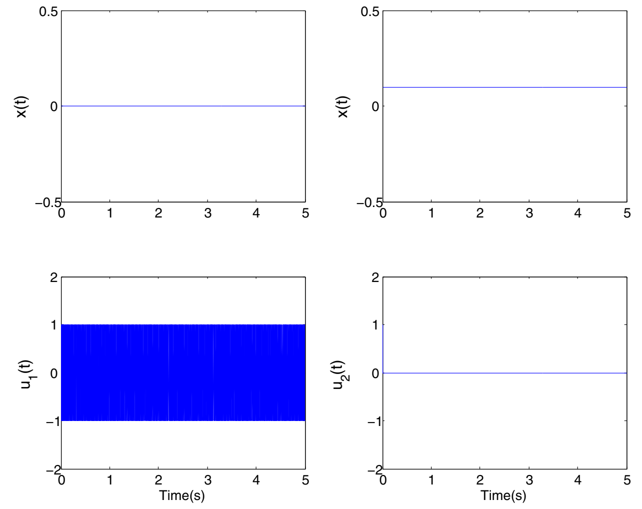

Fig. 1. The right-side plots show the evolutions of the position ,x, and the control law u_1 with the initial condition $(0, 0)$. The left-side plots show the evolution of x and u_2 with the initial condition $(0.1, 0)$ and $\varepsilon = 0.1$. The gain k is equal to 1 for both cases.

Here, we first study the existence of sliding mode solutions for u_1 and u_2. In [6], the necessary condition for the existence of the sliding mode solution on the surface $s(x) = x$ is that

$$\lim_{x \to 0^-} \dot{x} > 0, \qquad \lim_{x \to 0^+} \dot{x} < 0.$$

Since in the system dynamic (1) $\dot{x} = v$, the above condition is not satisfied for all initial conditions. Hence, we

conclude that the sliding mode solution for system (1) exists with neither u_1 nor u_2. We explain the reason behind the different behaviors of u_1 and u_2 as follows. Taking $u = u_1$, the vector field related to (1) is either $(v, -kv + 1)^T$ or $(v, -kv - 1)^T$ depends on the sign x. Hence, the norm of the vector field is bigger than zero while v is converging to zero. As a result, the system's trajectories start to move very fast between the right and the left half-planes in the vicinity of $(0, 0)$. Therefore, the control action exhibits chattering behavior.
In the case of u_2, the vector field related to (1) in the neighborhood of $x = \varepsilon$ is either $(v, -kv-1)^T$ or $(v, -kv)^T$. Similarly, the vector field in the neighborhood of $x = -\varepsilon$ is $(v, -kv)^T$ or $(v, -kv + 1)^T$. Provided that ε is big enough, the norm of the vector field goes to zero (from one side) as $v \to 0$. Therefore, the fast switching behavior in the vicinity of $v = 0$ can be prevented by adopting u_2 as the control law. Note that ε is a design choice and it should be chosen big enough, for example in the simulations, it is taken bigger than the relative tolerance of the integration method.

Remark 2 (Single integrator dynamics) It is worth to mention that a single integrator controlled by a discontinuous spring, $\dot{x} = -\text{sign}x$, converges to the origin in finite-time and the solutions of this system slide along $x = 0$.

Motivated by [5], the next section presents an application of the control law u_2 in formation control of a network of double-integrators.

III. APPLICATION OF A sign_ε BASED CONTROLLER TO FORMATION KEEPING PROBLEM

We consider n agents evolving in $\mathbb{R}$. The dynamics of each of the agents obeys (1). The way in which the agents exchange information is modeled with a connected undirected graph $G = (V, E)$, where the set of nodes V coincides with the set of agents (and hence $|V| = n$) and an edge $(i, j) \in E \subset V \times V$ models the fact that agents i and j can exchange information. Label one end of each edge in E with a positive sign and the other end with a negative sign. We define the relative position z_k between two agents i and j as follows

$$z_k = \begin{cases} x_i - x_j & \text{if node } i \text{ is the positive end of the edge k} \\ x_j - x_i & \text{if node } j \text{ is the positive end of the edge k.} \end{cases}$$

Let $m = |E|$ be the number of edges. We define B as the $(n \times m)$ incidence matrix associated to the graph G as follows

$$b_{ik} = \begin{cases} +1 & \text{if node } i \text{ is the positive end of the edge k} \\ -1 & \text{if node } i \text{ is the negative end of the edge k} \\ 0 & \text{otherwise.} \end{cases}$$

Define $z = B^T x$ as the vector of distances between agents, and $z^* \in \mathcal{R}(B^T)$ as the desired inter-agent position vector where $\mathcal{R}$ denotes the range of B^T. We define the concatenated vectors $x \triangleq [x_1 \dots x_n]^T \quad x \in \mathbb{R}^n \quad x_i \in \mathbb{R}$, $z \triangleq [z_1 \dots z_m]^T \quad z \in \mathbb{R}^m \quad z_k \in \mathbb{R}$ and $v \triangleq [v_1 \dots v_n]^T \quad v \in \mathbb{R}^n \quad v_i \in \mathbb{R}$.

A. Analysis

In this section we investigate the stability properties of the closed-loop system. We design the control law for each agent i as $u_i = -\sum_{k=1}^m b_{ik}\text{sign}_\varepsilon(z_k - z_k^*)$. We define the error vector $\tilde{z} = z - z^*$ and write the control laws u_i, $i = 1, 2, \dots, n$, in the following compact form

$$u = -B \ \text{sign}_\varepsilon \tilde{z} \tag{7}$$

where $\text{sign}_\varepsilon \ \tilde{z}$ represents the vector $(\text{sign}_\varepsilon \ \tilde{z}_1 \dots \text{sign}_\varepsilon \ \tilde{z}_m)^T$. The dynamics of the closed-loop error system has the following form

$$\begin{aligned} \dot{\tilde{z}} &= B^T v \\ \dot{v} &= -kv - B \ \text{sign}_\varepsilon \tilde{z}. \end{aligned} \tag{8}$$

The system (8) has a discontinuous right-hand side due to the discontinuity of the sign_ε function at $|\tilde{z}_k| = \varepsilon$. As the previous section, we analyze the system taking the solutions in a Krasovskii sense.
Define $X = (\tilde{z}, v)$ and let $F(X)$ be the set-valued map

$$F(X) = \begin{pmatrix} B^T \ v \\ -kv \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ B \end{pmatrix} \mathcal{K}\text{sign}_\varepsilon \tilde{z} \tag{9}$$

where $\mathcal{K} \ \text{sign}_\varepsilon \ \tilde{z} = \times_{k=1}^m \mathcal{K} \ \text{sign}_\varepsilon \ \tilde{z}_k$ and

$$\mathcal{K} \ \text{sign}_\epsilon \ \tilde{z}_k = \begin{cases} \{\text{sign}_\varepsilon \ \tilde{z}_k\} & if \quad \tilde{z}_k \neq \varepsilon \\ [-1, 0] & if \quad \tilde{z}_k = -\varepsilon \\ [0, +1] & if \quad \tilde{z}_k = \varepsilon. \end{cases} \tag{10}$$

Similar to the previous section, the existence of the solution for the above system can be guaranteed. Applying the tools of nonsmooth control theory, the following proposition can be proven based on an application of the nonsmooth La Salle's invariance principle.

Proposition 2 Any Krasovskii solution to (8) exists for all $t \geq 0$ and converges to the set

$$\{\forall i \in V, \forall k \in E : v_i = 0, |\tilde{z}_k| \leq \varepsilon\}.$$

Consider a locally Lipschitz Lyapunov function $V(\tilde{z}, v) = |\tilde{z}|_\varepsilon + \frac{1}{2}v^T v$. The proof of the above proposition is similar to the proof of Proposition 1 (for the lack of space, we skip the proof). Note that the velocities of the agents converge to zero asymptotically.

B. Fast switching of the control action

Similar to the previous section, the control law $u = -B$ sign $\tilde{z}$ results in fast switchings of the control action when the system converges to the desired formation [5]. Now, we argue that $u = -B \ \text{sign}_\varepsilon \ \tilde{z}$ can prevent the undesired fast switching behavior of the control action as the network converges to the desired formation. Here, we consider the state of the system $(\tilde{z}, v)$ and its corresponding vector field $(B^T v, -kv - B\text{sign}_\varepsilon \ \tilde{z})$. As stated in Proposition 2, the velocity of each agent converges to zero while the formation reaches the desired one. Therefore, we have $v_i = 0$ and $\dot{v}_i = 0$ where $\dot{v}_i = -kv_i - \sum_k b_{ik} \ \text{sign}_\varepsilon \tilde{z}_k$. First, consider a network of two agents connected via one link. The dynamics of the edge, $\dot{z} = v_2 - v_1$, resembles the double integrator controlled by u_2 in Section II-B. Hence, based on a similar reasoning we expect no chattering. Now, add one other node to the network and assume that both of the relative positions converge to $|z_k| = \varepsilon$, that is on the discontinuity surface (otherwise, if $|z_k| < \varepsilon$, we can conclude that u_i is zero and remains equal to zero). In this case, a small deviation from ε will result in small changes in the positions of the agents such that $|z_k| < \varepsilon$, then u_i will become zero. We can extend this reasoning for larger graphs to conclude that with $u = -B\, \text{sign}_\varepsilon\, \tilde{z}$ the control action does not switch very fast while the network is converging to the desired formation. Figure 2 shows the simulation results of a network of five double integrators in $\mathbb{R}$ communicating over a connected undirected graph. The results show the convergence of the network to a formation for which $|\tilde{z}_k| \leq \varepsilon$ while the velocity of each of the agents converges to zero. As expected, the control action does not show a fast switching behavior.

Despite a chattering-free convergence, the fast switching of the control action can initially occur for some graph topologies and specific initial conditions. Here, we provide an example.

Example 1 Consider a connected graph composed of three nodes and two links. Take $z^* = \mathbf{0}$, then we have $\tilde{z} = z$. Define $z_1 = x_2 - x_1$ and $z_2 = x_3 - x_2$ such that $z_1(0) = \varepsilon, z_2(0) > \varepsilon$ and $v(0) = \mathbf{0}$. Calculating the evolution of $\dot{z}_1$ on the small time interval $[0, \delta]$, we obtain

$$\dot{z}_1 = v_2 - v_1 = t - 2\, \text{sign}_\varepsilon(z_1(0))\, t.$$

Hence, $\lim_{z_1 \to \varepsilon^-} \dot{z}_1 = t$ and $\lim_{z_1 \to \varepsilon^+} \dot{z}_1 = -t$. The change of sign of $\dot{z}_1$ around $z_1 = \varepsilon$ implies initial oscillations. The oscillations will disappear while the system converges to the desired formation. Moreover, non-zero initial velocity for the agents can prevent the initial oscillations of the control action. Since, the system's trajectories will move faster over the discontinuity surfaces.

IV. CONCLUSIONS

In this paper we presented a comparison between two discontinuous control laws applied to a double integrator dynamics. The result was applied to formation keeping control of a network of double-integrators in $\mathbb{R}$ where each of two agents are connected by a virtual discontinuous spring. The results show the convergence of the formation to an ε-neighborhood of the desired formation. Despite the coarse communication scenario, the control action does not show fast oscillations while converging to the desired formation. However, the design is not robust with respect to initial oscillations resulting from specific initial conditions. Future avenues are to consider more robust control scenarios.

ACKNOWLEDGMENT

The author wishes to thank Claudio De Persis and Francesca Ceragioli for helpful insights and comments.

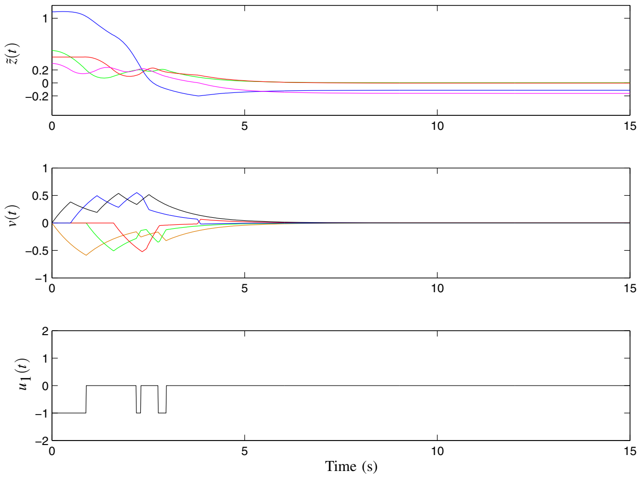

Fig. 2. The plots of relative position error, $\tilde{z}$, agents velocity,v, together with the controller of the first agent, u_1. The controller's parameters are $\varepsilon = 0.2$ and $k = 1$. The formation converges to $-0.2 < \tilde{z}_k < 0.2$. The controller does not show a fast switching behavior.

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