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A Nonlinear Observer via System Variables Dependent Sylvester Equation Approach with Applications

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Abstract— **In this note, we propose a new approach for nonlinear observer design based on Sylvester equations that depend on states and inputs. Using a state-dependent linear representation and generalizing the Luenberger observer formulation, a class of nonlinear observer with analytic expression is proposed. The proposed design method does not rely on numerical computation unlike existing methods such as the state-dependent Riccati equation approach, or extended Kalman filter. A numerical simulation for 2-dimensional pendulum control is illustrated to verify the effectiveness of the proposed observer.**

I. Introduction

Estimation of internal states in dynamical systems via (fewer) measurable output is a fundamental problem in science and technology. It is closely related with filtering as measurements are often corrupted with noise and the reconstruction of the states is in a stochastic sense. Modern control theory such as LQR often assumes that the states of the system are available for measurement and geometric nonlinear control theory [1], [2] is mostly based on the same assumption as well. In practice, however, it is not the case that system states are always available for measurement and therefore, observer/estimator plays an significant role in engineering. An observer is a dynamical system that takes measurements as input and gives estimated states as output. Luenberger[3] was the first to develop the estimation theory for linear systems and his observer is widely used not only in engineering but also in economics, biology and chemistry. Nonlinear extension of Luenberger theory has been a major challenge in nonlinear system theory and most studies focus on limited classes of nonlinear systems for assuring asymptotic convergence of error dynamics. One of the promising approaches for nonlinear observer is so-called state-dependent Riccati equation (SDRE) observer[4], in which the solution of Riccati equation constructed by linearization at each operating point is computed for observer gain. Although it can be applied for relatively large class of systems which takes state-dependent linear representation (SDLR). The implementation of SDRE observer requires to solve Riccati equation on-line and, in general, high performance computers are necessary, which can be a drawback for real applications.

As a matter of fact, Luenberger observer theory does not consider error system for estimation, but, it relies on an invariance of linear dynamical systems, which is described by a Sylvester equation. In [5], the authors proposed a new nonlinear observer using an SDLR form and solving a Sylvester equation that depends on system states. It is called a state-dependent Sylvester equation (SDSE) observer and its effectiveness and advantages are shown in [5]. The advantages include the property that observer gain is obtained with elementary algebraic computations without expensive on-line computation thanks to a formula by Wu *et. al.* [6] for solving Sylvester equations. However, the SDSE observer in [5] is limited to the class of systems without input terms.

In the present note, we propose a modification of SDSE observers to incorporate input terms so that SDSE observers can be used in a closed loop control. This modification requires the Sylvester equation to depend on input variable as well and we call this approach system variable dependent Sylvester equation (VDSE) approach. A numerical example of pendulum stabilization problem exhibits the effectiveness of the proposed approach.

II. Preliminaries

Consider a continuous-time nonlinear system of the following form

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^l$, $y \in \mathbb{R}^m$. x is the state vector, u is the input vector and y is the measurement output vector. $f(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$ and $h(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ are sufficiently smooth functions. Let the origin $x = 0$ be the equilibrium point of (1); $f(0,0) = 0,\ h(0) = 0$.

Assumption 1: It is possible to express the system (1) in the following form

$$\begin{cases} \dot{x} = F(x,u)x + v(u) \\ y = H(x)x, \end{cases} \tag{2}$$

where $F(x,u) \in \mathbb{R}^{n\times n},\ H(x) \in \mathbb{R}^{m\times n},\ v(\cdot) : \mathbb{R}^l \to \mathbb{R}^n$. Moreover, $F(x,u)$ and $H(x)$ are locally Lipschitz continuous for all $x \in \Omega$, a domain of $\mathbb{R}^n$ containing the origin.

Remark 1: The above representation (2) is called the state-dependent linear representation (SDLR) and is used in the previous observer design theory [4], [7], [5]. A slight modification is made in (2) in the sense that arbitrary input can be handled. More specifically, it is common to consider the following SDLR form in [4], [7]

$$\begin{cases} \dot{x} = F(x)x + Gu \\ y = H(x)x, \end{cases} \tag{3}$$

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where $F(x) \in \mathbb{R}^{n\times n}$. The difference of (2) and (3) makes the proposed method more beneficial in the convergence proof of error sysmtem as well as the performance of the observer.

Remark 2: Matrices $F(x,u)$ and $H(x)$ are not unique when $n \geq 2$. However, the following relationship holds [7]

$$F(0,0) = \frac{\partial f}{\partial x}(0,0),\ H(0) = \frac{\partial h}{\partial x}(0).$$

Therefore, $F(0,0)$ and $H(0)$ are the linearized matrices of $f(x,u)$ and $h(x)$ about the zero equilibrium, respectively.

III. Nonlinear observer design via VDSE approach

In this section, we introduce a new nonlinear observer design method by system variables dependent Sylvester equation (VDSE) approach. Originally, Luenberger observer design is based on the solution of Sylvester equation, which defines the invariance relation of observed system and observer [3]. A recent result by [6], which gives analytic solutions for Sylvester equations (see Theorem 1 in Appendix), and SDLR representations led us to a new nonlinear observer [5]. This observer was for systems without input and the present paper extends the result in [5] to systems with input. The approach produces observer gains analytically and is computationally advantageous in real implementation.

We first introduce an VDSE below for the system (2)

$$X(x,u)F(x,u) = A(x)X(x,u) + B(x)H(x). \tag{4}$$

$A(x) \in \mathbb{R}^{n\times n}$ and $B(x) \in \mathbb{R}^{n\times m}$ are free parameters for designing observer, but, they have to satisfy the following;

i) $A(x)$ has eigenvalues with negative real parts for each x,

ii) the pair $(A(x), B(x))$ is controllable for each x.

From the formula in Theorem 1, solution $X(x,u)$ for (4) is obtained by elementary algebraic calculations, which continuously depend on the states and inputs.

With the solution of VDSE (4), a VDSE observer is constructed as follows

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x},u) + L(\hat{x},u)(y - h(\hat{x})), \\ L(\hat{x},u) &= X(\hat{x},u)^{-1}B(\hat{x}). \end{aligned} \tag{5}$$

It is possible to prove that the error system between system (1) and observer (5) is locally asymptotically stable in Ω under Assumption 1. The detail is not presented here for the sake of space.

Remark 3: It is possible to handle more general functions $H(x,u)$, $A(x,u)$ and $B(x,u)$ in (2) instead of $H(x)$ $A(x)$ and $B(x)$ respectively. This may enhance the applicability of the method and leads a better observer design. The analytical method to obtain the solution of VDSE (Theorem 1), which is the key element in the proposed observer, brings some advantages. First, there is no need to use any kind of approximation or numerical approach [7] to design the observer. Secondly, not only the simplicity in constructing observer but also better performance in a larger region can be expected since the nolinearity is fully taken into account (see the comparison with a regular observer with linear gain in §IV). Thirdly, the VDLR representation (2) is possible for a large class of nonlinear systems. In [5], a state estimator is obtained for the Lorenz system and experimental verification is shown for the system realized by an electric circuit.

IV. An example for pendulum stabilization

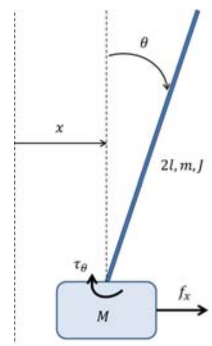

Fig. 1. Inverted pendulum.

In this section we demonstrate a VDSE observer based control for two dimensional pendulum system. The system considered here is derived from equations of motion for an inverted pendulum (Fig. 1).

Let l[m], m[kg] and J[m^2·kg] be pendulum's half length, mass and inertia around the rotational point, respectively. M[kg] is the mass of the cart on which the pendulum is attached and we assign g[m/s^2] gravitational acceleration. Then, equations of motion for the cart position x[m] and pendulum's rotational position θ[rad] are derived as follows

$$\begin{aligned} \ddot{x} =& \frac{Jml\sin\theta\dot{\theta}^2 - m^2l^2\cos\theta\sin\theta}{(M+m)J - m^2l^2\cos^2\theta} \\ &+ \frac{Jf_x - ml\cos\theta\tau_\theta}{(M+m)J - m^2l^2\cos^2\theta}, \\ \ddot{\theta} =& \frac{-m^2l^2\cos\theta\sin\theta\dot{\theta}^2 + (M+m)mgl\sin\theta}{(M+m)J - m^2l^2\cos^2\theta} \\ &+ \frac{-ml\cos\theta f_x + (M+m)\tau_\theta}{(M+m)J - m^2l^2\cos^2\theta}, \end{aligned} \tag{6}$$

where f_x[N] and τ_θ[N·m] are external force and torque. Let us assume $\tau_\theta = 0$.

We only focus on the pendulum dynamics, which means that the the cart position and velocity are not controlled. This derives a two dimensional model, which is suitable to show the procedure of the proposed observer design. We define state vector as $x = [\theta, \dot{\theta}]^T$ and input as $u = f_x$. Then, the following state equation can be derived from equation (6).

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ f_2(x) \end{bmatrix} + \begin{bmatrix} 0 \\ g_2(x) \end{bmatrix} u, \quad y = x_1 \tag{7}$$

where,

$$f_2(x) = f_{2,1}(x) + f_{2,2}(x)$$
$$f_{2,1}(x) = \frac{(M+m)mgl\sin x_1}{(M+m)J - m^2l^2\cos^2 x_1}$$
$$f_{2,2}(x) = \frac{-m^2l^2\cos x_1 \sin x_1 x_2^2}{(M+m)J - m^2l^2\cos^2 x_1}$$
$$g_2(x) = \frac{-ml\cos x_1}{(M+m)J - m^2l^2\cos^2 x_1}.$$

We first compute a stabilizing state feedback control law $u = Kx$ by the LQR method for the linearized system at the origin. The gain K is obtained for a cost function

$$J = \int_0^\infty x^T Qx + ru^2\, dt,$$

with $Q = I_2$, $r = 1$. This feedback law stabilizes the origin even for the pending position $x(0) = [\pi, 0]^T$.

Next, we design two observers; one is a VDSE observer and the other is a standard nonlinear observer with linear observer gain. We will see how the LQR control works when it is combined with two different observers. For the VDSE observer design, one transforms the system (7) into the form of (2). The one way of doing this is the following

$$\begin{cases} \dot{x} = F(x,u)x + \begin{bmatrix} 0 \\ g_2(0) \end{bmatrix} u \\ y = H(x)x, \end{cases}$$

where

$$F(x,u) = \begin{bmatrix} 0 & 1 \\ F_{21}(x,u) & \frac{f_{2,2}(x)}{x_2} \end{bmatrix},\ H(x) = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$F_{21}(x,u) = \begin{cases} \frac{f_{2,1}(x)}{x_1} + \frac{g_2(x) - g_2(0)}{x_1} u & (x_1 \neq 0) \\ \frac{(M+m)mgl}{(M+m)J - m^2l^2} & (x_1 = 0). \end{cases}$$

Notice that $F(x,u)$ is continuous function of x and u. The designing parameter $A(x)$, $B(x)$ in (4) satisfying the condition i), ii) can be taken, for example,

$$A(x) = \begin{bmatrix} 0 & 1 \\ -200 & -30 \end{bmatrix},\ B(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (8)$$

Observer gain is calculated via the solution of VDSE (4), which is solved by using the solution theory in [6] (Theorem 1 in Appendix)

$$L(x,u) = X(x,u)^{-1}B(x)$$
$$= \begin{bmatrix} 30 + \frac{f_{2,2}(x)}{x_2} \\ 200 + 30\frac{f_{2,2}(x)}{x_2} + F_{2,1}(x,u) + \frac{f_{2,2}(x)^2}{x_2^2} \end{bmatrix}.$$

A VDSE observer is set in the following form with observer state $\hat{x}_V$

$$\dot{\hat{x}}_V = f(\hat{x}_V, u) + L(\hat{x}_V, u)(y - h(\hat{x}_V)). \quad (9)$$

Next, we design a nonlinear observer with linear observer gain in the standard manner. Let the linearized matrices at the origin $F_0 = \frac{\partial f}{\partial x}(0,0), H_0 = \frac{\partial h}{\partial x}(0)$. We obtain the linear observer gain $L_0 \in \mathbb{R}^{2\times 1}$ such that the eigenvalues of $F_0 - L_0H_0$ coincide with those of $A(x)$ in (8), which corresponds to the linear term of the error dynamics of system (7) and observer (9). We notice that since $F_0 = F(0,0), H_0 = H(0)$, the relation $L_0 = L(0,0)$ holds. The nonlinear observer is then,

$$\dot{\hat{x}}_{LG} = f(\hat{x}_{LG}, u) + L_0(y - h(\hat{x}_{LG})), \quad (10)$$

where $\hat{x}_{LG}$ is the observer states.

Numerical simulations were run for two cases to compare the performances of the estimators with the same feedback law. The initial states for the system is $x(0) = [\pi, 0]^T$, pending position, and the initial states for the estimators are $\hat{x}_V(0) = \hat{x}_{LG}(0) = [0,0]^T$. Figs.2 $\sim$ 7 show that stabilization with the linear gain observer (10) fails while it is successful with the proposed observer. The reason of this difference is that the nonlinear observer with linear gain cannot capture the total nonlinearity of the system and the pendulum states are not well-estimated with this observer (Figs. 2, 3). On the other hand, the LQR control with the proposed observer performs well with the same initial condition.

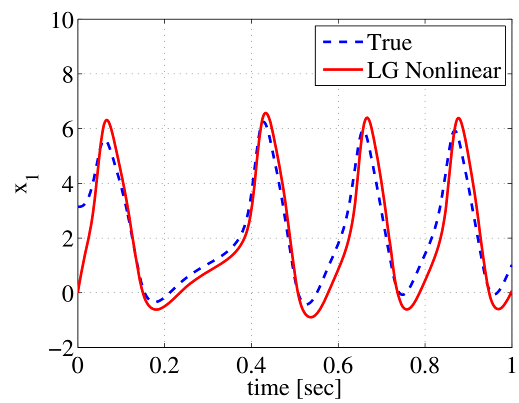

Fig. 2. State x_1 of LQR with the linear gain observer. ($u = K\hat{x}_{LG}$)

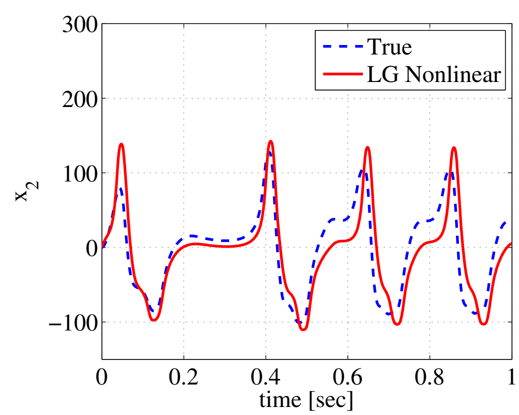

Fig. 3. State x_2 of LQR with the linear gain observer. ($u = K\hat{x}_{LG}$)

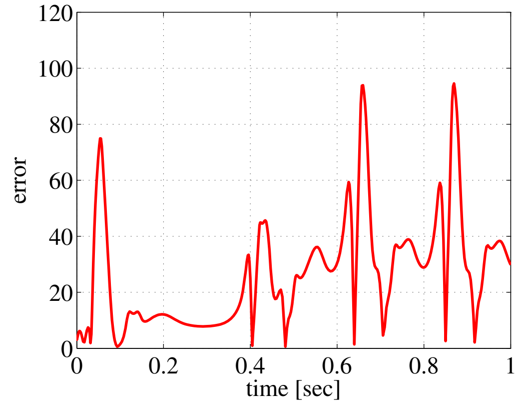

Fig. 4. Estimation error of the linear gain observer. ($u = K\hat{x}_{LG}$)

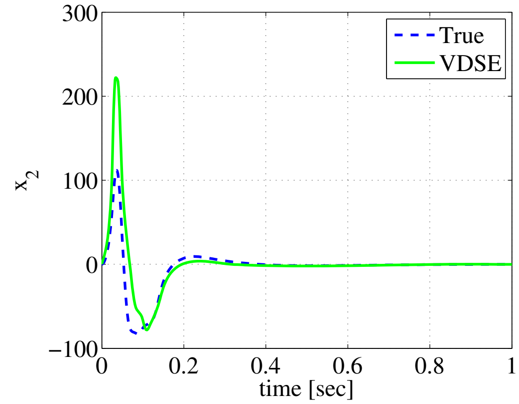

Fig. 6. State x_2 of LQR and the VDSE observer. ($u = K\hat{x}_V$)

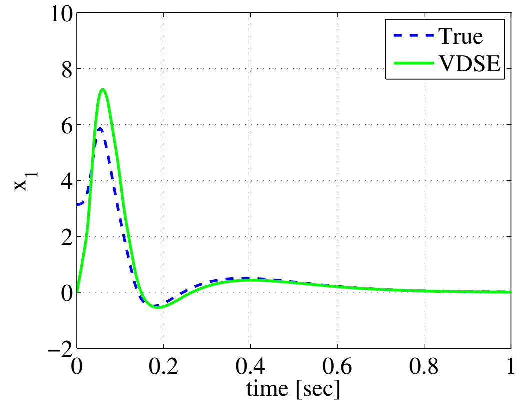

Fig. 5. State x_1 of LQR with the VDSE observer. ($u = K\hat{x}_V$)

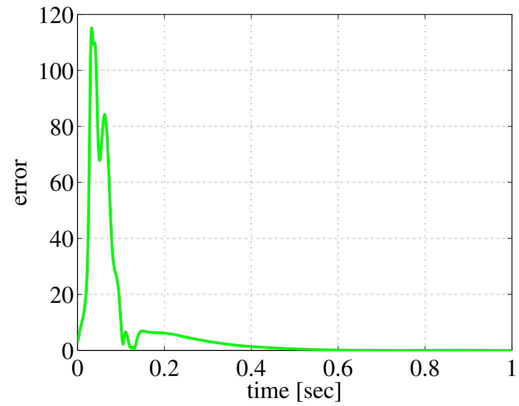

Fig. 7. Estimation error of the VDSE observer. ($u = K\hat{x}_V$)

V. Conclusion

In this note, we proposed a new approach for a nonlinear observer based on Sylvester equation as an extension of the result in [5]. Utilizing the state dependent linear representation and analytic solution for Sylvester equations that depend on system states and inputs, we obtained a class of nonlinear observer that requires less computational power and easier to implement in real applications. The effectiveness of the observer has been verified by simulations for a 2-dimensional pendulum swing up and stabilization with an LQR controller.

Appendix

Theorem 1: [6] When $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $F \in \mathbb{R}^{p\times p}$ are given, Sylvester equation

$$XF = AX + BH$$

has the following matrices for its solution.

$$X = \sum_{k=0}^{n-1} R_k BZF^k, \quad H = \sum_{k=0}^{n} q_k ZF^k = Zq(F),$$

where

$$R_k = \begin{cases} I & k = n-1 \\ AR_{k+1} + q_{k+1} I & k = n-2, \cdots, 1, 0 \end{cases}$$

$$q_k = \begin{cases} 1 & k = n \\ -\frac{1}{k}\mathrm{tr}(AR_k) & k = n-1, \cdots, 1, 0 \end{cases}$$

and $Z \in \mathbb{R}^{r\times p}$ is a free parameter.

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