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Online statistical change detection for a multivariable system in closed-loop using recursive estimation of a high-order model*

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Abstract— **In this paper, we will develop an online statistical change detection method for a multivariable system in closedloop. The proposed method is based on online hypotheses testing by observing a test signal generated by a recursive least squares algorithm for estimation of a bunch of selected matrixvalued Markov parameters of a high-order ARX model. A test signal generated sequentially by the recursive algorithm is studied and its asymptotic whiteness is proved explicitly. The proposed online change detection method is based on a likelihood ratio test to examine the change in the covariance of the aforementioned test signal.**

I. INTRODUCTION

There has been a great demand for monitoring a multivariable system in order to detect and to give early warning of unwelcome changes (or faults) in the system [1]. The literature [2], [3] is the excellent summary of conventional statistical change detection methods. The book [4] is a good introductory textbook for this area. For more than a decade, subspace-based statistical change detection methods have been studied by several research groups, e.g., [5], [6], [7], [8], [9], [10], [11] and the references therein. It is a strong point that subspace-based change detection methods are applicable to multivariable systems. Recursive algorithms based on subspace model identification work better as whitening filters than conventional least squares and least mean squares algorithms do [6], [7], [8]. One of the reasons may be that, essentially, subspace model identification implements estimation under a high-order model structure followed by model reduction [12], [13]. The former corresponds to projection onto the space spanned by rows of input sequences, while the latter does to singular value decomposition [14], [15], [16].

In this paper, we will develop an online statistical change detection method which is useful for multivariable systems in closed-loop. The contributions of the paper are as follows. First, we derive a matrix-inversion-lemma-based recursive algorithm for updating matrix-valued parameters, and clarify the mechanism of sequentially generating a test signal used for statistical change detection from the proposed recursivef algorithm. Our recursive update algorithm is derived from the least squares estimate of a bunch of the selected matrixvalued Markov parameters of a predictor form, which is related to predictor-based subspace identification [17]. Second, we show the asymptotic whiteness of the test signal explicitly. Third, using the test signal, we develop an online statistical change detection method based on a likelihood ratio test and a GMA-based recursive algorithm [3].

We must emphasize that the proposed recursive algorithm is derived from the recursification of the solution to a matrixvalued least-squares problem with the Frobenius norm introduced for measuring the distance between two matrices [18], and it is different from the recursive algorithm presented in [19]. We must mention that the preceding studies [9], [10], [11] have considered subspace based fault detection for systems in closed-loop. We can say that, essentially, their test statistics are also derived from estimation of a high-order model whose order is related to the length of the past horizen. However, they are generated by batch-processing using data from a sliding window of a fixed interval, which is different from our approach. Their hypotheses testing is to observe a change in the mean value of their test signal, while ours is to observe a change in the variance of our test signal.

The paper is organized as follows. The problem is formulated in section II, followed by notations in section III. A recursive algorithm of closed-loop subspace model identification is derived in section IV, followed by analysis of asymptotic whiteness of a sequence generated sequentially by the proposed recursive algorithm. An online statistical change detection method for systems in closed-loop is developed in section V. The aforementioned sequence is adopted as the test signal for the online change detection. The proposed method is based on a likelihood ratio test to examine the change in the covariance of the test signal. A numerical simulation is presented in section VI to demonstrate the effectiveness of the proposed method.

II. PROBLEM FORMULATION

Let us suppose that a system under surveillance can be described by an innovation-type model with an abrupt change in the coefficients at an unknown time instant k_c as follows:

$$x_{k+1} = (A + A_\Delta)x_k + (B + B_\Delta)u_k + Ke_k, \tag{1a}$$

$$y_k = (C + C_\Delta)x_k + Du_k + e_k, \tag{1b}$$

$$\begin{cases} (A_\Delta, B_\Delta, C_\Delta) \equiv (0,0,0), & k < k_c, \\ (A_\Delta, B_\Delta, C_\Delta) \not\equiv (0,0,0), & k > k_c. \end{cases}$$

where $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^l$, $u_k \in \mathbb{R}^m$. A, B, C, D, K are unknown real matrices with appropriate dimensions. Henceforth, for the sake of simplicity, $D = 0$ is assumed to ensure the well-posedness of the feedback loop [15], [20]. Assume that $\bar{A} := A - KC$ is stable, i.e., the spectral radius of $\bar{A}$, denoted

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by $\lambda_{\max}(\bar{A})$, satisfies $\lambda_{\max}(\bar{A}) < 1$. Assume that $(\bar{A}, C)$ is observable, and that $(\bar{A}, [B\ K])$ is reachable. Note that A_Δ, B_Δ and C_Δ correspond to a change in the dynamics, an actuator fault and a sensor fault, respectively. Define $\bar{A}_\Delta := A_\Delta - KC_\Delta$. The innovation $e_k \in \mathbb{R}^l$ is a zero-mean independently identically distributed Gaussian noise with its covariance matrix equal to $\sigma_0^2 I$, that is, $\mathbf{E}[e_k] = 0$ and $\mathbf{E}[e_s e_t^T] = \delta_{st}\sigma_0^2 I$, where $\mathbf{E}$ denotes expectation and δ_{st} Kronecker's delta. The coefficient matrices are assumed to be real matrices with appropriate dimensions. Feedback from the output y_k to the input u_k is allowed. The signals are assumed to be bounded, pseudo-stationary [21] and satisfies a relevant persistence of excitation condition defined later.

The goal is to develop an online change detection method based on whiteness testing to detect, as soon as possible, the abrupt change in the system which occurs at unknown time instant k_c.

III. Notations

For a finite integer N, define the matrix, denoted by $U_{t,N}$, whose columns consist of a finite sequence of u_t as follows:

$$U_{t,N} = \begin{bmatrix} u_t & u_{t+1} & \cdots & u_{t+N-1} \end{bmatrix}. \tag{2}$$

The matrices $X_{t,N}$, $Y_{t,N}$ and $E_{t,N}$ are defined in the same manner as $U_{t,N}$ in (2).

For $s > n$ with relevant conditions [22], define the following block Hankel matrix, $\mathscr{U}_{t,N}$, whose block rows consist of a bunch of s consecutive matrices $U_{k,N}$ $(t \le k \le t+s-1)$ as follows,

$$\mathscr{U}_{t,N} = \begin{bmatrix} u_t & u_{t+1} & \cdots & u_{t+N-1} \\ u_{t+1} & u_{t+2} & \cdots & u_{t+N} \\ \vdots & \vdots & & \vdots \\ u_{t+s-2} & u_{t+s-1} & \cdots & u_{t+N+s-3} \\ u_{t+s-1} & u_{t+s} & \cdots & u_{t+N+s-2} \end{bmatrix} = \begin{bmatrix} U_{t,N} \\ U_{t+1,N} \\ \vdots \\ U_{t+s-2,N} \\ U_{t+s-1,N} \end{bmatrix}$$
$$= \begin{bmatrix} u_{s(t)} & u_{s(t+1)} & \cdots & u_{s(t+N-1)} \end{bmatrix},$$

where the size of $\mathscr{U}_{t,N}$ is of $ms \times N$, and $u_{s(t+k)}$ denotes the vector whose block entries consist of a bunch of s consecutive data samples from u_{t+k} to $u_{t+k+s-1}$. Similarly, define the matrix $\mathscr{U}_{t,N}^-$ with the size of $m(s-1) \times N$ as follows:

$$\mathscr{U}_{t,N}^- = \begin{bmatrix} u_t & u_{t+1} & \cdots & u_{t+N-1} \\ u_{t+1} & u_{t+2} & \cdots & u_{t+N} \\ \vdots & \vdots & & \vdots \\ u_{t+s-2} & u_{t+s-1} & \cdots & u_{t+N+s-3} \end{bmatrix} = \begin{bmatrix} U_{t,N} \\ U_{t+1,N} \\ \vdots \\ U_{t+s-2,N} \end{bmatrix}$$
$$= \begin{bmatrix} u_{s-1(t)} & u_{s-1(t+1)} & \cdots & u_{s-1(t+N-1)} \end{bmatrix}.$$

The block Hankel matrices $\mathscr{Y}_{t,N}$, $\mathscr{E}_{t,N}$, $\mathscr{Y}_{t,N}^-$ and $\mathscr{E}_{t,N}^-$ are defined similarly to $\mathscr{U}_{t,N}$ and $\mathscr{U}_{t,N}^-$, respectively.

The joint input-output data matrices, denoted by $\mathscr{Z}_{t,N}$ and $\mathscr{Z}_{t,N}^-$, are respectively defined as follows:

$$\mathscr{Z}_{t,N} = \begin{bmatrix} \mathscr{U}_{t,N} \\ \hdashline \mathscr{Y}_{t,N} \end{bmatrix} = \begin{bmatrix} u_{s(t)} & u_{s(t+1)} & \cdots & u_{s(t+N-1)} \\ \hdashline y_{s(t)} & y_{s(t+1)} & \cdots & y_{s(t+N-1)} \end{bmatrix}$$
$$= \begin{bmatrix} z_{s(t)} & z_{s(t+1)} & \cdots & z_{s(t+N-1)} \end{bmatrix},$$
$$\mathscr{Z}_{t,N}^- = \begin{bmatrix} \mathscr{U}_{t,N}^- \\ \hdashline \mathscr{Y}_{t,N}^- \end{bmatrix} = \begin{bmatrix} u_{s-1(t)} & u_{s-1(t+1)} & \cdots & u_{s-1(t+N-1)} \\ \hdashline y_{s-1(t)} & y_{s-1(t+1)} & \cdots & y_{s-1(t+N-1)} \end{bmatrix}$$
$$= \begin{bmatrix} z_{s-1(t)} & z_{s-1(t+1)} & \cdots & z_{s-1(t+N-1)} \end{bmatrix},$$

where joint input-output vectors are defined $z_{s(k)} = [u_{s(k)}^T\ y_{s(k)}^T]^T$, $z_{s-1(k)} = [u_{s-1(k)}^T\ y_{s-1(k)}^T]^T$.

For future reference the following matrices are defined, where $\bar{A} := A - KC$:

$$\begin{aligned}
\bar{\Theta} &= \begin{bmatrix} \bar{\Theta}_B & \bar{\Theta}_K \end{bmatrix}, \\
\bar{\Theta}_B &= \begin{bmatrix} C\bar{A}^{2s-2}B & C\bar{A}^{2s-3}B & \cdots & C\bar{A}^{s-1}B \end{bmatrix}, \\
\bar{\Theta}_K &= \begin{bmatrix} C\bar{A}^{2s-2}K & C\bar{A}^{2s-3}K & \cdots & C\bar{A}^{s-1}K \end{bmatrix}, \\
\bar{\mathscr{H}} &= \begin{bmatrix} \bar{\mathscr{H}}_B & \bar{\mathscr{H}}_K \end{bmatrix}, \\
\bar{\mathscr{H}}_B &= \begin{bmatrix} C\bar{A}^{s-2}B & C\bar{A}^{s-3}B & \cdots & CB \end{bmatrix}, \\
\bar{\mathscr{H}}_K &= \begin{bmatrix} C\bar{A}^{s-2}K & C\bar{A}^{s-3}K & \cdots & CK \end{bmatrix},
\end{aligned}$$

IV. A recursive least square algorithm

Throughout this section, consider the case of no change, $(A_\Delta, B_\Delta, C_\Delta) \equiv (0,0,0)$, that is,

$$x_{k+1} = Ax_k + Bu_k + Ke_k, \tag{3a}$$
$$y_k = Cx_k + e_k. \tag{3b}$$

Again, we emphasize that feedback from y_k to u_k is allowed.

A. Least squares estimation

Substitute $e_k = y_k - Cx_k$ for e_k in (3a), and we have the predictor form as follows [23]:

$$x_{k+1} = \bar{A}x_k + \begin{bmatrix} B & K \end{bmatrix} \begin{bmatrix} u_k \\ y_k \end{bmatrix}, \qquad y_k = Cx_k + e_k. \tag{4}$$

Using the forward shift operator q, e.g., $qu_k = u_{k+1}$, we have the following input-output relation:

$$\begin{aligned}
y_k &= C(qI - \bar{A})^{-1}Bu_k + C(qI - \bar{A})^{-1}Ky_k + e_k \\
&= \sum_{i=0}^{\infty} C\bar{A}^i Bq^{-(i+1)}u_k + \sum_{i=0}^{\infty} C\bar{A}^i Kq^{-(i+1)}y_k + e_k
\end{aligned} \tag{5}$$

Truncation of the sums on the right-hand side of (5) up to $2s-2$ gives a high-order ARX model as follows:

$$\begin{aligned}
y_k &= \sum_{i=0}^{2s-2} C\bar{A}^i Bq^{-(i+1)}u_k + \sum_{i=0}^{2s-2} C\bar{A}^i Kq^{-(i+1)}y_k + e_k + \Delta^s \\
&\approx \sum_{i=0}^{2s-2} C\bar{A}^i Bq^{-(i+1)}u_k + \sum_{i=0}^{2s-2} C\bar{A}^i Kq^{-(i+1)}y_k + e_k
\end{aligned} \tag{6}$$

where Δ^s denotes the truncation error, which satisfies, for $\exists M > 0, |\Delta^s| \le M(\lambda_{\max}(\bar{A}))^s \to 0$ as $s \to \infty$. Using the notations defined in Section III, (6) can be rewritten as

$$y_{k+s-1} = \bar{\Theta} z_{s(k-s)} + \bar{\mathscr{H}} z_{s-1(k)} + e_k. \tag{7}$$

For a finite number of samples of $\{(u_k, y_k)\}$ with k from 1 to $N+2s-1$, (7) brings the following relation:

$$Y_{2s,N} = \begin{bmatrix} \bar{\Theta} & \bar{\mathscr{H}} \end{bmatrix} \begin{bmatrix} \mathscr{Z}_{1,N} \\ \mathscr{Z}^-_{s+1,N} \end{bmatrix} + E_{2s,N}. \tag{8}$$

Assumption 1: Given a positive integer N_0. The closed-loop signals are persistently exciting in the sense that the following relations hold for $\forall N > N_0$ including infinity:

$$\frac{1}{N}\mathscr{Z}^-_{s+1,N}\mathscr{Z}^-_{s+1,N}{}^T > 0,$$
$$\frac{1}{N}\begin{bmatrix} \mathscr{Z}_{1,N} \\ \mathscr{Z}^-_{s+1,N} \end{bmatrix}\begin{bmatrix} \mathscr{Z}^T_{1,N} & \mathscr{Z}^-_{s+1,N}{}^T \end{bmatrix} > 0.$$

Consider the following cost function w.r.t. $\bar{\Theta}$ and $\bar{\mathscr{H}}$:

$$J(\bar{\Theta}, \bar{\mathscr{H}}) = \left\| Y_{2s,N} - \begin{bmatrix} \bar{\Theta} & \bar{\mathscr{H}} \end{bmatrix} \begin{bmatrix} \mathscr{Z}_{1,N} \\ \mathscr{Z}^-_{s+1,N} \end{bmatrix} \right\|_F^2, \tag{9}$$

where the Frobenius norm of a matrix X is defined as $\|X\|_F = \sqrt{\mathrm{trace}(X^T X)}$ [18]. Under the assumption 1, similarly to the open-loop case [24], the minimizer of the cost function $J(\bar{\Theta}, \bar{\mathscr{H}})$, denoted by $(\hat{\bar{\Theta}}_N, \hat{\bar{\mathscr{H}}}_N)$, can be given as follows:

$$\hat{\bar{\Theta}}_N = \frac{1}{N} Y_{2s,N}\Pi^{\perp}_{\mathscr{Z}^-_{s+1,N}}\mathscr{Z}^T_{1,N}\left(\frac{1}{N}\mathscr{Z}_{1,N}\Pi^{\perp}_{\mathscr{Z}^-_{s+1,N}}\mathscr{Z}^T_{1,N}\right)^{-1}, \tag{10}$$
$$\hat{\bar{\mathscr{H}}}_N = \frac{1}{N}\left(Y_{2s,N} - \hat{\bar{\Theta}}_N \mathscr{Z}_{1,N}\right)\mathscr{Z}^-_{s+1,N}{}^T\left(\frac{1}{N}\mathscr{Z}^-_{s+1,N}\mathscr{Z}^-_{s+1,N}{}^T\right)^{-1},$$

where $\Pi^{\perp}_{\mathscr{Z}^-_{s+1,N}}$ denotes a projection matrix defined as

$$\Pi^{\perp}_{\mathscr{Z}^-_{s+1,N}} = I - \frac{1}{N}\mathscr{Z}^-_{s+1,N}{}^T\left(\frac{1}{N}\mathscr{Z}^-_{s+1,N}\mathscr{Z}^-_{s+1,N}{}^T\right)^{-1}\mathscr{Z}^-_{s+1,N}. \tag{11}$$

Note that the inverse matrices as above exist due to assumption 1.

Remark 1: Using the QR factorization of

$$\begin{bmatrix} \mathscr{Z}^-_{s+1,N} \\ \mathscr{Z}_{1,N} \\ Y_{2s,N} \end{bmatrix} = \begin{bmatrix} L_{11} & & \\ L_{21} & L_{22} & \\ L_{31} & L_{32} & L_{33} \end{bmatrix}\begin{bmatrix} Q_1^T \\ Q_2^T \\ Q_3^T \end{bmatrix}, \tag{12}$$

(10) can be rewritten as follows [25]:

$$\hat{\Theta}_N = L_{32}L_{22}^{-1}. \tag{13}$$

Remark 2: According to the reference [22], let the oblique projection of $Y_{2s,N}$ along the space generated by the rows of $\mathscr{Z}^-_{s+1,N}$ onto the space generated by the rows of $\mathscr{Z}_{1,N}$ be denoted by $\hat{\mathbf{E}}_{||\mathscr{Z}^-_{s+1,N}}[Y_{2s,N}|\mathscr{Z}_{1,N}]$. Some calculation using the definition of the oblique projection and eqs. (10) and (12) leads to the relation

$$\hat{\bar{\Theta}}_N \mathscr{Z}_{1,N} = \hat{\mathbf{E}}_{||\mathscr{Z}^-_{s+1,N}}[Y_{2s,N}|\mathscr{Z}_{1,N}]. \tag{14}$$

B. Recursive formula of updating $\hat{\bar{\Theta}}_N$

In this subsection, we will derive a recursive update formula of $\hat{\bar{\Theta}}_N$. Taking it into account that a finite number of samples are used for system identification in practice, the following equations are used instead of eqs. (10) and (11):

$$\hat{\bar{\Theta}}_N = Y_{2s,N}\Pi^{\perp}_{\mathscr{Z}^-_{s+1,N}}\mathscr{Z}^T_{1,N}\left(\mathscr{Z}_{1,N}\Pi^{\perp}_{\mathscr{Z}^-_{s+1,N}}\mathscr{Z}^T_{1,N}\right)^{-1}, \tag{15}$$
$$\Pi^{\perp}_{\mathscr{Z}^-_{s+1,N}} = I - \mathscr{Z}^-_{s+1,N}{}^T\left(\mathscr{Z}^-_{s+1,N}\mathscr{Z}^-_{s+1,N}{}^T\right)^{-1}\mathscr{Z}^-_{s+1,N}.$$

Note that the matrices, $Y_{2s,N}$, $\mathscr{Z}^-_{s+1,N}$, $\mathscr{Z}_{1,N}$ are composed of the data sampled during the interval $[1,\ N+2s-1]$. Note also that, using the data sampled during the interval $[1,\ N+2s]$, we can describe the matrices $Y_{2s,N+1}$, $\mathscr{Z}^-_{s+1,N+1}$, $\mathscr{Z}_{1,N+1}$ as follows:

$$Y_{2s,N+1} = \begin{bmatrix} Y_{2s,N} & y_{N+2s} \end{bmatrix},$$
$$\mathscr{Z}_{1,N+1} = \begin{bmatrix} \mathscr{Z}_{1,N} & z_{s(N+1)} \end{bmatrix},$$
$$\mathscr{Z}^-_{s+1,N+1} = \begin{bmatrix} \mathscr{Z}^-_{s+1,N} & z_{s-1(N+s+1)} \end{bmatrix}.$$

Now, a recursive algorithm of closed-loop subspace model identification is given as follows:

Proposition 1: The derivation of the latest estimate, $\hat{\bar{\Theta}}_{N+1}$, from the previous estimate, $\hat{\bar{\Theta}}_N$, and the vectors including the latest data, y_{N+2s}, $z_{s(N+1)}$ and $z_{s-1(N+s+1)}$, is given by

$$\hat{\bar{\Theta}}_{N+1} = \hat{\bar{\Theta}}_N - b_{N+1}\varepsilon_{N+2s|N+2s-1}\xi^T_{N+1}\Psi_N^{-1} \tag{16}$$

where the notations are defined as follows:

$$\Psi_N = \mathscr{Z}_{1,N}\Pi^{\perp}_{\mathscr{Z}^-_{s+1,N}}\mathscr{Z}^T_{1,N},$$
$$\varepsilon_{N+2s|N+2s-1} = y_{N+2s} - Y_{2s,N}\zeta_{N+1} + \hat{\bar{\Theta}}_N\xi_{N+1} \tag{17}$$

$$\xi_{N+1} = \mathscr{Z}_{1,N}\zeta_{N+1} - z_{s(N+1)}$$
$$\zeta_{N+1} = \mathscr{Z}^-_{s+1,N}{}^T\left(\mathscr{Z}^-_{s+1,N}\mathscr{Z}^-_{s+1,N}{}^T\right)^{-1}z_{s-1(N+s+1)},$$
$$a_{N+1} = \left(1 + \zeta^T_{N+1}\zeta_{N+1}\right)^{-1},$$
$$b_{N+1} = \left(a^{-1}_{N+1} + \xi^T_{N+1}\Psi_N^{-1}\xi_{N+1}\right)^{-1}.$$

The recursive updating equations auxiliary to (16) are given as follows:

$$\Psi^{-1}_{N+1} = \Psi^{-1}_N - b_{N+1}\Psi^{-1}_N\xi_{N+1}\xi^T_{N+1}\Psi^{-1}_N,$$

$$\begin{aligned}\left(\mathscr{Z}^-_{s+1,N+1}\mathscr{Z}^-_{s+1,N+1}{}^T\right)^{-1} &= \left(\mathscr{Z}^-_{s+1,N}\mathscr{Z}^-_{s+1,N}{}^T\right)^{-1} \\ &- a_{N+1}\left(\mathscr{Z}^-_{s+1,N}\mathscr{Z}^-_{s+1,N}{}^T\right)^{-1}z_{s-1(N+s+1)} \\ &\cdot z^T_{s-1(N+s+1)}\left(\mathscr{Z}^-_{s+1,N}\mathscr{Z}^-_{s+1,N}{}^T\right)^{-1}\end{aligned}$$

$$Y_{2s,N+1}\mathscr{Z}^-_{t,N+1}{}^T = Y_{2s,N}\mathscr{Z}^-_{s+1,N}{}^T + y_{N+2s}z^T_{s-1(N+s+1)}$$
$$\mathscr{Z}_{1,N+1}\mathscr{Z}^-_{t,N+1}{}^T = \mathscr{Z}_{1,N}\mathscr{Z}^-_{s+1,N}{}^T + z_{s(N+1)}z^T_{s-1(N+s+1)}$$

Proof: This proof is omitted due to limitations of space. The proposed recursive formula is based on the matrix inversion lemma, and its derivation is very similar to one presented in [24]. ■

Remark 3: One may prefer the recursive update of the QR-factorization-based estimate (13) to that of the estimate based on the matrix inversion lemma as above. In such a case, similarly to the literature, e.g., [26], [27], [28], pivoting using Givens rotations [18] are helpful for derivation of a relevant recursive updating formula.

Remark 4: Similarly to the conventional recursive least squares algorithm, an exponential forgetting factor can easily be installed in the recursive formula in Proposition 1 by $Y_{2s,N+1} = \begin{bmatrix} \nu Y_{2s,N} & y_{N+2s} \end{bmatrix}$, $\mathscr{Z}_{1,N+1} = \begin{bmatrix} \nu \mathscr{Z}_{1,N} & z_{s(N+1)} \end{bmatrix}$ and $\mathscr{Z}^-_{s+1,N+1} = \begin{bmatrix} \nu \mathscr{Z}^-_{s+1,N} & z_{s-1(N+s+1)} \end{bmatrix}$ with $0 < \nu < 1$.

C. On asymptotic whiteness of a sequence

When the system (3) to be identified stays at a stationary point and N is sufficiently large, the recursion formula (16) implies that the second term on the right hand side of (16) is a perturbation which is driven by $\varepsilon_{N+2s|N+2s-1}$. Here, we must emphasize that the vector quantity $\varepsilon_{N+2s|N+2s-1}$ can be computed on-line from the sampled data at every sampling period by the recursive algorithm composed of (16), (17) and their auxiliary update equations. In this subsection, asymptotic whiteness of the sequence $\{\varepsilon_{N+2s|N+2s-1}\}_{N>N_0}$ derived from the recursive algorithm is investigated.

Assumption 2: An integer s is sufficiently large but finite.

Assumption 3 (Inverse boundness): For $\forall N > N_0$ including infinity there exists a positive number $\beta < \infty$ such that

$$\left\| \left(\frac{1}{N} \mathscr{Z}^-_{s+1,N} \mathscr{Z}^{-\,T}_{s+1,N} \right)^{-1} \right\|_2 \leq \beta.$$

Assumption 4: There exists a stationary point of the Markov parameters $(\bar{\Theta}_*, \bar{\mathscr{H}}_*)$ such that for $\forall N > N_0$ the following equations hold:

$$Y_{2s,N} = \bar{\Theta}_* \mathscr{Z}_{1,N} + \bar{\mathscr{H}}_* \mathscr{Z}^-_{s+1,N} + E_{2s,N}, \tag{18}$$

$$y_{N+2s} = \bar{\Theta}_* z_{s(N+1)} + \bar{\mathscr{H}}_* z_{s-1(N+s+1)} + e_{N+2s}. \tag{19}$$

Substitute eqs. (18) and (19) into (17), and for sufficiently large N we have

$$\begin{aligned} \varepsilon_{N+2s|N+2s-1} &= y_{N+2s} - Y_{2s,N}\zeta_{N+1} + \bar{\Theta}_* \xi_{N+1} \\ &= e_{N+2s} - \frac{1}{N} E_{2s,N} \mathscr{Z}^{-\,T}_{s+1,N} \\ &\quad \cdot \left(\frac{1}{N} \mathscr{Z}^-_{s+1,N} \mathscr{Z}^{-\,T}_{s+1,N} \right)^{-1} z_{s-1(N+s+1)} \end{aligned} \tag{20}$$

Note that, in spite of the existence of the feedback, we have

$$\lim_{N\to\infty} \frac{1}{N} E_{2s,N} \mathscr{Z}^{-\,T}_{s+1,N} = 0,$$

since the relatively future innovation is uncorrelated[1] with the relatively past input and output. Henceforth, $\varepsilon_{N+2s|N+2s-1}$ is called the residual. The second term on the right hand side of (20) vanishes as N tends to infinity. Hence, the residual $\varepsilon_{N+2s|N+2s-1}$ can be thought to be asymptotically equivalent to the innovation e_{N+2s}.

In summary of this section, we have the following theorem, which is one of the main contribution of this paper:

Theorem 1: For $\forall \delta > 0$,

$$\lim_{N\to\infty} \mathbf{P}\left(\| e_{N+2s} - \varepsilon_{N+2s|N+2s-1} \| \geq \delta \right) = 0,$$

where $\mathbf{P}(\Omega)$ denotes the probability of an event Ω occurring.

Proof: From (20) and assumption 3,

$$\begin{aligned} &\lim_{N\to\infty} \mathbf{P}\left(\| e_{N+2s} - \varepsilon_{N+2s|N+2s-1} \| \geq \delta \right) \\ &= \lim_{N\to\infty} \mathbf{P}\left(\left\| \frac{1}{N} E_{2s,N} \mathscr{Z}^{-\,T}_{s+1,N} \cdot \left(\frac{1}{N} \mathscr{Z}^-_{s+1,N} \mathscr{Z}^{-\,T}_{s+1,N} \right)^{-1} z_{s-1(N+s+1)} \right\| \geq \delta \right) \\ &\leq \lim_{N\to\infty} \mathbf{P}\left(\left\| \frac{1}{N} E_{2s,N} \mathscr{Z}^{-\,T}_{s+1,N} \right\|_2 \cdot \left\| \left(\frac{1}{N} \mathscr{Z}^-_{s+1,N} \mathscr{Z}^{-\,T}_{s+1,N} \right)^{-1} \right\|_2 \| z_{s-1(N+s+1)} \| \geq \delta \right) \\ &= 0 \end{aligned}$$

Thus, the proof is completed. ■

V. ONLINE CHANGE DETECTION OF A SYSTEM IN CLOSED-LOOP

A. Discussion of the test statistics

As is discussed in subsection IV-C, the residual $\varepsilon_{N+2s|N+2s-1}$ is a promising candidate for a test signal used for whiteness testing. Taking account of similarity in derivation from recursive subspace model identification between the open-loop [7], [8] and closed-loop cases, and taking account of the fact that system identification using high-order model structure followed by model reduction gives no worse result, sometimes a better result[2], than system identification using true-order model structure [12], [13], the proposed residual is also expected to be superior in accuracy to conventional test signals generated by, e.g., conventional RLS and RLMS algorithms with low-order models. To convince ourselves of its usefulness, it is investigated here what happens to the residual when an abrupt change in the system occurs. Suppose $k_c = N + 2s$, in other words, an abrupt change in the system under surveillance occurs between $N+2s-1$ and $N+2s$. Assume that N is sufficiently large, and that the signal to noise ratio, $\{(u_k, y_k)\}$ to $\{e_k\}$ is significant. Then, from eqs. (20) and (1), we have

$$\varepsilon_{N+2s|N+2s-1} \approx e_{N+2s} + \Delta_1 \bar{A}^{s-2} x_{N+s+1} + \Delta_2 z_{s-1(N+s+1)} \tag{21}$$

where $\Delta_1 := C\bar{A}_\Delta + C_\Delta \bar{A} + C_\Delta \bar{A}_\Delta$ and

$$\begin{aligned} \Delta_2 &:= \begin{bmatrix} \Delta_3 & \Delta_4 \end{bmatrix}, \\ \Delta_3 &:= \begin{bmatrix} \Delta_1 \bar{A}^{s-3} B & \cdots & \Delta_1 B & CB_\Delta + C_\Delta B + C_\Delta B_\Delta \end{bmatrix}, \\ \Delta_4 &:= \begin{bmatrix} \Delta_1 \bar{A}^{s-3} K & \cdots & \Delta_1 K & C_\Delta K \end{bmatrix}. \end{aligned}$$

Due to the abrupt change at $k_c = N + 2s$, the presence of the second and third terms of (21) leads to the covariance of

[1]Note that the word "*uncorrelated*" is used here with a slight abuse of terminology ([29], [30].)

[2]It happens only when the low order model structure does not contain the true system.

$\varepsilon_{k_c|k_c-1}$ significantly greater than that of e_{k_c}, i.e., $\sigma_0^2 I$. Similarly, for $\varepsilon_{k_c+i|k_c-1+i}$ with $i \geq 1$, the asymptotic equivalence between $\varepsilon_{k_c+i|k_c-1+i}$ and e_{k_c+i} does not hold any more after the abrupt change occurs at k_c.

The discussion as above supports the usefulness of the proposed residual as a test signal, with high accuracy, for a whiteness test.

B. Likelihood ratio test

In this subsection, making use of the real-time observation of the residual, denoted briefly by $\varepsilon_k := \varepsilon_{k|k-1}$ henceforth, an on-line change detection scheme is developed. The key point of the change detection scheme presented here is to decide *whether or not the covariance of the residual* $\{\varepsilon_k\}$ *is significantly larger than that of* $\{e_k\}$ *(i.e.,* $\sigma_0^2 I$*)*. In other words, we decide that the system is in fact unchanged if no significant increase of the covariance can be observed. Note that, thanks to the similarity between this case and the open-loop case [7], [8], the proposed change detection scheme is quite similar to that presented in these references.

Now, suppose

$$\varepsilon_1, \cdots, \varepsilon_M \;\sim\; \mathbf{N}(\mu, \sigma^2 I),\ i.i.d. \tag{22}$$

The change detection scheme tests between the two following hypotheses:

$$\begin{cases} \mathbf{H}_0: \ \theta \in S_0 := \{(0, \sigma_0^2)\} \\ \mathbf{H}_1: \ \theta \in S_1 := \{(\mu, \sigma^2) \mid \sigma^2 > \sigma_0^2,\ \mu \text{ is arbitrary.}\} \end{cases}$$

where, according to (22), the parameter θ is defined as $\theta = (\mu, \sigma^2)$. Note that both mean and covariance are known under $\mathbf{H}_0$ while they are unknown under $\mathbf{H}_1$. The logarithm of the likelihood ratio for the sequence $\{\varepsilon_k\}_{k=1}^M$ is given by

$$\begin{aligned} l_{\max} &= \ln \frac{\max_{\theta \in S_1} \frac{1}{\sigma^{Ml}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{k=1}^{M} (\varepsilon_k - \mu)^T (\varepsilon_k - \mu)\right\}}{\frac{1}{\sigma_0^{Ml}} \exp\left(-\frac{1}{2\sigma_0^2}\sum_{k=1}^{M} \varepsilon_k^T \varepsilon_k\right)} \\ &= \frac{Ml\tilde{\sigma}^2}{2\sigma_0^2} - \frac{Ml}{2}\left(1 + \ln \frac{\tilde{\sigma}^2 - \widehat{\mu}^T \widehat{\mu}/l}{\sigma_0^2}\right), \end{aligned} \tag{23}$$

where

$$\tilde{\sigma}^2 := \frac{1}{Ml}\sum_{k=1}^{M} \varepsilon_k^T \varepsilon_k, \qquad \widehat{\mu} := \frac{1}{M}\sum_{k=1}^{M} \varepsilon_k. \tag{24}$$

Note that the maximization in (23) is accomplished when $\mu = \widehat{\mu}$, $\sigma^2 = \frac{1}{Ml}\sum_{k=1}^{M}(\varepsilon_k - \widehat{\mu})^T(\varepsilon_k - \widehat{\mu})$. Using an appropriate threshold $c > 0$, the decision rule is given by

$$l_{\max} > c \implies \text{reject } \mathbf{H}_0. \tag{25}$$

We now derive the threshold c. From (24), the conditional probability under the null hypothesis, $\mathbf{P}_{(0,\sigma_0^2)}(l_{\max} > c)$, is given by

$$\begin{aligned} &\mathbf{P}_{(0,\sigma_0^2)}(l_{\max} > c) \\ &= \mathbf{P}_{(0,\sigma_0^2)}\left(\frac{Ml\tilde{\sigma}^2}{2\sigma_0^2} - \frac{Ml}{2}\left(1 + \ln \frac{\tilde{\sigma}^2 - \widehat{\mu}^T\widehat{\mu}/l}{\sigma_0^2}\right) > c\right) \\ &\approx \mathbf{P}_{(0,\sigma_0^2)}\left(\frac{Ml\tilde{\sigma}^2}{2\sigma_0^2} - \frac{Ml}{2} > c\right) \\ &= \mathbf{P}_{(0,\sigma_0^2)}\left(\frac{Ml\tilde{\sigma}^2}{\sigma_0^2} > 2\left(c + \frac{Ml}{2}\right)\right), \end{aligned}$$

where the approximation is because $\tilde{\sigma}^2 - \widehat{\mu}^T\widehat{\mu}/l \approx \sigma_0^2$ for sufficiently large M under the null hypothesis. Note that $Ml\tilde{\sigma}^2/\sigma_0^2$ has a χ^2 distribution with $(Ml-1)$ degrees of freedom. Therefore, if $\chi_\alpha^2(Ml-1)$ denotes the $100(1-\alpha)$-th percentile of the χ^2 distribution with $(Ml-1)$ degrees of freedom, from

$$2\left(c + \frac{Ml}{2}\right) = \chi_\alpha^2(Ml-1),$$

the threshold at the α significance level is given by

$$c = \frac{1}{2}\left(\chi_\alpha^2(Ml-1) - Ml\right). \tag{26}$$

In summary, from (23) and (26), the likelihood ratio test with respect to $\mathbf{H}_0$ vs. $\mathbf{H}_1$ is obtained by

$$\frac{\tilde{\sigma}^2}{\sigma_0^2} - \ln \frac{\tilde{\sigma}^2 - \frac{\widehat{\mu}^T\widehat{\mu}}{l}}{\sigma_0^2} > \frac{\chi_\alpha^2(Ml-1)}{Ml} \Rightarrow \text{reject } \mathbf{H}_0. \tag{27}$$

C. Online change detection

In this subsection, an online change detection method is presented. GMA-based recursive algorithms for updating $\tilde{\sigma}^2$ and $\widehat{\mu}$ in (24) are derived and they are applied to the likelihood ratio test (27). GMA is the abbreviation for Geometric Moving Average ([3]).

Replace $\tilde{\sigma}^2$ and $\widehat{\mu}$ in (27), respectively, by

$$s_N = \frac{1}{l}\sum_{i=0}^{\infty} \gamma_i \varepsilon_{N-i}^T \varepsilon_{N-i}, \quad m_N = \sum_{i=0}^{\infty} \gamma_i \varepsilon_{N-i}, \tag{28}$$

where $\gamma_i := \lambda(1-\lambda)^i$ and $0 < \lambda \leq 1$ is an exponential forgetting factor. Note that effective data-length can be approximated by $M \approx 1/\lambda$. Then, recursive algorithms for updating s_N and m_N in (28) are, respectively, given by

$$s_{N+1} = (1-\lambda)s_N + \lambda\frac{\varepsilon_{N+1}^T\varepsilon_{N+1}}{l}, \tag{29}$$

$$m_{N+1} = (1-\lambda)m_N + \lambda\varepsilon_{N+1}. \tag{30}$$

The proposed online change detection method can be summarized as follows ([7]):

Online change detection method : Suppose s_N and m_N in (28) are given. Then, ε_{N+1}, s_{N+1} and m_{N+1} are computed according to (29) and (30), respectively, and then, the following likelihood ratio test is carried out:

$$\frac{s_{N+1}}{\sigma_0^2} - \ln \frac{s_{N+1} - \frac{m_{N+1}^T m_{N+1}}{l}}{\sigma_0^2} > \frac{\chi_\alpha^2(l/\lambda - 1)}{l/\lambda} \tag{31}$$

$$\implies \text{reject } \mathbf{H}_0.$$

The left hand side of inequality (31) is called the *decision function*.

VI. Numerical simulation

Consider a closed-loop system composed of a system $P(s)$ to be observed and a stabilizing feedback controller $K(s)$, depicted in Fig. 1, where

$$P(s) = \frac{0.05}{s^2 + cs + 0.48}, \qquad K(s) = \frac{3}{s+2}. \tag{32}$$

Assume that, at an unknown time instant, the parameter c changes abruptly as follows:

$$c = \begin{cases} 0.13 & \text{before the change,} \\ 0.21 & \text{after the change.} \end{cases}$$

Note that the closed-loop system is internally stable both before and after the change. The closed-loop system has three external inputs, namely, the persistently exciting signal r, the disturbance v and the reference. The reference is set to 0. The signals u and y are available for change detection. Assume that the disturbance v is an unmeasurable colored signal.

In the numerical simulation, $P(s)$ and $K(s)$ are discretized using the bilinear transform with the unit sampling period. Fig. 2 illustrates the Bode diagrams of $P(s)$ before and after the abrupt change. Let the abrupt change occur at the sampling instant $k_c = 2000$. Note that k_c is not known nor available for change detection. The persistently exciting signal r is a zero mean Gaussian signal with the variance 0.16. The disturbance v is generated as an output of the filter

$$\frac{1 - 1.56z^{-1} + 1.045z^{-2} - 0.3338z^{-3}}{1 + 0.85z^{-1} - 0.31z^{-2} - 0.6675z^{-3}}$$

whose input is a zero mean Gaussian signal with the variance $(0.025)^2$. The profiles of (u, r) and (y, b) are depicted respectively in Figs. 3 and 4. Note that the signal y to noise v ratio (SNR) is about 0.4dB.

The problem considered in this simulation is to detect the abrupt change in the system $P(s)$, which occurs at $k_c = 2000$, as early as possible using the proposed online change detection method with observations of u and y. The block size of block Hankel matrices, s, is set at 10. Using the first 500 samples, the initial values of the recursive algorithm in Proposition 1 is calculated. The recursive algorithm runs from the sampling instant $k = 501$ to generate the signal for the hypothesis testing ε. The exponential forgetting factor λ in (29) and (30) is set to 0.2. The level of significance is set to $\alpha = 0.1\%$.

The result of the numerical simulation is illustrated in Fig. 5. The proposed change detection method makes an alarm at $k = 2005$, which is 5 steps after the instance where the change occurs. The proposed method achieves early detection of the abrupt change in the system under surveillance. As illustrated in Fig. 4, note that SNR, y to v, is about 0.4dB, that is, the noise level is almost comparable to the output signal level.

VII. Conclusion

In this paper, we have developed an online change detection scheme based on whiteness testing, which is used for detection of changes in a system under surveillance in closed-loop. A recursive algorithm of closed loop subspace model identification has been presented, which is based on the matrix inversion lemma. A test signal online generated by the recursive algorithm is studied and its asymptotic whiteness has explicitly been proved. The introduction of statistical hypotheses testing has brought us a statistically solid decision rule based on a significant level to be designed.

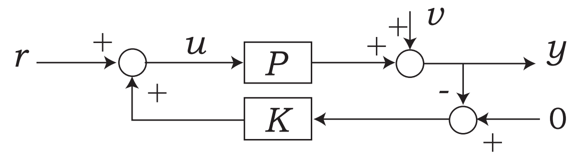

Fig. 1. A feedback system. P is to be under surveillance.

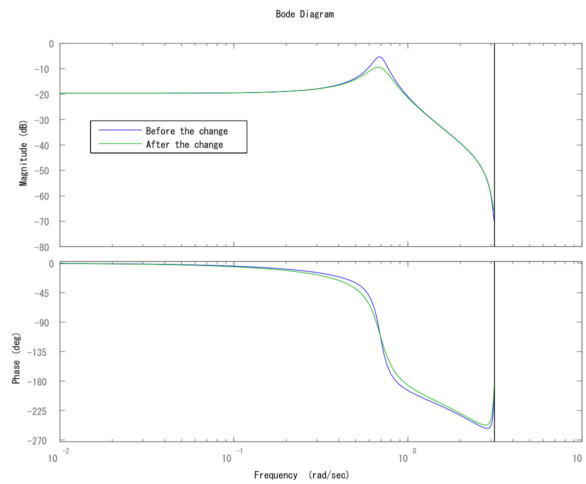

Fig. 2. Bode diagrams of the system under surveillance before and after the abrupt change at the sampling instant 2000.

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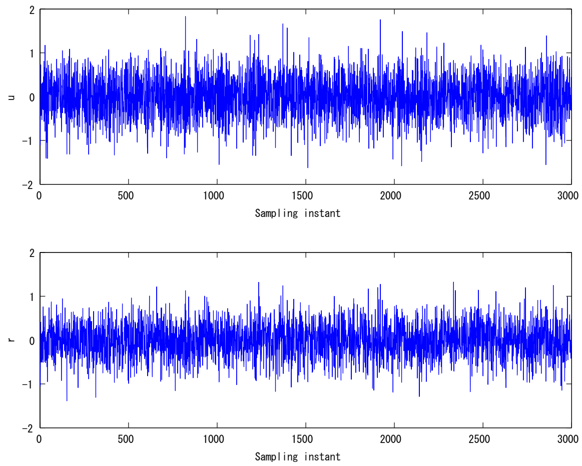

Fig. 3. The input u and the external persistently exciting signal r.

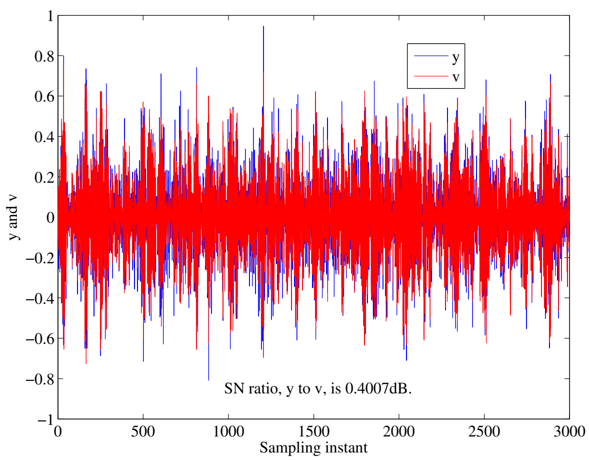

Fig. 4. The output y and the disturbance v. The signal to noise ratio, y to v is 0.4dB.

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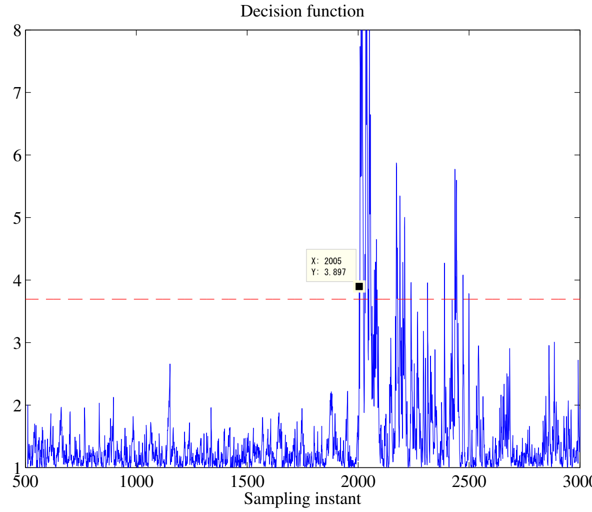

Fig. 5. Decision function (solid line) and a threshold (dashed line) in (31).

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Appendix

For three tail matrices, F, G and H, the oblique projection of F onto the space generated by the rows of G along the space generated by the rows of H, denoted by $\hat{\mathrm{E}}_{||H}[F|G]$, is defined as follows:

$$\hat{\mathrm{E}}_{||H}[F|G] = \hat{\Sigma}_{FG|H}\hat{\Sigma}_{GG|H}^{-1}G,$$

where $\hat{\Sigma}_{FG} = \frac{1}{N}FG^T$, and $\hat{\Sigma}_{FH}$ and $\hat{\Sigma}_{GH}$ are defined in the same manner as $\hat{\Sigma}_{FG}$ is defined. We define

$$\hat{\Sigma}_{FG|H} = \hat{\Sigma}_{FG} - \hat{\Sigma}_{FH}\hat{\Sigma}_{HH}^{-1}\hat{\Sigma}_{HG},$$
$$\hat{\Sigma}_{GG|H} = \hat{\Sigma}_{GG} - \hat{\Sigma}_{GH}\hat{\Sigma}_{HH}^{-1}\hat{\Sigma}_{HG}.$$