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A max-plus dual space fundamental solution semigroup for operator differential Riccati equations

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Abstract— **Recent work concerning the development of fundamental solution semigroups for specific classes of integro-differential equations is generalized via the consideration of a class of operator differential Riccati equations. In particular, a max-plus dual space representation for the fundamental solution semigroup for this more general class of equations is constructed by exploiting max-plus linearity and semiconvexity properties of an associated optimal control problem. A class of time-indexed max-plus integral operators is shown to be instrumental in this construction, and in the evolution of the fundamental solution semigroup obtained.**

I. Introduction

Recent investigations of the algebraic properties of dynamic programming have led to the development of a range of promising new *max-plus methods* for solving optimal control and related problems, see for example [1], [2], [3], [4], [5], [6]. By exploiting max-plus linearity and semiconvexity properties of specific dynamic programming evolution operators, these methods are founded on the existence and construction of a *max-plus fundamental solution semigroup* for the associated optimal control problem. This fundamental solution semigroup describes how a dynamic programming evolution operator propagates the associated value function for *any* terminal cost or payoff. Where propagation of this value function can be described by specific integro-differential equations, these fundamental solution semigroups can be used to develop new computationally very efficient methods for solving those equations. Fast computational methods have been developed in this way for the solution of classes of advection and diffusion integro-differential equations [3], [4], [5], and for finite dimensional differential and difference Riccati equations [1], [6].

In the infinite dimensional setting, the *max-plus dual space fundamental solution semigroups* developed in [3], [4] facilitate propagation of functional solutions of the attendant integro-differential equations via a semiconvex dual space. In actuality, these semigroups consist of time horizon indexed *max-plus linear max-plus integral* operators that propagate the semiconvex dual of value functionals in this semiconvex dual space. However, by embedding the integro-differential equations of interest in the evolution of these value functionals (via a suitable optimal control problem), the aforementioned semigroups can be used to propagate the solutions of interest.

In this paper, the fundamental solution semigroup construction for the two classes of integro-differential equations considered in [3], [4] is generalized. In particular, a max-plus dual space fundamental solution semigroup is developed for a single class of operator differential Riccati equations that naturally generalizes both classes of equations considered in [3], [4]. This generalization is achieved by noting that a solution of either integro-differential equation can be interpreted as defining the kernel of an integral operator that solves a more general operator differential Riccati equation [7], [8]. Indeed, by focussing on this operator differential equation, the max-plus dual space fundamental solution semigroup obtained facilitates the propagation of its solutions *without reference to any specific operator representation*. It is noted that the construction of [3], [4] can be recovered by assuming the specific operator representations adopted there.

In terms of organization, some preliminary details concerning the operator spaces required are set out in Section II, followed by the introduction of the class of Riccati equations of interest in Section III. The max-plus dual space fundamental solution is constructed in Section IV, followed by some brief concluding remarks in Section V.

II. Preliminaries

Let $\mathscr{X}$ and $\mathcal{L}(\mathscr{X})$ denote respectively a Banach space, and the corresponding space of bounded linear operators defined on it. An operator-valued function $\mathcal{F}:\mathbb{R}\to\mathcal{L}(\mathscr{X})$ is (uniformly) continuous on an interval $I\subset\mathbb{R}$ if it is continuous at every $t_0\in I$. That is, given $\epsilon\in\mathbb{R}_{>0}$ there exists an $\delta\in\mathbb{R}_{>0}$ such that $|t-t_0|<\delta\Rightarrow\|\mathcal{F}(t)-\mathcal{F}(t_0)\|_{\mathcal{L}(\mathscr{X})}<\epsilon$, in which $\|\mathcal{F}(t)\|_{\mathcal{L}(\mathscr{X})}\doteq\sup_{\|x\|=1}\|\mathcal{F}(t)\,x\|$ denotes the induced operator norm of $\mathcal{F}(t)\in\mathcal{L}(\mathscr{X})$, and $\|\cdot\|$ is the norm on $\mathscr{X}$. The space of operator-valued functions defined on $I\subset\mathbb{R}$ is denoted by $C(I;\mathcal{L}(\mathscr{X}))$.

Similarly, an operator-valued function $\mathcal{F}:\mathbb{R}\to\mathcal{L}(\mathscr{X})$ is strongly continuous on an interval $I\subset\mathbb{R}$ if it is strongly continuous for every $t_0\in I$. That is, for every $x\in\mathscr{X}$, the function $\mathcal{F}(\cdot)\,x:I\to\mathscr{X}$ is continuous. The space of strongly continuous operator-valued functions defined on $I\subset\mathbb{R}$ is denoted by $C_0(I;\mathcal{L}(\mathscr{X}))$. In summary,

$$C(I;\mathcal{L}(\mathscr{X}))\doteq\left\{\mathcal{F}:I\to\mathcal{L}(\mathscr{X})\;\middle|\;\begin{array}{c}\mathcal{F}\text{ is}\\\text{continuous}\end{array}\right\},\quad(1)$$

$$C_0(I;\mathcal{L}(\mathscr{X}))\doteq\left\{\mathcal{F}:I\to\mathcal{L}(\mathscr{X})\;\middle|\;\begin{array}{c}\mathcal{F}\text{ is strongly}\\\text{continuous}\end{array}\right\}.\quad(2)$$

These spaces are related via the following classical lemma.

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Lemma 1: For any compact interval $I \subset \mathbb{R}$,

$$C(I;\mathcal{L}(\mathscr{X})) \subset C_0(I;\mathcal{L}(\mathscr{X})) \equiv \mathcal{L}(\mathscr{X};C(I;\mathscr{X}))\,, \quad (3)$$

in which $\|f\|_{C(I;\mathscr{X})} \doteq \sup_{t\in I}\|f(t)\|$.
The spaces (1), (2), abbreviated for convenience to $C\{I\}$, $C_0\{I\}$, may be equipped with the respective norms

$$\begin{aligned}\|\mathcal{F}\|_{C\{I\}} &\doteq \sup_{t\in I}\|\mathcal{F}(t)\|_{\mathcal{L}(\mathscr{X})}, \qquad \mathcal{F}\in C(I;\mathcal{L}(\mathscr{X})),\\ \|\mathcal{F}\|_{C_0\{I\}} &\doteq \sup_{\|x\|=1}\|\mathcal{F}(\cdot)\,x\|_{C(I;\mathscr{X})},\ \mathcal{F}\in C_0(I;\mathcal{L}(\mathscr{X})).\end{aligned} \quad (4)$$

Lemma 2: $\|\cdot\|_C$ may be extended to $C_0(I;\mathcal{L}(\mathscr{X}))$, whereupon it is equivalent to $\|\cdot\|_{C_0}$.

Proof: Fix $\mathcal{F}\in C_0(I;\mathcal{L}(\mathscr{X}))$. By definition (4),

$$\begin{aligned}\|\mathcal{F}\|_{C_0\{I\}} &= \sup_{\|x\|=1}\|\mathcal{F}(\cdot)\,x\|_{C(I;\mathscr{X})} = \sup_{\|x\|=1}\sup_{t\in I}\|\mathcal{F}(t)\,x\|\\ &= \sup_{t\in I}\sup_{\|x\|=1}\|\mathcal{F}(t)\,x\| = \sup_{t\in I}\|\mathcal{F}(t)\|_{\mathcal{L}(\mathscr{X})} \equiv \|\mathcal{F}\|_{C\{I\}}\,.\end{aligned}$$

■

It is well-known that the normed spaces defined by (1), (2), and (4) are Banach spaces.

Lemma 3: Given a compact $I\subset\mathbb{R}$, the normed spaces $(C(I;\mathcal{L}(\mathscr{X})),\|\cdot\|_{C\{I\}})$ and $(C_0(I;\mathcal{L}(\mathscr{X})),\|\cdot\|_{C_0\{I\}})$, defined via (1), (2), and (4), are Banach spaces.

A family $\{\mathcal{T}(t)\}_{t\in\mathbb{R}_{\ge0}}$ of bounded linear operators $\mathcal{T}(t)\in\mathcal{L}(\mathscr{X})$ is a *semigroup of bounded linear operators* on $\mathscr{X}$ if $\mathcal{T}(0)=\mathcal{I}$ and $\mathcal{T}(t)\,\mathcal{T}(s)=\mathcal{T}(t+s)$ for all $s,t\in\mathbb{R}_{\ge0}$, where $\mathcal{I}\in\mathcal{L}(\mathscr{X})$ denotes the identity, see [9]. The *infinitesimal generator* [9] of this semigroup is the linear operator $\mathcal{A}:\mathsf{dom}(\mathcal{A})\subset\mathscr{X}\to\mathscr{X}$ defined by

$$\begin{aligned}\mathcal{A}\,x &\doteq \lim_{t\to0^+}\frac{\mathcal{T}(t)\,x-x}{t}\,, \qquad x\in\mathsf{dom}(\mathcal{A})\,,\\ \mathsf{dom}(\mathcal{A}) &\doteq \left\{x\in\mathscr{X}\;\middle|\;\lim_{t\to0^+}\frac{\mathcal{T}(t)\,x-x}{t}\text{ exists}\right\}.\end{aligned} \quad (5)$$

Operator $\mathcal{A}$ generates a uniformly continuous semigroup if and only if $\mathsf{dom}(\mathcal{A})\equiv\mathscr{X}$, in which case $\mathcal{A}$ is bounded, see [9, Theorem 1.2, p.2]. Alternatively, if $\mathcal{A}$ generates a strongly continuous (C_0-) semigroup, then $\mathsf{dom}(\mathcal{A})$ is merely dense in $\mathscr{X}$. Where $\mathscr{X}\setminus\mathsf{dom}(\mathcal{A})\neq\emptyset$, $\mathcal{A}$ must be unbounded (but closed), see [9, Corollary 2.5, p.5]. By definition (1), the space $C(I;\mathcal{L}(\mathscr{X}))$ includes all uniformly continuous semigroups of bounded linear operators, see p.1 of [9]. Similarly, the space $C_0(I;\mathcal{L}(\mathscr{X}))$ of (2) includes all strongly continuous (C_0-) semigroups of bounded linear operators. Adopting the notation of [8], an element of either type of semigroup is denoted by $\mathcal{T}(t)=e^{\mathcal{A}\,t}\in\mathcal{L}(\mathscr{X})$ for $t\in\mathbb{R}_{\ge0}$.

III. Operator differential Riccati equation

Attention is initially restricted to the operator differential Riccati equation of interest. Two auxiliary operator differential equations of utility later are considered subsequently.

A. Riccati equation

Consider the operator differential Riccati equation posed with respect to Hilbert spaces $\mathscr{X}$ and $\mathscr{W}$ by

$$\dot{\mathcal{P}}(t)=\mathcal{A}'\,\mathcal{P}(t)+\mathcal{P}(t)\,\mathcal{A}+\mathcal{P}(t)\,\sigma\,\sigma'\,\mathcal{P}(t)+\mathcal{C}\,, \quad (6)$$

in which $\mathcal{A}:\mathsf{dom}(\mathcal{A})\subset\mathscr{X}\to\mathscr{X}$ is unbounded and densely defined on $\mathscr{X}$, $\sigma\in\mathcal{L}(\mathscr{W};\mathscr{X})$, $\mathcal{C}\in\mathcal{L}(\mathscr{X})$ is self-adjoint and non-negative, and $\mathcal{A}'$ and σ' denote the respective adjoints of $\mathcal{A}$ and σ.

Assumption 4: $\mathcal{A}$ generates a C_0-semigroup of bounded linear operators.

In order to define a suitable notion of solution for (6), it is convenient to define a vector space $\Sigma(\mathscr{X})$ and a set $\Sigma_{\mathcal{M}}(\mathscr{X})$ of bounded linear operators defined on $\mathscr{X}$ by

$$\Sigma(\mathscr{X})\doteq\left\{\mathcal{P}\in\mathcal{L}(\mathscr{X})\;\middle|\;\mathcal{P}\text{ is self-adjoint}\right\}, \quad (7)$$

$$\Sigma_{\mathcal{M}}(\mathscr{X})\doteq\left\{\mathcal{P}\in\Sigma(\mathscr{X})\;\middle|\;\begin{array}{c}\mathcal{P}-\mathcal{M}\text{ is coercive}\\ \text{on }\mathsf{dom}(\mathcal{A})\end{array}\right\}. \quad (8)$$

In (8), an operator $\mathcal{F}:\mathsf{dom}(\mathcal{F})\subset\mathscr{X}\to\mathscr{X}$ is coercive if there exists an $\epsilon\in\mathbb{R}_{>0}$ such that $\langle x,\,\mathcal{F}\,x\rangle\ge\epsilon\,\|x\|^2$ for all $x\in\mathsf{dom}(\mathcal{F})$. Consistent with definitions (1) and (2), the corresponding sets of continuous and strongly continuous operator-valued functions defined on $[0,T]\subset\mathbb{R}_{\ge0}$, $T\in\mathbb{R}_{>0}$, are denoted by $C([0,T];\Omega)$ and $C_0([0,T];\Omega)$ respectively, where $\Omega\in\{\Sigma(\mathscr{X}),\,\Sigma_{\mathcal{M}}(\mathscr{X})\}$. A *mild* solution of (6) on a time interval $[0,T]$, $T\in\mathbb{R}_{>0}$, is any operator-valued function $\mathcal{P}\in C_0([0,T];\Sigma(\mathscr{X}))$ that satisfies

$$\mathcal{P}(t)\,x=\gamma(\mathcal{P})(t)\,x\,, \quad (9)$$

for all $x\in\mathscr{X}$, $t\in[0,T]$, where operator γ applied to $\mathcal{O}\in C_0([0,T];\Sigma(\mathscr{X}))$ is defined by

$$\begin{aligned}&[\gamma(\mathcal{O})(t)]\,x\doteq e^{\mathcal{A}'\,t}\,\mathcal{O}(0)\,e^{\mathcal{A}\,t}\,x\\ &\quad+\int_0^t e^{\mathcal{A}'\,(t-s)}\left[\mathcal{O}(s)\,\sigma\,\sigma'\,\mathcal{O}(s)+\mathcal{C}\right]e^{\mathcal{A}\,(t-s)}\,x\,ds\end{aligned} \quad (10)$$

for all $t\in[0,T]$, $x\in\mathscr{X}$, where $e^{\mathcal{A}'\,\cdot}$ denotes the C_0-semigroup generated by the operator adjoint $\mathcal{A}'$ (see, for example, Theorem 2.2.6 of [7]). As per [8], it is convenient to introduce an analogous operator differential Riccati equation to (6), defined with respect to the Yosida approximation $\mathcal{A}_n\in\mathcal{L}(\mathscr{X})$ of $\mathcal{A}$ for all $n\in\mathbb{N}$. In particular [5],

$$\dot{\mathcal{P}}_n(t)=\mathcal{A}_n'\,\mathcal{P}_n(t)+\mathcal{P}_n(t)\,\mathcal{A}_n+\mathcal{P}_n(t)\,\sigma\,\sigma'\,\mathcal{P}_n(t)+\mathcal{C}\,. \quad (11)$$

A mild solution of (11) on a time interval $[0,T]$, $T\in\mathbb{R}_{>0}$, is any operator-valued function $\mathcal{P}_n\in C([0,T];\Sigma(\mathscr{X}))$ that satisfies the corresponding equation to (9), i.e.,

$$\mathcal{P}_n(t)\,x=\gamma_n(\mathcal{P}_n)(t)\,x\,, \quad (12)$$

for all $x\in\mathscr{X}$, $t\in[0,T]$, where operator γ_n applied to $\mathcal{O}\in C([0,T];\Sigma(\mathscr{X}))$ is defined by

$$\begin{aligned}&\gamma_n(\mathcal{O})(t)\,x\doteq e^{\mathcal{A}_n'\,t}\,\mathcal{O}(0)\,e^{\mathcal{A}_n\,t}\,x\\ &\quad+\int_0^t e^{\mathcal{A}_n'\,(t-s)}\left[\mathcal{O}(s)\,\sigma\,\sigma'\,\mathcal{O}(s)+\mathcal{C}\right]e^{\mathcal{A}_n\,(t-s)}\,x\,ds\end{aligned} \quad (13)$$

for all $t \in [0,T]$, $x \in \mathscr{X}$, and $e^{\mathcal{A}_n' \cdot}$ denotes the uniformly continuous semigroup generator by the adjoint of the Yoshida approximation $\mathcal{A}_n \in \mathcal{L}(\mathscr{X})$.

Theorem 5 ([5]): Given any $\mathcal{P}_0 \in \Sigma(\mathscr{X})$, there exists a $\tau \in \mathbb{R}_{>0}$ such that the operator differential Riccati equations (6), (11) exhibit respective unique mild solutions $\mathcal{P} \in C_0([0,\tau];\Sigma(\mathscr{X}))$, $\mathcal{P}_n \in C([0,\tau];\Sigma(\mathscr{X}))$ satisfying $\mathcal{P}(0) = \mathcal{P}_0 = \mathcal{P}_n(0)$. Furthermore, for all $x \in \mathscr{X}$,

$$\lim_{n\to\infty} \mathcal{P}_n(\cdot)\, x = \mathcal{P}(\cdot)\, x\,, \tag{14}$$

where the limit is defined with respect to the Banach space $(C([0,\tau];\mathscr{X}),\, \|\cdot\|_{C([0,\tau];\mathscr{X})})$.

Remark 6: Note that the limit (14) is a statement of strong (operator) convergence on $C([0,\tau];\mathscr{X})$. This is strictly weaker than uniform (operator) convergence defined on $C([0,\tau];\mathcal{L}(\mathscr{X}))$ via the norm $\|\cdot\|_{C[0,\tau]}$ (see, for example, p.263 of [10]). As $(C([0,\tau];\mathcal{L}(\mathscr{X})),\, \|\cdot\|_{C[0,\tau]})$ defines a Banach space, this weaker form of convergence allows the limit to reside in $C_0([0,\tau];\mathcal{L}(\mathscr{X}))\setminus C([0,\tau];\mathcal{L}(\mathscr{X}))$ should $\mathcal{A}$ be unbounded.

Remark 7: It may also be noted that, by an analogous argument to Proposition 2.1 of [8] (see p. 391), $\mathcal{P} \in C_0([0,\tau];\mathcal{L}(\mathscr{X}))$ is a mild solution of (9) if and only if it is a *weak* solution (see [8], Definition 2.1, p. 390).

Assumption 8: There exists an operator $\mathcal{M} \in \Sigma(\mathscr{X})$ such that the unique mild solution $\mathcal{P} \in C_0([0,\tau_0];\Sigma(\mathscr{X}))$ of (6) satisfying $\mathcal{P}(0) = \mathcal{M}$ that exists for some $\tau_0 \in \mathbb{R}_{>0}$ by Theorem 5 is such that $\mathcal{P}(t) - \mathcal{M}$ is coercive for all $t \in (0,\tau_0]$. That is,

$$\mathcal{P} \in C_0([0,\tau_0];\Sigma(\mathscr{X})) \cap C_0((0,\tau_0];\Sigma_{\mathcal{M}}(\mathscr{X})). \tag{15}$$

Theorem 9 ([5]): Given any $\mathcal{M} \in \Sigma(\mathscr{X})$ and $\tau_0 \in \mathbb{R}_{>0}$ satisfying Assumption 8, and any $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$, there exists a $\tau_1 \in (0,\tau_0]$ such that a unique mild solution

$$\widetilde{\mathcal{P}} \in C_0([0,\tau_1];\Sigma(\mathscr{X})) \cap C_0((0,\tau_1];\Sigma_{\mathcal{M}}(\mathscr{X})) \tag{16}$$

of (6) satisfying $\widetilde{\mathcal{P}}(0) = \widetilde{\mathcal{M}}$ exists.

Remark 10: The integral form (9) of the operator Riccati equation (6) plays a key role in the corresponding notion of mild solution [9], [8], [5]. In order to see how (9) may be formally derived from (6), let $\mathcal{P} \in C_0^1([0,t];\Sigma^{\mathcal{A}}(\mathscr{X}))$, $t \in \mathbb{R}_{>0}$, denote a (strict) solution [8] of (9), where $C_0^1([0,t];\Sigma^{\mathcal{A}}(\mathscr{X}))$ is the set of strongly Frechet differentiable operator-valued functions mapping $[0,t]$ to $\Sigma^{\mathcal{A}}(\mathscr{X})$, where $\Sigma^{\mathcal{A}}(\mathscr{X}) \doteq \{\mathcal{P} \in \Sigma(\mathscr{X}) \,|\, \mathcal{P}\, x \in \mathsf{dom}(\mathcal{A}),\, x \in \mathsf{dom}(\mathcal{A})\}$. Define $\pi_t : [0,t] \to \Sigma^{\mathcal{A}}(\mathscr{X})$ by $\pi_t(s)\, x \doteq e^{\mathcal{A}'\,(t-s)}\, \mathcal{P}(s)\, e^{\mathcal{A}\,(t-s)}\, x$, $s \in [0,t]$, $x \in \mathscr{X}$. By definition, $\pi(\cdot)\, x$ is Fréchet differentiable for each $x \in \mathscr{X}$. Hence, recalling [9, Theorem 2.4, p.4], and the chain rule for Fréchet differentiation,

$$\tfrac{d[\pi_t(s)\, x]}{ds} = e^{\mathcal{A}'(t-s)}[-\mathcal{A}'\, \mathcal{P}(s) + \dot{\mathcal{P}}(s) + \mathcal{P}(s)\, \mathcal{A}]\, e^{\mathcal{A}(t-s)}\, x.$$

The integral equation (9) follows by substitution for $\dot{\mathcal{P}}(s)$ in the right-hand side using (6), followed by integration with respect to s from 0 to t, and application of the identities $\pi_t(0)\, x = e^{\mathcal{A}'\, t}\, \mathcal{P}(0)\, e^{\mathcal{A}\, t}\, x$ and $\pi_t(t)\, x = \mathcal{P}(t)\, x$ that follow by definition of π_t.

B. Auxiliary operator differential equations

In proposing a max-plus dual space fundamental solution to the differential operator Riccati equation (6), two (additional) auxiliary operator differential equations are of interest. These equations, also defined with respect to Hilbert spaces $\mathscr{X}$ and $\mathscr{W}$, are given by

$$\dot{\mathcal{Q}}(t) = \mathcal{A}'\, \mathcal{Q}(t) + \mathcal{P}(t)\, \sigma\, \sigma'\, \mathcal{Q}(t)\,, \tag{17}$$

$$\dot{\mathcal{R}}(t) = \mathcal{Q}'(t)\, \sigma\, \sigma'\, \mathcal{Q}(t)\,, \tag{18}$$

in which $\mathcal{A} : \mathsf{dom}(\mathcal{A}) \subset \mathscr{X} \to \mathscr{X}$ and $\sigma \in \mathcal{L}(\mathscr{W};\mathscr{X})$ are defined as per (6). Also as per (6), any operator-valued functions $\mathcal{Q} \in C_0([0,\tau];\mathcal{L}(\mathscr{X}))$ and $\mathcal{R} \in C_0([0,\tau];\Sigma(\mathscr{X}))$ satisfying

$$\mathcal{Q}(t)\, x = e^{\mathcal{A}'\, t}\, \mathcal{Q}(0)\, x + \int_0^t e^{\mathcal{A}'\,(t-s)}\, [\mathcal{P}(s)\, \sigma\, \sigma'\, \mathcal{Q}(s)]\, x\, ds\,,$$

$$\mathcal{R}(t)\, x = \mathcal{R}(0)\, x + \int_0^t \mathcal{Q}(s)'\, \sigma\, \sigma'\, \mathcal{Q}(s)\, x\, ds\,, \tag{19}$$

for all $x \in \mathscr{X}$, $t \in [0,\tau]$, $\tau \in \mathbb{R}_{>0}$, are defined to be *mild solutions* of (17) and (18) (respectively) on $[0,\tau]$. With regard to the range of $\mathcal{Q}$, note $\mathcal{Q}(t) \in \mathcal{L}(\mathscr{X})$ is not self-adjoint by inspection of (17) or (19). Hence, $\mathcal{Q} \in C_0([0,\tau];\mathcal{L}(\mathscr{X}))$ (rather than $C_0([0,\tau];\Sigma(\mathscr{X}))$). On the other hand, $\mathcal{R}(t) \in \Sigma(\mathscr{X})$ is self-adjoint by inspection of (18) or (19). Existence of unique mild solutions of (17) and (18) can also be established for specific initial conditions, see [5].

Theorem 11: Given any $\mathcal{M} \in \Sigma(\mathscr{X})$, and $\tau \in \mathbb{R}_{>0}$, $\mathcal{P} \in C_0([0,\tau];\Sigma(\mathscr{X}))$, $\mathcal{P}_n \in C([0,\tau];\Sigma(\mathscr{X}))$ as specified by Theorem 5 with $\mathcal{P}_0 = \mathcal{M}$, there exists a $\tau_2 \in (0,\tau]$ such that the operator differential equations (17), (18), and their corresponding Yosida approximations [5] exhibit unique mild solutions

$$\mathcal{Q} \in C_0([0,\tau_2];\mathcal{L}(\mathscr{X}))\,,\ \ \mathcal{Q}_n \in C([0,\tau_2];\mathcal{L}(\mathscr{X}))\,,$$
$$\mathcal{R} \in C_0([0,\tau_2];\Sigma(\mathscr{X}))\,,\ \ \mathcal{R}_n \in C([0,\tau_2];\Sigma(\mathscr{X}))\,,$$

satisfying $\mathcal{Q}(0) = -\mathcal{M} = \mathcal{Q}_n(0)$, $\mathcal{R}(0) = \mathcal{M} = \mathcal{R}_n(0)$ for all $n \in \mathbb{N}$. Furthermore, for all $x \in \mathscr{X}$,

$$\lim_{n\to\infty} \mathcal{Q}_n(\cdot)\, x = \mathcal{Q}(\cdot)\, x\,,\ \ \lim_{n\to\infty} \mathcal{R}_n(\cdot)\, x = \mathcal{R}(\cdot)\, x\,, \tag{20}$$

with the limits defined with respect to the Banach space $(C([0,\tau_2];\mathscr{X}),\, \|\cdot\|_{C([0,\tau_2];\mathscr{X})})$.

C. Common horizon of existence of mild solutions

For the remainder, it is convenient to define a common horizon $\tau^* \in \mathbb{R}_{>0}$ of existence for the unique mild solutions $\mathcal{P}$, $\widetilde{\mathcal{P}}$ of the operator differential Riccati equation (6), $\mathcal{Q}$, $\mathcal{R}$ of the auxiliary operator differential equations (17), (18), and $\mathcal{P}_n$, $\mathcal{Q}_n$, $\mathcal{R}_n$ of the corresponding Yosida approximation operator differential equations, for example (11). In particular, with $\tau_1(\mathcal{M},\widetilde{\mathcal{M}}) \doteq \tau_1 \in \mathbb{R}_{>0}$ as fixed by applying Theorem 9 for any $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$ with $\mathcal{M} \in \Sigma(\mathscr{X})$ fixed as per Assumption 8, and $\tau_2(\mathcal{M}) \doteq \tau_2 \in \mathbb{R}_{>0}$ as fixed by applying Theorem 11 for the same $\mathcal{M} \in \Sigma(\mathscr{X})$, define

$$\tau^* = \tau^*(\mathcal{M},\widetilde{\mathcal{M}}) \doteq \tau_1(\mathcal{M},\widetilde{\mathcal{M}}) \wedge \tau_2(\mathcal{M}) \in \mathbb{R}_{>0}\,, \tag{21}$$

where $\wedge$ denotes the $\min$ operation. That is, Theorems 5 and 11 guarantee existence of unique $\mathcal{P}$, $\widetilde{\mathcal{P}}$, $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{P}_n$, $\mathcal{Q}_n$, $\mathcal{R}_n$ on $[0,\tau^*] \subset \mathbb{R}_{\geq 0}$.

IV. MAX-PLUS FUNDAMENTAL SOLUTION SEMIGROUP

A max-plus dual space fundamental solution semigroup for the operator differential Riccati equation (6) is proposed. Its construction exploits the semigroup property that attends the dynamic programming evolution operator of a related optimal control problem. By employing the Legendre-Fenchel transform, it is shown that a max-plus integral operator defined in the corresponding *max-plus dual space* inherits the aforementioned semigroup property, and in fact allows the realization of any solution of the operator differential Riccati equation (6) from one particular solution. This construction generalizes that of [3], [4] as it does not assume an explicit representation for the operators involved.

A. Optimal control problem

With $\tau^* \in \mathbb{R}_{>0}$ as per (21), an optimal control problem of interest is defined with respect to the abstract Cauchy problem [8], [7], [9]

$$\dot{\xi}(t) = \mathcal{A}\,\xi(t) + \sigma\, w(t)\,, \tag{22}$$

where $\xi(t) \in \mathsf{dom}(\mathcal{A}) \subset \mathscr{X}$ denotes the state at time $t \in [0,\tau^*]$, evolved from an initial state $\xi(0) = x \in \mathsf{dom}(\mathcal{A})$ in the presence of an input signal $w \in \mathscr{L}_2([0,t];\mathscr{W})$. A function $\xi \in C([0,t];\mathscr{X})$ is a *mild* solution of (22) on $[0,t]$ if given $w \in \mathscr{L}_2([0,t];\mathscr{W})$ it satisfies

$$\xi(t) = e^{\mathcal{A}\,t}\,x + \int_0^t e^{\mathcal{A}\,(t-s)}\,w(s)\,ds\,, \tag{23}$$

(see for example [8], p.129), where $e^{\mathcal{A}\,\cdot}$ denotes the C_0-semigroup of operators in $\mathcal{L}(\mathscr{X})$ generated by $\mathcal{A}$. Note that $\xi \in C([0,t);\mathscr{X})$ is in fact implied by (23), see for example Lemma 3.1.5 of [7]. Indeed, given any $\xi(0) = x \in \mathscr{X}$ and $w \in \mathscr{L}_2([0,t];\mathscr{W})$, (22) has a unique *strong* solution which is also the mild solution (see for example Definition 3.1 and Proposition 3.1 on pp.129 – 130 of [8]). The optimal control problem of interest is defined via the value functional $W^z : [0,\tau^*] \times \mathscr{X} \to \mathbb{R}$, $z \in \mathscr{X}$, where

$$W^z(t,x) \doteq \sup_{w \in \mathscr{L}_2([0,t];\mathscr{W})} J_{\psi(\cdot,z)}(t,x;w)\,, \tag{24}$$

in which payoff $J_\Psi : [0,t] \times \mathscr{X} \times \mathscr{L}_2([0,t];\mathscr{W}) :\to \mathbb{R}$ defined with respect to a generic terminal payoff $\Psi : \mathscr{X} \to \mathbb{R}$ by

$$J_\Psi(t,x;w) \doteq \int_0^t \tfrac{1}{2}\langle \xi(s),\, \mathcal{C}\,\xi(s)\rangle - \tfrac{1}{2}\|w(s)\|^2\,ds + \Psi(\xi(t))\,. \tag{25}$$

(Here, $\xi(\cdot)$ denotes the mild solution (23) with $\xi(0) = x$.) A specific terminal payoff $\Psi(\cdot) = \psi(\cdot, z) : \mathscr{X} \to \mathbb{R}$ of interest is defined for each $z \in \mathscr{X}$ via the operator $\mathcal{M} \in \Sigma(\mathscr{X})$ of Assumption 8, with

$$\psi(x,z) \doteq \tfrac{1}{2}\langle x - z,\, \mathcal{M}\,(x-z)\rangle \tag{26}$$

for all $x, z \in \mathscr{X}$.

Solutions of the operator differential Riccati equation (6), and the auxiliary operator differential equations (17) and (18), are fundamentally related to the optimal control problem of (24). To explore and exploit this relationship, define the operator-valued map $\mathcal{K}_t : [0,t] \to \mathcal{L}(\mathscr{X};\mathscr{W})$ by

$$\mathcal{K}_t(s)\,x \doteq \sigma'\,(\mathcal{P}(t-s)\,x + \mathcal{Q}(t-s)\,z)\,, \tag{27}$$

where $\mathcal{P} \in C_0([0,\tau^*];\Sigma(\mathscr{X}))$ and $\mathcal{Q} \in C_0([0,\tau^*];\mathcal{L}(\mathscr{X}))$ denote the unique mild solutions of (6) and (17) satisfying $\mathcal{P}(0) = \mathcal{M}$ and $\mathcal{Q}(0) = -\mathcal{M}$ respectively (as per Theorems 5 and 11). This map (27) can be regarded as a feedback for the abstract Cauchy problem (22), yielding the closed-loop abstract Cauchy problem

$$\dot{\xi}(s) = (\mathcal{A} + \sigma\,\mathcal{K}_t(s))\,\xi(s)\,, \quad s \in [0,t]\,, \tag{28}$$

where $\xi(0) = x \in \mathscr{X}$ and $t \in [0,\tau^*]$.

Theorem 12: Given any $t \in [0,\tau^*]$, the closed-loop abstract Cauchy problem (28) has a unique mild solution $\xi^* \in C([0,t];\mathscr{X})$. Furthermore, the input $w^* \in C([0,t];\mathscr{W})$ defined by

$$w^*(s) \doteq K_t(s)\,\xi^*(s) = \sigma'(\mathcal{P}(t-s)\,\xi^*(s) + \mathcal{Q}(t-s)\,z) \tag{29}$$

is optimal with respect to (24), (25), with

$$\begin{aligned} J_{\psi(\cdot,z)}(t,x;w) &\leq J_{\psi(\cdot,z)}(t,x;w^*) = W^z(t,x) \\ &= \tfrac{1}{2}\langle x,\, \mathcal{P}(t)\,x + \langle x,\, \mathcal{Q}(t)\,z\rangle + \tfrac{1}{2}\langle z,\, \mathcal{R}(t)\,z\rangle \end{aligned} \tag{30}$$

for all $w \in \mathscr{L}_2([0,t];\mathscr{W})$, $x \in \mathscr{X}$.

Proof: Fix any $t \in [0,\tau^*]$. The abstract Cauchy problem (28) exhibits a unique mild solution $\xi^* \in C([0,t];\mathscr{X})$ via a straightforward modification of Proposition 6.1 on p.409 of [8]. The fact that input w^* defined by (29) is optimal follows by a modification of the proof of Proposition 6.2 on p. 409 of [8]. In particular, let $\mathcal{P}_n, \mathcal{R}_n \in C([0,\tau^*];\Sigma(\mathscr{X}))$ and $\mathcal{Q}_n \in C([0,\tau^*];\mathcal{L}(\mathscr{X}))$ denote the unique mild solution of (11) and the corresponding integral equations (19) with $\mathcal{A}$ replaced with the Yoshida approximation $\mathcal{A}_n$, see [5]. Let $\xi_n \in C([0,t];\mathscr{X})$ denote the unique mild solution of the corresponding abstract Cauchy problem (22) for arbitrary $w \in \mathscr{L}_2([0,t];\mathscr{W})$. Define $\pi_n : [0,t] \to \mathbb{R}$ by

$$\pi_n(s) \doteq p_n(s) + q_n(s) + r_n(s) \tag{31}$$

where $p_n, q_n, r_n : [0,t] \to \mathbb{R}$ are given by

$$p_n(s) \doteq \tfrac{1}{2}\langle \xi_n(s),\, \mathcal{P}_n(t-s)\,\xi_n(s)\rangle\,, \tag{32}$$

$$q_n(s) \doteq \langle \xi_n(s),\, \mathcal{Q}_n(t-s)\,z\rangle\,, \tag{33}$$

$$r_n(s) \doteq \tfrac{1}{2}\langle z,\, \mathcal{R}_n(t-s)\,z\rangle\,. \tag{34}$$

Differentiating and applying (11), along with (17), (18) with $\mathcal{A}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ replaced with the Yosida approximations $\mathcal{A}_n$, $\mathcal{P}_n$, $\mathcal{Q}_n$, $\mathcal{R}_n$ analogously to (11), it follows that

$$\begin{aligned} \dot{p}_n(s) = &-\tfrac{1}{2}\langle \xi_n(s),\, [\mathcal{C} + \mathcal{P}_n(t-s)\,\sigma\,\sigma'\,\mathcal{P}_n(t-s)]\;\xi_n(s)\rangle \\ &+ \langle w(s),\, \sigma'\mathcal{P}_n(t-s)\,\xi_n(s)\rangle\,, \end{aligned} \tag{35}$$

$$\begin{aligned} \dot{q}_n(s) = &-\langle \xi_n(s),\, \mathcal{P}_n(t-s)\,\sigma\,\sigma'\,\mathcal{Q}_n(t-s)\,z\rangle \\ &+ \langle w(s),\, \sigma'\mathcal{Q}_n(t-s)\,z\rangle\,, \end{aligned} \tag{36}$$

$$\dot{r}_n(s) = -\tfrac{1}{2}\langle z,\, \mathcal{Q}_n(t-s)'\,\sigma\,\sigma'\,\mathcal{Q}_n(t-s)\,z\rangle \tag{37}$$

Differentiation of (31), substitution of (35), (36), (37), and completion of squares yields

$$\dot{\pi}_n(s) = -\left[\tfrac{1}{2}\langle \xi_n(s)\, \mathcal{C}\, \xi_n(s)\rangle - \tfrac{1}{2}\|w(s)\|^2\right] - \tfrac{1}{2}\|w(s) - \overline{w}(s)\|^2\,, \tag{38}$$

where $\overline{w}(s) \doteq \sigma'\left(\mathcal{P}_n(t-s)\,\xi_n(s) + \mathcal{Q}_n(t-s)\,z\right)$ for all $s \in [0,t]$. As $\mathcal{P}_n \in C([0,t];\Sigma(\mathscr{X}))$ and $\mathcal{Q}_n \in C([0,t];\mathcal{L}(\mathscr{X}))$, note that $\overline{w} \in C([0,t];\mathscr{W}) \subset \mathscr{L}_2([0,t];\mathscr{W})$ by definition. Note also by (31), and that $\mathcal{P}_n(0) = \mathcal{M} = \mathcal{R}_n(0)$ and $\mathcal{Q}_n(0) = -\mathcal{M}$, so that

$$\begin{aligned}\pi_n(t) &= \tfrac{1}{2}\langle \xi_n(t),\, \mathcal{M}\,\xi_n(t)\rangle - \langle \xi_n(t),\, \mathcal{M}\, z\rangle + \tfrac{1}{2}\langle z,\, \mathcal{M}\, z\rangle \\ &= \psi(\xi(t), z)\,,\end{aligned} \tag{39}$$

where ψ is the terminal payoff (26). Similarly, as $\xi_n(0) = x$,

$$\pi_n(0) = \tfrac{1}{2}\langle x,\, \mathcal{P}_n(t)\, x\rangle + \langle x,\, \mathcal{Q}_n(t)\, z\rangle + \tfrac{1}{2}\langle z,\, \mathcal{R}_n(t)\, z\rangle\,.$$

So, integrating (38) with respect to $s \in [0,t]$ and applying (39) yields that

$$\begin{aligned}\pi_n(0) = \int_0^t \tfrac{1}{2}\langle \xi_n(s),\, \mathcal{C}\,\xi_n(s)\rangle - \tfrac{1}{2}\|w(s)\|^2\, ds + \psi(\xi(t),\, z) \\ + \tfrac{1}{2}\int_0^t \|w(s) - \overline{w}(s)\|^2\, ds\,,\end{aligned}$$

Taking the limit as $n \to \infty$, setting $\pi_\infty \doteq \lim_{n\to\infty} \pi_n$, and applying (25),

$$\pi_\infty(0) - \tfrac{1}{2}\int_0^t \|w(s) - \overline{w}(s)\|^2\, ds = J_{\psi(\cdot,z)}(t,x;w)\,.$$

Finally, taking the supremum over $w \in \mathscr{L}_2([0,t];\mathscr{W})$ yields

$$\begin{aligned}W^z(t,x) &= \sup_{w\in\mathscr{L}_2([0,t];\mathscr{W})} J_{\psi(\cdot,z)}(t,x;w) \\ &= \pi_\infty(0) = \tfrac{1}{2}\langle x,\, \mathcal{P}(t)\, x\rangle + \langle x,\, \mathcal{Q}\, z\rangle + \tfrac{1}{2}\langle z,\, \mathcal{R}\, z\rangle\,,\end{aligned}$$

in which the optimal input is $w^* = \overline{w}$, as per (29). ∎

Theorem 12 is crucial to the development of a max-plus fundamental solution to the operator differential Riccati equation (6). In particular, it demonstrates that the unique mild solution $\mathcal{P} \in C_0([0,\tau^*];\Sigma(\mathscr{X}))$ of (6) may be propagated forward in time via propagation of the value function $W^z(t,\cdot)$ of (24) with respect to its time horizon $t \in [0,\tau^*]$. This is significant as propagation of $W^z(t,\cdot)$ is possible via the dynamic programming [11], [12] evolution operator. In particular, W^z may be written as

$$W^z(t,x) = (\mathcal{S}_t\, \psi(\cdot,z))(x) \tag{40}$$

for all $t \in [0,\tau^*]$, $x \in \mathscr{X}$, where $\mathcal{S}_t$ denotes the aforementioned dynamic programming evolution operator. This operator is defined by

$$(\mathcal{S}_t\, \Psi)(x) \doteq \sup_{w\in\mathscr{L}_2([0,t];\mathscr{W})} J_\Psi(t,x;w)\,. \tag{41}$$

and satisfies the semigroup property

$$\mathcal{S}_{t+s} = \mathcal{S}_t\, \mathcal{S}_s \tag{42}$$

for all $s,t \in [0,\tau^*]$, $s+t \in [0,\tau^*]$. It is this semigroup property, in combination with (40), that allows $W^z(t,\cdot)$ to be propagated to longer time horizons via the dynamic programming evolution operator of (41).

B. Max-plus dual space fundamental solution semigroup

With Theorem 12 in place, the max-plus dual space fundamental solution semigroup for the operator differential Riccati equation (6) can be constructed analogously to that of [3], [4]. This construction is summarized here for completeness. To this end, recall that the max-plus algebra is a commutative semi-field over $\mathbb{R}^- \doteq \mathbb{R}\cup\{-\infty\}$, equipped with addition and multiplication operations defined respectively by $a \oplus b \doteq \max(a,b)$ and $a \otimes b = a + b$. Max-plus integration of a functional $f : \mathscr{X} \to \mathbb{R}^-$ is defined as $\int_{\mathscr{X}}^{\oplus} f(x)\, dx \doteq \sup_{x\in\mathscr{X}} f(x)$. A space of semiconvex functionals of interest is defined for $-\mathcal{K} > \mathcal{M}$ by

$$\mathscr{S}^{\mathcal{K}}(\mathscr{X}) \doteq \left\{ f : \mathscr{X} \to \mathbb{R}^- \;\middle|\; \begin{array}{c} f \text{ is closed on } \mathscr{X}, \\ f + \tfrac{1}{2}\langle\cdot,\, \mathcal{K}\,\cdot\rangle \\ \text{is convex on } \mathscr{X} \end{array} \right\},$$

where $\mathcal{M}$ is as per Assumption 8. Any functional $\phi \in \mathscr{S}^{\mathcal{K}}(\mathscr{X})$ admits a semiconvex dual [13], [14] defined with respect to ψ of (26). Specifically,

$$\phi(x) = (\mathcal{D}_\psi^{-1}\, a)(x)\,, \quad a(z) = (\mathcal{D}_\psi\, \phi)(z)\,, \tag{43}$$

where

$$\mathcal{D}_\psi\, \phi \doteq -\int_{\mathscr{X}}^{\oplus} \psi(x,\cdot) \otimes (-\phi(x))\, dx\,, \tag{44}$$

$$\mathcal{D}_\psi^{-1}\, a \doteq \int_{\mathscr{X}}^{\oplus} \psi(\cdot,z) \otimes a(z)\, dz\,. \tag{45}$$

As highlighted in [3], [4], the semiconvex duality pairing (43) facilitates the definition of a semigroup $\{\mathcal{B}_t\}$ of max-plus linear operators. In particular,

$$\mathcal{B}_t\, a \doteq \int_{\mathscr{X}}^{\oplus} B_t(\cdot,z) \otimes a(z)\, dz\,, \tag{46}$$

where kernel $B_t : \mathscr{X}^2 \to \mathbb{R}^-$ is defined via (41), (43) by

$$B_t(y,z) \doteq (\mathcal{D}_\psi\, \mathcal{S}_t\, \psi(\cdot,z))(y)\,, \tag{47}$$

$$(\mathcal{S}_t\, \psi(\cdot,z))(x) = (\mathcal{D}_\psi^{-1}\, B_t(\cdot,z))(x) \tag{48}$$

for all $x,y,z \in \mathscr{X}$. The fact that $\{\mathcal{B}_t\}_{t\in\mathbb{R}_{\ge 0}}$ defines a semigroup follows from (42), and that for any $\tilde{\psi} \in \mathscr{S}^{\mathcal{K}}(\mathscr{X})$,

$$\mathcal{S}_t\, \tilde{\psi} = \mathcal{D}_\psi^{-1}\, \mathcal{B}_t\, \mathcal{D}_\psi\, \tilde{\psi}\,, \tag{49}$$

for all $t \in [0,\tau^*]$, see [3], [4]. In particular, given $\tilde{a} \doteq \mathcal{D}_\psi\, \tilde{\psi}$, and any $\tau \in [0,\tau^*]$, $t+\tau \in [0,\tau^*]$, it follows that

$$\begin{aligned}\mathcal{B}_{\tau+t}\, \tilde{a} &= \mathcal{D}_\psi\, \mathcal{D}_\psi^{-1}\, \mathcal{B}_{\tau+t}\, \mathcal{D}_\psi\, \tilde{\psi} = \mathcal{D}_\psi\, \mathcal{S}_{\tau+t}\, \tilde{\psi} = \mathcal{D}_\psi\, \mathcal{S}_\tau\, \mathcal{S}_t\, \tilde{\psi} \\ &= \mathcal{D}_\psi\, \mathcal{D}_\psi^{-1}\, \mathcal{B}_\tau\, \mathcal{D}_\psi\, \mathcal{D}_\psi^{-1}\, \mathcal{B}_t\, \mathcal{D}_\psi\, \tilde{\psi} = \mathcal{B}_\tau\, \mathcal{B}_t\, \tilde{a}\,.\end{aligned} \tag{50}$$

In view of (46), (50), and Theorem 12, $\{\mathcal{B}_t\}_{t\in\mathbb{R}_{\ge 0}}$ is referred to here as the *max-plus dual space fundamental solution semigroup*. By selecting

$$\tilde{\psi}(x) \doteq \tfrac{1}{2}\langle x,\, \widetilde{\mathcal{M}}\, x\rangle \tag{51}$$

for any $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$, this fundamental solution semigroup can be used to propagate the value

$$(\mathcal{S}_{t_0}\, \tilde{\psi})(x) = \tfrac{1}{2}\langle x,\, \widetilde{\mathcal{P}}(t_0)\, x\rangle\,, \quad x \in \mathscr{X}\,,\ t_0 \in [0,\tau^*]\,,$$

and hence the operator differential Riccati equation solution $\widetilde{\mathcal{P}}(t_0)$ corresponding to $\widetilde{\mathcal{P}}(0) = \widetilde{\mathcal{M}}$, to any time horizon $t \in [\underline{t_0}, \tau^*]$. The propagation of this semigroup is independent of $\widetilde{\mathcal{M}}$. Indeed, it is straightforward to see that the kernels $\{B_t\}$ also inherit a semigroup property, with

$$B_{\tau+t}(y,z) = \int_{\mathscr{X}}^{\oplus} B_\tau(y,\mu) \otimes B_t(\mu,z)\, d\mu\,. \tag{52}$$

Furthermore, Theorem 12 and (47) imply that $B_t(y,z)$ is quadratic in $y, z \in \mathscr{X}$. Using the notation of [4],

$$B_t(y,z) = \tfrac{1}{2}\langle x,\, \mathcal{B}_t^{1,1}\, x\rangle + \langle x,\, \mathcal{B}_t^{1,2}\, z\rangle + \tfrac{1}{2}\langle z,\, \mathcal{B}_t^{2,2}\, z\rangle \tag{53}$$

where the operators within the inner products are defined by

$$\begin{aligned} \mathcal{B}_t^{1,1} &\doteq -\mathcal{M} - \mathcal{M}\,(\mathcal{P}(t) - \mathcal{M})^{-1}\,\mathcal{M}\,, \\ \mathcal{B}_t^{1,2} &\doteq -\mathcal{M}\,(\mathcal{P}(t) - \mathcal{M})^{-1}\,\mathcal{Q}(t)\,, \\ \mathcal{B}_t^{2,2} &\doteq -\mathcal{Q}(t)'\,(\mathcal{P}(t) - \mathcal{M})^{-1}\,\mathcal{Q}(t) + \mathcal{R}(t)\,. \end{aligned} \tag{54}$$

Note that these operators are well-defined for $t \in (0, \tau^*]$, as $\mathcal{P}(t) - \mathcal{M}$ is boundedly invertible on that interval (by the self-adjoint and coercivity requirements of Assumption 8, see for example p.609 of [7]). Furthermore, applying (53) yields a semigroup property for these operators [4], with

$$\begin{aligned} \mathcal{B}_{\tau+t}^{1,1} &= \mathcal{B}_\tau^{1,1} - \mathcal{B}_\tau^{1,2}\left(\mathcal{B}_\tau^{2,2} + \mathcal{B}_t^{1,1}\right)^{+} (\mathcal{B}_\tau^{1,2})'\,, \\ \mathcal{B}_{\tau+t}^{1,2} &= -\mathcal{B}_t^{1,2}\left(\mathcal{B}_\tau^{2,2} + \mathcal{B}_t^{1,1}\right)^{+} \mathcal{B}_\tau^{1,2}\,, \\ \mathcal{B}_{\tau+t}^{2,2} &= \mathcal{B}_t^{2,2} - (\mathcal{B}_t^{1,2})'\left(\mathcal{B}_\tau^{2,2} + \mathcal{B}_t^{1,1}\right)^{+} \mathcal{B}_t^{1,2}\,. \end{aligned} \tag{55}$$

Remark 13: The superscript $^+$ used in (55) denotes the Moore-Penrose operator inverse. The existence of this inverse is guaranteed by the fact that the operators $\mathcal{P}(t)$, $\mathcal{Q}(t)$, $\mathcal{R}(t)$ exist and are bounded for all $t \in [0, \tau^*]$. This implies that the value functional $W^z(t,\cdot)$ is finite for t in the same interval, so that propagation via $\mathcal{S}_t$ of (41) must be possible up to time τ^*. This in turn implies that the dual space iterations must also be well-defined.

Remark 14: By inspection of (46), (47), it is important to note that the fundamental solution semigroup $\{\mathcal{B}_t\}$ is entirely determined by the unique mild solution $\mathcal{P}$ of the operator differential Riccati equation (6) that satisfies $\mathcal{P}(0) = \mathcal{M}$ as per Assumption 8. *Consequently, if this solution $\mathcal{P}$ (or a piece of it) is known, then all solutions $\widetilde{\mathcal{P}}$ of (6) with initial data in $\Sigma_{\mathcal{M}}(\mathscr{X})$ can be computed.* A recipe that summarizes such computations then follows immediately, see [4].

C. Special cases [3], [4]

The max-plus dual space fundamental solution described above has been considered previously in two special cases, see [3], [4]. The former case assumes a space of vector-valued functions $\mathscr{X} \doteq \mathscr{L}_2(\Lambda; \mathbb{R}^n)$, where Λ is a compact interval in $\mathbb{R}$. The operator $\mathcal{A} \doteq A - \partial$ is employed, with $A \in \mathcal{L}(\mathscr{X})$ restricted to be finite dimensional, and ∂ selected as the (unbounded) differentiation operator. The latter case assumes a space of scalar-valued functions $\mathscr{X} \doteq \mathscr{L}_2(\Lambda; \mathbb{R})$ using an exponentially weighted norm. The operator $\mathcal{A} = \partial^2 - \alpha\,\partial + A$ is employed, where $\alpha, A \in \mathbb{R}$ are fixed scalars, and ∂^2 denotes the second order differentiation operator. In both cases [3], [4], a specific integral operator representation is assumed for the solution $\mathcal{P}$ of the operator differential Riccati equation corresponding to (6). For example, in [3],

$$\mathcal{P}(t)\, x = \int_\Lambda P_t(\cdot,\zeta)\, x(\zeta)\, d\zeta, \quad \Lambda \doteq (0,\ell),\ \ell \in \mathbb{R}_{>0}, \tag{56}$$

for all $t \in [0, \tau^*]$, $x \in \mathscr{X}$, with $P_t \in \mathscr{L}_2(\Lambda^2; \mathbb{R})$ denoting an absolutely continuous kernel satisfying $P_t(\ell,\cdot) = 0 = P_t(\ell,\cdot)$ and $\partial P_t \in \mathscr{L}_2(\Lambda^2; \mathbb{R})$, in which ∂ denotes the partial derivative with respect to either variable defined on Λ. This assumption allows the operator differential Riccati equation (6) to be considered formally, with its solutions defined via the kernel P_t. Consequently, the fundamental solution developed there is restricted to the corresponding integro-differential equation for the kernel P_t induced by (6) and (56). For further details, see [3], [4].

V. Conclusion

Using the semigroup property that attends the dynamic programming principle, a max-plus dual space fundamental solution semigroup is developed for a general form of operator differential Riccati equation. The development generalizes previous work [3], [4] by the authors by not assuming any specific representation for the operator solution to this type of Riccati equation.

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