21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

A Complete LMI/Riccati Theory from Stochastic Modeling*

Augusto Ferrante† and Giorgio Picci†

***Abstract*— We present a theory of stochastic modeling for stationary discrete-time processes in terms of general acausal realizations which involve a possibly unstable A matrix. This leads to a unified derivation of many known results on spectral factorization, Linear Matrix Inequalities and the Riccati inequality/equation in a coherent framework.**

I. Introduction

We consider discrete-time m-dimensional purely nondeterministic (p.n.d.) wide-sense stationary processes $\mathbf{y}$ with a rational spectral density matrix. It is well-known that such processes admit linear state-space representations of the form

$$\begin{cases} x(t+1) = Ax(t) + Bw(t) \\ y(t) \quad\;\; = Cx(t) + Dw(t) \end{cases} \tag{1}$$

where $\{w(t)\}$ is a p-dimensional normalized white noise; i.e., $\mathbb{E}\{w(t)w(s)^\top\} = I\delta_{ts} \quad \mathbb{E}\{w(t)\} = 0$, and the state, $\{x(t)\}_{t\in\mathbb{Z}}$ is a n-dimensional (wide-sense) Markov process. In this model both x and w are part of the representation and are *hidden variables*; i.e. variables which are not observed, and can be chosen in different ways. It is customary in representing stationary processes to assume that A is a matrix with all the eigenvalues in the open unit disc (which we shall call an *asymptotically stable matrix* for short). It is a fact that most of the analysis of stationary stochastic systems is done under this assumption. This has led to a theory of stochastic modeling involving spectral factorization, a Linear Matrix Inequality, the associated Riccati inequality and equation, based on a body of mixed frequency-domanin and algebraic techniques which formally parallels the stationary Linear Quadratic Gaussian (so-called LQG) theory, which can be found in textbooks on control and filtering theory. The apparent difference is that while the former seems to be valid only under the assumption of asymptotic stability of A, in the latter setting, A is not required to be stable. Although in some cases in control one may have the possibility to re-allocate poles, a possible instability of A is an important characteristic which is physically realistic in many applications, say Kalman filtering just to mention one.

One may then wonder how special are the results obtained in stochastic modeling and how may they relate to the apparently more general LQG setup. The scope of this paper is to show that, still remaining within the same circle of ideas of stationary realization of stochastic processes, but giving up the requirement of stability of A, one can re-obtain virtually all known results on spectral factorization, the Linear Matrix Inequality, the Riccati inequality and so on. In particular we obtain a parallel of the recent comprehensive treatment of the discrete time Riccati equation in [12].

*This research was supported by the University of Padova under project "Progetto di ricerca di Ateneo".

† Department of Information Engineering, University of Padova, via Gradenigo 6/B, 35131 Padova, Italy; e-mail: augusto@dei.unipd.it, picci@dei.unipd.it

II. Acausal linear stochastic models

Let us start from the observation, that a stochastic process has no privileged direction of time built in and therefore must admit many representations which are neither causal nor anticausal. Assume that (1) is a minimal realization of the process $\mathbf{y}$ (this condition can be defined in purely geometric terms and is therefore independent of which coordinates and which specific models are used to represent the process [6]).

If, and only if, A has no eigenvalues on the unit circle the rows of the transfer function $(zI - A)^{-1}B$ belong to L^2 of the unit circle and are therefore Fourier transforms of square summable sequences on $\mathbb{Z}$. Hence the inverse Fourier transform of the relation $d\hat{x} = (zI - A)^{-1}B\, d\hat{w}$ is a convolution sum of the form

$$x(t) = \sum_{k=-\infty}^{+\infty} G(t-k)w(k) \tag{2}$$

where G is the *Green function* of the system. The rows of $G(t)$ are square summable sequences on $\mathbb{Z}$ and must tend to zero when $t \to \pm\infty$. In fact, by a partial fractions decomposition of $(zI - A)^{-1}B$, it is seen that G can be split into a causal and anticausal parts supported on the positive and negative time axis. This decomposition can be also obtained by splitting the state space into the direct sum of the stable and unstable invariant manifolds of A and accordingly splitting the state-space model of x into a causal and an anticausal components evolving on these invariant manifolds. This gives a meaning to the formula (2) in terms of familiar causal plus anticausal state response formulas.

Introduce a change of basis so that the model (1) splits into a pair

$$\begin{cases} x_+(t+1) = A_+x_+(t) + B_+w(t) \\ x_-(t+1) = A_-x_-(t) + B_-w(t) \end{cases} \tag{3}$$

where, since A has no eigenvalues on the unit circle, we choose A_+ and A_- to have all eigenvalues strictly outside and strictly inside of the unit disc; in short, $|\lambda(A_+)| > 1$ and $|\lambda(A_-)| < 1$, respectively. It is obvious that reachability of the overall system (1) holds if and only if both subsystems in (3) are reachable. The first model in (3) can be solved backwards in time to yield

$$x_+(t) = -\sum_{k=t}^{+\infty} A_+^{t-k-1}B_+w(k).$$

ISBN: 978-90-367-6321-9

which shows that $x_+(t) \in \mathbf{H}_t^+(w)$ (the future space at time t of w) while, by stability of A_- we have $x_-(t) \in \mathbf{H}_t^-(w)$, that is, $x_-(t) \perp x_+(t)$ for all t. Hence in this basis the state covariance matrix is block-diagonal; i.e. $P = \operatorname{diag}\{P_+,\, P_-\}$ with P_+ solving the Lyapunov equation:

$$P_+ = A_+ P_+ A_+' - B_+ B_+'$$

which has a unique positive definite solution. In general, the overall state covariance does not satisfy a single Lyapunov equation but needs to be computed by combining the covariances of the stable and antistable components.

III. Generalization of rational spectral factorization and of the Linear Matrix inequality

Consider a rational $m \times m$ matrix function $\Phi(z)$ which is *parahermitian*; i.e. satisfies the symmetry condition

$$\Phi(z^{-1}) = \Phi(z)',$$

and is such that $\Phi(e^{j\theta})$ is integrable on the unit circle, which, by rationality implies that $\Phi(z)$ has no poles on the unit circle. Clearly, every such rational L^1-parahermitian function has a Laurent expansion in a neighborhood of the unit circle,

$$\Phi(z) = \sum_{k=-\infty}^{+\infty} \Lambda_k z^{-k} \tag{4}$$

where Λ is an $m \times m$ summable matrix sequence satisfying the symmetry relation $\Lambda_{-k} = \Lambda_k'$.

When $\Phi(z)$ is also positive semidefinite on the unit circle; i.e. $\Phi(e^{j\theta}) \geq 0$, it obviously becomes a *bona-fide* spectral density of a stationary process $\mathbf{y}$. In this case one may compute Λ_k by standard formulas, using a causal realization ($|\lambda(A)| < 1$), see e.g. [1].

Let us now consider the following problem. Given a rational parahermitian function Φ in L^1 of the unit circle, consider the problem of finding minimal spectral factors W of Φ and the corresponding (minimal) realizations

$$W(z) = H(zI - F)^{-1}B + D. \tag{5}$$

Note that the existence of spectral factors is not automatically guaranteed and is in fact *equivalent to the positivity of* $\Phi(e^{j\theta})$ since, irrespective of analiticity, if a spectral factor W exists, then $\Phi(e^{j\theta}) = W(e^{j\theta})W(e^{-j\theta})' \geq 0$.

Now the poles of a parahermitian matrix function $\Phi(z)$ have reciprocal symmetry; i.e. if $\Phi(z)$ has a pole in $z = p_k$ then $1/p_k$ must also be a pole (be it finite or not) of the same multiplicity. Besides, by integrability, they cannot be in the unit circle. Hence the set of poles of a L^1-parahermitian function $\Phi(z)$ of McMillan degree $2n$ can be split in two reciprocal subsets σ_1 and σ_2 each containing n complex numbers (repeated according to multiplicity), such that σ_2 contains exactly the reciprocal elements of σ_1, which for short we write $\sigma_2 = 1/\sigma_1$. This decomposition of the singularities of $\Phi(z)$ yields, by partial fraction expansion, a rational additive decomposition of the type

$$\Phi(z) = Z(z) + Z(z^{-1})', \tag{6}$$

where $Z(z)$ is a rational matrix function which we shall write as $C(zI - A)^{-1}\bar{C}' + \Lambda_0/2$, with poles in σ_1 while those of $Z(z^{-1})'$ are then necessarily in $\sigma_2 = 1/\sigma_1$. One can hence have many additive decompositions of a parahermitian function $\Phi(z)$ of the type (6) in particular one where $Z(z)$ is analytic; without however any a priori guarantee of positivity.

We say that a real $n \times n$ matrix A has *unmixed spectrum* or for short, that A is *unmixing*, if its spectrum, $\sigma(A)$, with elements listed according to algebraic multiplicity, does not contain reciprocal pairs. In particular, A cannot have eigenvalues of modulus one. Assuming minimality of $(C,\, A,\, \bar{C}')$, then A has unmixed spectrum if and only if the selected pole set $\sigma_1 \equiv \sigma(A)$ has no self-reciprocal elements. It is obvious that this happens if and only if $\sigma_1 \cap \sigma_2 = \emptyset$. When $\sigma_1 \equiv \sigma(A)$ has self-reciprocal elements there may be ambiguities in forming the decomposition $\Phi(z) = Z(z) + Z(z^{-1})'$. Actually in some case the decomposition may not even exist.

Note that the unmixed spectrum condition is exactly the condition insuring that the Lyapunov equation $P - APA' = Q$ has a (unique) solution for arbitrary Q, see [13].

The rational spectral factorization problem alluded at before can be stated in purely algebraic terms without any reference to probabilistic concepts. Let us agree to call (6) an *unmixing additive decomposition* of Φ, if A has unmixed spectrum.

Problem 1: Given an unmixing additive decomposition (6) *of a rational parahermitian matrix* $\Phi(z)$*, where* $Z(z) := C(zI - A)^{-1}\bar{C}' + 1/2\Lambda_0$*, look for rational spectral factors of* Φ *of the form* (5) *whose pair* (H, F) *is similar to* (C, A)*; concisely: having the same pole structure of* $Z(z)$*.*

Following a procedure which has appeared in different forms in the literature, first observe that the additive decomposition (6) can be rewritten in matrix form as

$$\Phi(z) = \begin{bmatrix} C(zI - A)^{-1} & I \end{bmatrix} \begin{bmatrix} 0 & \bar{C}' \\ \bar{C} & \Lambda_0 \end{bmatrix} \begin{bmatrix} (z^{-1}I - A')^{-1}C' \\ I \end{bmatrix}, \tag{7}$$

and then use the following lemma,

Lemma 2: Let P *be any symmetric* $n \times n$ *matrix and form the array*

$$N(P) = \begin{bmatrix} P - APA' & -APC' \\ -CPA' & -CPC' \end{bmatrix},$$

then

$$\begin{bmatrix} C(zI - A)^{-1} & I \end{bmatrix} N(P) \begin{bmatrix} (z^{-1}I - A')^{-1}C' \\ I \end{bmatrix} \equiv 0$$

identically.

Thereby (7) can be rewritten as

$$\Phi(z) = \begin{bmatrix} C(zI - A)^{-1} & I \end{bmatrix} M(P) \begin{bmatrix} (z^{-1}I - A')^{-1}C' \\ I \end{bmatrix} \tag{8}$$

where

$$M(P) := \begin{bmatrix} P - APA' & \bar{C}' - APC' \\ \bar{C} - CPA' & \Lambda_0 - CPC' \end{bmatrix}. \tag{9}$$

From this the following sufficient condition for spectral factorization of $\Phi(z)$ follows.

Theorem 3: Let the rational parahermitian matrix $\Phi(z)$ admit an additive decomposition (6) *with $Z(z) := C(zI - A)^{-1}\bar{C}' + 1/2\Lambda_0$. If there exists a real symmetric matrix $P = P'$ solving the* Linear Matrix Inequality (LMI)

$$M(P) \geq 0 \tag{10}$$

then $\Phi(z)$ admits spectral factors. In fact, let B, D be defined by the factorization

$$M(P) = \begin{bmatrix} B \\ D \end{bmatrix} \begin{bmatrix} B' & D' \end{bmatrix} , \tag{11}$$

then $W(z) = C(zI - A)^{-1}B + D$ is a spectral factor and $\Phi(z)$ is a spectral density.

Hence, in particular, we can choose $F = A$ and $H = C$ in (5). Note that no stability of A nor minimality are required. In this respect, one has

Proposition 4: If A is unmixing and $C(zI - A)^{-1}\bar{C}'$ is a minimal realization then $W(z) = C(zI - A)^{-1}B + D$ is also a minimal realization of a minimal spectral factor. The converse is also true.

The following is a converse of Theorem 3. For analytic spectral factors ($|\lambda(A)| < 1$) the statement of the theorem is very well-known.

Theorem 5: Let $W(z) = C(zI-A)^{-1}B+D$ be a rational spectral factor of $\Phi(z)$ with an unmixing A matrix. Then there is a corresponding additive decomposition (6) *with $Z(z) = C(zI - A)^{-1}\bar{C}' + 1/2\Lambda_0$ and a unique $P = P'$ satisfying the linear matrix equality* (11). *In particular, if $|\lambda(A)| < 1$ and (A, B) is reachable, then $P > 0$.*

Let $M : \mathbb{R}^{n\times n} \to \mathbb{R}^{(n+m)\times(n+m)}$ be the linear map defined by (14) in terms of an arbitrary unmixing additive decomposition of $\Phi(z)$ and let $\mathcal{P}$ denote the set of all symmetric matrices P satisfying the linear matrix inequality

$$M(P) \geq 0\,. \tag{12}$$

Theorems 5 and 3 imply in particular the following *positivity test*: Φ is a spectral density; that is $\Phi(e^{j\theta}) \geq 0$, if and only if $\mathcal{P}$ is nonempty.

Theorems 5 and 3 together also provide a necessary and sufficient condition for spectral factorization and provide, at least in principle, a way to compute spectral factors via the following procedure: First compute an unmixing additive decomposition of $\Phi(z)$; then look at the corresponding linear Matrix inequality (12) and check if $\mathcal{P}$ is nonempty. Any solution $P = P'$ will then provide a spectral factor by factoring $M(P)$ as in (11) to obtain B and D. To avoid redundancy we shall always require that $\begin{bmatrix} B \\ D \end{bmatrix}$ is full column rank. Then the factorization (11) yields a pair (B, D), which is unique modulo an orthogonal transformation.

Note that when in Theorem 5 $W(z) = C(zI-A)^{-1}B+D$ is assumed to be a minimal realization, the matrix $\bar{C} = CPA' + DB'$ must be the same for all other (minimal) spectral factors of the form $W(z) = C(zI - A)^{-1}B + D$, irrespective of which P (and hence which B, D) are selected. This is so since $\bar{C}'$ is the "B" parameter of a minimal realization of $Z(z)$ in the decomposition (6); hence it cannot depend on which spectral factor is chosen to form $\Phi(z)$. In other words, $\bar{C}$ is an *invariant over the family of all minimal spectral factors expressed with a fixed* (C, A) *pair*. This is well-known in case of a stable A matrix.

If we assume minimality the correspondence $P \leftrightarrow W$ is one-to-one.

Theorem 6: Let $(A, C, \bar{C}, \Lambda_0)$ be a minimal realization of $Z(z)$, in an unmixing additive decomposition of a spectral density matrix Φ. Then there is a one-to-one correspondence between minimal spectral factors of Φ with the same pole structure of $Z(z)$ and symmetric $n \times n$ matrices P solving the Linear Matrix Inequality (10) *in the following sense: Corresponding to each solution $P = P'$ of* (10), *there is a minimal spectral factor $W(z) = C(zI - A)^{-1}B + D$ where A and C are as given above and $\begin{bmatrix} B \\ D \end{bmatrix}$ is the unique (modulo orthogonal transformations) full-rank factor of $M(P)$. Conversely, for each minimal spectral factor* (5) *of $\Phi(z)$ with the same pole structure of $Z(z)$ there is a change of basis such that $F = A$, $H = C$ and a unique $P \in \mathcal{P}$ solving the equations* (11) *in the basis of the realization $(A, C, \bar{C}, \Lambda_0)$. In partcular, if $Z \equiv \Phi_+$ then the correspondence is between solutions $P = P'$ of* (10) *and minimal analytic spectral factors.*

Specializing to the case when A is asymptotically stable gives the famous Kalman-Yakubovich-Popov Lemma [5], [9], [14] which characterizes positive real functions and has played a fundamental role in many areas of systems and control theory.

IV. Duality and discrete-time rational all pass functions

Theorems 5 and 3 have a dual version which turns out to generalize and place in a proper perspective the forward/backward modeling discussed in [7], [8].

Let us again assume an unmixing additive decomposition (6) of a rational parahermitian matrix $\Phi(z)$ and consider the problem, which will be referred to as the *dual* of Problem 1, of finding rational spectral factors $\bar{W}(z)$ of $\Phi(z)$ with the same pole structure of $Z(z^{-1})' = \bar{C}(z^{-1}I - A')^{-1}C' + 1/2\Lambda_0$. Precisely, we shall look for rational matrices of the form $\bar{W}(z) = \bar{C}(z^{-1}I - A')^{-1}\bar{B} + \bar{D}$, such hat

$$\Phi(z) = \bar{W}(z)\bar{W}(z^{-1})'\,. \tag{13}$$

This problem can be addressed by using the same apparatus developed in Section III leading to the following sufficient condition for dual spectral factorization of $\Phi(z)$.

Theorem 7: Let the rational parahermitian matrix $\Phi(z)$ admit an additive decomposition (6) *with $Z(z^{-1})' = \bar{C}(z^{-1}I - A')^{-1}C' + 1/2\Lambda_0$. If there exists a real symmetric matrix $Q = Q'$ solving the dual Linear Matrix Inequality*

$$\bar{M}(Q) := \begin{bmatrix} Q - A'QA & C' - A'Q\bar{C}' \\ C - \bar{C}QA & \Lambda_0 - \bar{C}'Q\bar{C} \end{bmatrix} \geq 0 \tag{14}$$

then $\Phi(z)$ admits dual spectral factors. In fact, let $\bar{B}$, $\bar{D}$ be defined by the factorization

$$\bar{M}(Q) = \begin{bmatrix} \bar{B} \\ \bar{D} \end{bmatrix} \begin{bmatrix} \bar{B}' & \bar{D}' \end{bmatrix} , \tag{15}$$

then $\bar{W}(z) = \bar{C}(z^{-1}I - A')^{-1}\bar{B} + \bar{D}$ *is a spectral factor and* $\Phi(z)$ *is actually a spectral density.*
If A *is unmixing and* $\bar{C}(z^{-1}I - A')^{-1}C'$ *is a minimal realization then* $\bar{W}(z) = \bar{C}(z^{-1}I - A')^{-1}\bar{B} + \bar{D}$ *is also a minimal realization of a minimal spectral factor. The converse is also true.*
There is also a necessary condition for existence of spectral factors $\bar{W}$ in the same spirit of Theorem 5.

Theorem 8: Let $\bar{W}(z) = \bar{C}(z^{-1}I - A')^{-1}\bar{B} + \bar{D}$ *be a dual rational spectral factor of* $\Phi(z)$ *with an unmixing* A *matrix. Then there is a* $\tilde{C}$*, a corresponding additive decomposition* (6) *with* $Z(z) = \tilde{C}(zI - A)^{-1}\bar{C}' + 1/2\Lambda_0$ *and a unique* $Q = Q'$ *satisfying the linear matrix equality* (15).
Let $\mathcal{Q}$ denote the set of all symmetric matrices Q satisfying the linear matrix inequality

$$\bar{M}(Q) \geq 0\,. \tag{16}$$

Theorems 8 and 7 imply in particular that Φ is a spectral density; that is $\Phi(e^{j\theta}) \geq 0$, if and only if $\mathcal{Q}$ is nonempty.

Since $\bar{W}(z)$ is a spectral factor, it can be interpreted as the transfer function of a stochastic realization of y of the form

$$\begin{cases} \bar{x}(t-1) = A'\bar{x}(t) + \bar{B}\bar{w}(t) \\ y(t) \qquad\; = \bar{C}\bar{x}(t) + \bar{D}\bar{w}(t), \end{cases} \tag{17}$$

which is exactly of the same form as the well-known backward system encountered in stochastic realization [6]. It is however only when $|\lambda(A)| < 1$ that this realization has the backward property described in [7], [8]. Also, for a general unmixing A the matrix $\bar{P} = P^{-1}$ will no longer be interpretable as the covariance matrix of the state process $\bar{x}$. Nevertheless we have the following remarkable fact.

Theorem 9: Let $(A, C, \bar{C}', \Lambda_0)$ *be a minimal realization of* $Z(z)$*, in an unmixing additive decomposition of a spectral density matrix* Φ*. Then the solutions of the two linear matrix inequalities* (12) *and* (16) *are non singular matrices pairwise related by the equation*

$$Q = P^{-1} \tag{18}$$

which can be interpreted in the following sense: if P *solving* (12) *generates the minimal spectral factor* $W(z) = C(zI - A)^{-1}B + D$*, then there is a minimal* $\bar{W}(z)$*, solution of the dual spectral factorization* (13) *such that the corresponding solution* Q *of the linear matrix inequality* (16) *(Theorem 8) is the inverse of* P*, and conversely.*
Hence, if $(A, C, \bar{C}', \Lambda_0)$ is a minimal realization with A unmixing, the two solution sets $\mathcal{P}$ of (12) and $\mathcal{Q}$ of (16) are reciprocal sets, which can symbolically be written as

$$\mathcal{Q} = \mathcal{P}^{-1}\,.$$

More importantly, the mapping $P \to P^{-1}$ establishes a one-to-one correspondence between minimal spectral factor W and their dual $\bar{W}$. Below we shall show that this correspondence can be concretely represented as multiplication by an all-pass function which relates the white noise inputs to the two systems.

A general characterization of rational all-pass functions (in discrete time) is given in Theorem 10 below. This characterization is the discrete-time version of [4, Theorem 5.1] which, to our best knowledge has never been obtained directly. It is a nice application of Theorems 5 and 3 to the factorization of the spectral density matrix $\Phi(z) \equiv I$. It actually holds with minor modifications also for rectangular all-pass functions and turns out to be useful in many circumstances.

Theorem 10: Let $Q(z) := H(zI - A)^{-1}G + V$ *with* $A \in \mathbb{R}^{n\times n}, G \in \mathbb{R}^{n\times m}, H \in \mathbb{R}^{m\times n}, V \in \mathbb{R}^{m\times m}$ *be a minimal realization and let* A *be unmixing. Then*

1) *If* $Q(z)$ *is all-pass, there exist* $P = P'$ *and* $Q = Q'$ *such that*

$$\begin{cases} P = APA' + GG' \\ APH' + GV' = 0 \\ VV' + HPH' = I \end{cases} \tag{19}$$

$$\begin{cases} Q = A'QA + H'H \\ A'QG + H'V = 0 \\ V'V + G'QG = I \end{cases} \tag{20}$$

Moreover, P *and* Q *are non-singular and* $PQ = I$.

2) *Let* (A, G, H, V) *be as before (no minimality, invertibility or unmixing are now assumed). If there exists* $P = P'$ *satisfying (19) then* $Q(z)$ *is all-pass.*
Similarly, if there exists $Q = Q'$ *satisfying (20) then* $Q'(z^{-1})$ *(and in fact also* $Q(z)$*) is all-pass.*

Now let $(W, \bar{W})$ be a dual pair of minimal spectral factors represented in the basis corresponding to a fixed minimal realization of $Z(z) = C(zI - A)^{-1}\bar{C}' + 1/2\Lambda_0$ of an unmixing additive decomposition of a spectral density matrix Φ. Define

$$K(z) := \bar{B}'(zI - A)^{-1}B + V, \tag{21}$$

where V is an, as yet undefined, $m \times m$ matrix. Note that $(A,\, B,\, \bar{B}^{\top})$ is clearly also a minimal realization. We want K to be an all-pass function, namely

$$K(z)K(z^{-1})' = I \tag{22}$$

Apply Theorem 10, letting $G = B$ in the first equation (19) and $H = \bar{B}'$ in the first equation (20). It is clear that there exist solutions P and $Q = P^{-1}$, in fact equal to the solutions of the two dual Lyapunov equations which parametrize the dual minimal spectral factors $(W, \bar{W})$ corresponding via the map (18). The second and third equations in (19), (20), after having made the identifications $G = B$ and $H = \bar{B}'$, lead to a set of conditions on V like:

$$\begin{cases} AP\bar{B} + BV' \quad = 0, \\ VV' + \bar{B}'P\bar{B} \quad = I \end{cases} \tag{23a}$$

$$\begin{cases} A'P^{-1}B + \bar{B}V' \quad = 0, \\ V'V + B'P^{-1}B \quad = I \end{cases} \tag{23b}$$

which we need to show have a solution.

Proposition 11: Let $A \in \mathbb{R}^{n\times n}, G \in \mathbb{R}^{n\times m}, H \in \mathbb{R}^{m\times n}$ *be given. If there exists a pair* $P = P'$ *and* $Q = Q'$ *such* *that*

$$\left\{\begin{array}{l} P = APA' + BB' \\ Q = A'QA + H'H \\ PQ = I \end{array}\right. \tag{24}$$

then there exists a matrix $V \in \mathbb{R}^{m\times m}$ *such that* $Q(z) = H(zI - A)^{-1}G + V$ *is all-pass. If* A *is non-singular, then also* V *can be chosen to be non singular.*

Now once $K(z)$ is all pass, we can introduce the pair of white noise inputs $(w, \bar{w})$.

Theorem 12: Let $(W, \bar{W})$ *be a dual pair of spectral factors both parametrized in a fixed basis as defined above and let* $K(z)$ *in* (21) *be all-pass. Then*

$$\bar{W} = WK^*\,. \tag{25}$$

Conversely, if K *satisfies* (25) *it must be all-pass and admit a minimal realization as in* (21).

V. Equivalent representations of Markov processes

As anticipated in the introduction, there are many different dynamical models describing a Markov process $\mathbf{x}$. We shall describe all such equivalent dynamical representations and their invariance properties.

One such invariance relation is with respect to duality. Given a reference minimal stochastic realization of the form (1) with an unmixing A matrix and corresponding transfer function W, let $\bar{W}$ be the dual minimal spectral factor associated to W having minimal realization (17). This spectral factor is related to W by the (generalized) structural function $K(z)$ of equation (21). We define the input noise to this model to be the process having Fourier transform

$$d\hat{\bar{w}} := K(z)d\hat{w}$$

so that, $\bar{w}$ is a normalized p-dimensional white noise which generates the same Hilbert space of w, namely $\mathbf{H}(\bar{w}) = \mathbf{H}(w)$ see e.g. [6]. Since $d\hat{w} = K(z)^* d\hat{\bar{w}}$ it follows from Theorem 12 that the process y admits also a representation as the output of the system with transfer function $\bar{W}$ and input $\bar{w}$. We have in fact

Proposition 13: With the above choice of input noise $\bar{w}$, *the state* $\bar{x}$ *of the stochastic system* (17) *is related to the state* x *of the stochastic realization* (1) *by the equation*

$$\bar{x}(t) = P^{-1}x(t+1) \tag{26}$$

where P *is the solution of the Lyapunov equation* $P = APA' + BB'$. *Moreover the noise processes are related by*

$$\bar{w}(t) = \bar{B}'x(t) + Vw(t), \tag{27a}$$

$$w(t) = B'\bar{x}(t) + V'\bar{w}(t). \tag{27b}$$

In particular when $|\lambda(A)| < 1$, *one re-obtains the classical forward-backward relations of [7], [8].*

Also in the present more general context, the dual realizations (1) and (17) describe (modulo a time shift of one unit) the same Markovian state space.

Another invariance property is with respect to the dynamical representation of the state process. There is a whole family of difference equation representations of the same (wide sense) Markov process involving dynamic matrices A of a different structure. To investigate this family let us assume that the same n-dimensional stationary process x is represented by two difference equation models, say

$$x(t+1) = Ax(t) + Bw(t) \tag{28}$$

$$x(t+1) = Fx(t) + Gv(t) \tag{29}$$

where A and F have no eigenvalues of modulus one, both models are reachable and B and G are $n \times m$ with linearly independent columns. The first assumption is for ensuring stationarity while the remaining guarantee minimality of the representations and entail no loss of generality.

Note that subtracting (29) from (28) we get

$$v(t) = G^{-L}Bw(t) + G^{-L}(A - F)x(t) := Vw(t) + Cx(t)$$

where we have denoted $V := G^{-L}B$ and $C := G^{-L}(A-F)$. It is clear that $V + C(zI - A)^{-1}B$, being the transfer function of a filter mapping w into v, both normalized m-dimensional white noise processes, must be an all-pass function. Hence (29) is a minimal representation of the same Markov process described by (28) if and only if $V + C(zI - A)^{-1}B$ is all-pass. Hence, to study the family of equivalent representations (29) we can invoke the second statement of Theorem 10 and study the condition for solvability of the matrix equations

$$\left\{\begin{array}{l} Q = A'QA + C'C \\ A'QB + C'V = 0 \\ V'V + B'QB = I \end{array}\right. \tag{30}$$

which leads to the following theorem.

Theorem 14: The model (29) *is a minimal representation of the Markov process* x *described by* (28) *if and only if the process* v *and the matrices* (F, G) *are generated by the feedback transformation*

$$v(t) = Cx(t) + Vw(t) \tag{31a}$$

$$F = A + BC \tag{31b}$$

$$GV = B \tag{31c}$$

where the matrices Q, V *and* C *are solutions of the linear equations* (30). *If* A *is non singular then* V *is also non singular,* $C = -B'QAV^{-\prime}$ *and all solutions* Q *of* (30) *solve the* Homogeneous Algebraic Riccati Equation

$$Q = A'QA + A'QB(I - B'QB)^{-1}B'QA \tag{32}$$

Hence there are as many (minimal) state models, describing the dynamics of a fixed basis x in a fixed minimal Markovian state space $\mathbf{X}$, as there are solutions of the homogeneous Riccati equation (32). As we shall see in Section VI the state feedback law (31b) corresponding to each solution Q of (32), flips some of the eigenvalues of A to their reciprocals. Note that Q need not be invertible. In fact, to the solution $Q = 0$ of (32) there correspond $C = 0$, $F = A$ and $v = w$ *mod* $\mathcal{O}$, that is the original model.

In essence, the equivalence between dynamic representations of a finite dimensional stationary Markov process $x(t)$ can be described by the action of a finite subgroup of the general feedback group acting on the pair (A, B) and on the input noise w. This group action is explicitly parametrized by the solutions of the Riccati equation (32). This equation is a particular case of a more general Riccati equation which will be discussed in the next section. For reasons of space we shall not be able to discuss it in this paper.

VI. THE RICCATI INEQUALITY, THE ARE AND ZERO FLIPPING

We have shown that the family of minimal spectral factors can be parameterized by the solutions of the linear matrix inequality (10). In a vast majority of cases there is a more compact characterization of the set $\mathcal{P}$, namely by means of a *Riccati inequality*, which is of dimension n instead of $n+m$ as with the linear matrix inequality. This is of course classical, see [1], [2], [11].

Recall that in characterizing minimal spectral factors, we keep the poles fixed, the spectral factors differing by their zero structure, since zeros may be flipped to the reciprocal locations in the complex plane to create new spectral factors. While zeros located either at $z = 0$ or at $z = \infty$ can be handled by the linear matrix inequality, this will not be possible if we want to do parametrization by a Riccati inequality. The Riccati inequality is a device capable of flipping only *finite* zeros to their *finite* reciprocals. Hence to proceed we need to exclude the presence of zeros either at $z = 0$ or at $z = \infty$ in the spectral factors. The condition that precludes this is a condition on the spectral density matrix of the process, called *regularity*, which is studied in [3] and is diffusely discussed in [6]. A process y having a spectral density Φ with this property is called a *regular process* and is described by the condition

$$\Delta(P) := \Lambda_0 - CPC' > 0 \quad \text{for all } P \in \mathcal{P}. \tag{33}$$

It follows from the representation (8) that a regular process must be full rank, but the converse is not necessarily true.

$W(z)$ is $m \times p$, where $p := \operatorname{rank} M(P)$. Then, if regularity holds, a straight-forward calculation yields that $M(P) \geq 0$ if and only if $R(P) \geq 0$ where $R(\cdot)$ is the *Riccati operator*, defined for all $P = P'$ satisfying condition (33), by

$$R(P) = P - APA' - (\bar{C}' - APC')\Delta(P)^{-1}(\bar{C}' - APC')'. \tag{34}$$

Hence, $P \in \mathcal{P}$ if and only if it satisfies the *Riccati inequality*

$$R(P) \geq 0. \tag{35}$$

with the usual meaning of the inequality $\geq$ for matrices. We can thereby describe the set $\mathcal{P}$ as the inverse image through the map R of the cone $\mathbb{S}_+^{n\times n}$ of symmetric $n \times n$ positive semidefinite matrices; that is

$$\mathcal{P} = R^{-1}(\mathbb{S}_+^{n\times n})\,. \tag{36}$$

Moreover,

$$p = \operatorname{rank}\, M(P) = m + \operatorname{rank}\, R(P) \geq m.$$

If P satisfies the *algebraic Riccati equation (ARE)* $R(P) = 0$, i.e.,

$$P = APA' + (\bar{C}' - APC')\Delta(P)^{-1}(\bar{C}' - APC')', \tag{37}$$

$\operatorname{rank} M(P) = m$. Then the spectral factor W is $m\times m$. The family of P corresponding to square spectral factors form a subfamily $\mathcal{P}_0$ of $\mathcal{P}$. If $P \notin \mathcal{P}_0$, W is rectangular. We have the following sharpening of Theorem 6.

Corollary 15: Let $(A, C, \bar{C}')$ be a minimal realization of the matrix function $Z(z)$ in an unmixed additive decomposition (6) *of a regular spectral density $\Phi(z)$. Then there is a one-to-one correspondence, $P \to W(z) \mod \mathcal{O}$ between the set $\mathcal{P}$ of symmetric solutions P of the ARI $R(P) \geq 0$ and the minimal spectral factors of $\Phi(z)$ written in the representation $W(z) = C(zI - A)^{-1}B + D$, were $\begin{bmatrix} B' & D' \end{bmatrix}'$ is a full column rank factor of $M(P)$.*

Let the dual Riccati operator $\bar{R}$ *be defined as*

$$\bar{R}(Q) := Q - A'QA - (C' - A'Q\bar{C}')\bar{\Delta}(Q)^{-1}(C - \bar{C}QA) \tag{38}$$

where $\bar{\Delta}(Q) := \Lambda_0 - \bar{C}Q\bar{C}'$. Then the statement has a dual counterpart in terms of a one-to-one mapping $Q \to \bar{W}(z) \mod \mathcal{O}$ between the set $\mathcal{Q} = \mathcal{P}^{-1}$ of symmetric solutions Q of the ARI $\bar{R}(Q) \geq 0$ and the minimal dual spectral factors of $\Phi(z)$ written in the representation $\bar{W}(z) = \bar{C}(z^{-1}I - A')^{-1}\bar{B} + \bar{D}$.

Being defined by a matrix inequality, it is obvious that $\mathcal{P}$ is a closed and convex set. If regularity holds, $\mathcal{P}$ is also bounded with a minimum and maximum element.

From now on we shall assume that y is a regular process and fix an unmixing additive decomposition (6) of its (regular) spectral density $\Phi(z)$.

Lemma 16: Let $W(z) = C(zI - A)^{-1}B + D$ be a minimal realization of a minimal spectral factor of a regular spectral density $\Phi(z)$. Then D has full row rank m. In particular, if $W(z)$ is square then D is an invertible $m \times m$ matrix.

Then all solutions of the Algebraic Riccati equation correspond 1:1 to square minimal spectral factors and hence to minimal stochastic realizations, of the form (1) where (A is just unmixing and) the input white noise process w has the same dimension m of y. Hence for any square spectral factor the input noise process w can be uniquely recovered from y by inverting W, obtaining a (minimal) realization of the whitening filter mapping $y \to w$. By regularity this filter can be written as

$$\begin{cases} x(t+1) = \Gamma x(t) + BD^{-1}y(t) \\ w(t) \quad\;\; = -D^{-1}Cx(t) + D^{-1}y(t) \end{cases} \tag{39}$$

where the matrix

$$\Gamma = A - BD'\Delta(P)^{-1}C \tag{40}$$

describes the so-called *zero dynamics* of the system. The general formula (40) holds for non-square spectral factors and reduces to $\Gamma = A - BD^{-1}C$ when D is square since in this case $D'\Delta(P)^{-1} = D'(DD')^{-1} = D^{-1}$. We call Γ the *numerator matrix* of the model (1) since for square spectral factors the eigenvalues of Γ are exactly the poles of the inverse $W(z)^{-1}$ counting multiplicity. That in this setting we can actually eliminate the noise w from the state equation in the model (39), has the consequence that the state process x of a minimal realization of a square spectral factor belongs to $\mathbf{H}(y)$; that is, it is the state of an internal realization. The transfer function $y \to x$ is in turn parametrized by the numerator matrix Γ and hence its dynamics depends only on the *zeros* of the corresponding spectral factors. One may say that there is a 1:1 correspondence between the zero structure and each state space $\mathbf{X}$ in the family of minimal internal Markovian splitting subspaces, the particular choice of the A matrix being actually irrelevant.

Recall now that the matrices B, D are uniquely parametrized (mod $\mathcal{O}$) by solutions P of the LMI 10 through the relations

$$BD' = \bar{C}' - APC' \tag{41a}$$
$$DD' = \Lambda_0 - CPC' \tag{41b}$$

In particular all B, D corresponding to solutions $P \in \mathcal{P}$ of the algebraic Riccati inequality (35) make the numerator matrix a function of P which shall occasionally be denote as Γ_P. From the regularity condition (33) we extract the following useful fact.

Proposition 17: All numerator matrices Γ_P, $P \in \mathcal{P}_0$ are non singular and

$$\sigma(\Gamma_P) \cup \sigma(\Gamma_P^{-1}) = \text{ zeros of } \Phi(z)\,,$$
$$\sigma(\Gamma_P) \cap \sigma(\Gamma_P^{-1}) \supset \text{ zeros of } \Phi(z) \text{ on the unit circle}\,.$$

counting multiplicity. If Γ_P has no reciprocal pairs of eigenvalues outside of the unit circle, the last relation is actually an equality.

Therefore there are two numerator matrices Γ_+ and Γ_- whose spectrum is contained in the region $\bar{\mathcal{D}}_+ := \{z; |z| \geq 1\}$ and $\bar{\mathcal{D}}_- := \{z; |z| \leq 1\}$. These are the numerators of the maximum and minimum phase spectral factors. Our goal in what follows will be to characterize the corresponding solutions P_+, P_- of the ARE by showing that they are the maximal and minimal solutions in the set $\mathcal{P}$. To this end we shall need to consider differences $P - P_0$ of elements of $\mathcal{P}$ with a fixed $P_0 \in \mathcal{P}_0$. Let us denote for simplicity $\Sigma := P - P_0$ and by Γ, Γ_0 the corresponding numerator matrices. To proceed we shall need to define several intermediate quantities such as the $n \times m$ matrix

$$K := \Gamma\Sigma C'\Delta(P_0)^{-1} = \Gamma\Sigma C'(\Lambda_0 - CP_0C')^{-1} \tag{42}$$

and a "gain" matrix inspired to the steady state Kalman filter gain[1],

$$G := K + BD'\Delta(P)^{-1} \tag{43}$$

so that $\Gamma - KC = A - GC$ and the action of feedback by the matrix K (it would actually be more accurate to call this *output injection*) on the numerator matrix Γ is the same as that of G on the matrix A. We shall then prove a useful identity.

[1]In which case $P_0 = P_-$.

Lemma 18: Let B and D be solutions of (41) with $P \in \mathcal{P}$; then

$$G = (A\Sigma C' + BD')\Delta(P_0)^{-1} = B_0D_0'\Delta(P_0)^{-1} \tag{44}$$

For square spectral factors the action of feedback on the numerator matrix has a remarkable effect.

Theorem 19: Let $W_i(z) = C(zI - A)^{-1}B_i + D_i$ $i = 1, 2$ be two minimal square spectral factors parametrized by two elements P_1, P_2 in $\mathcal{P}_0$ and let Γ_i, $i = 1, 2$ be the corresponding numerator matrices. Set $\Sigma := P_1 - P_2$ and let K_1 be defined as in (42) with $P_0 \equiv P_2$; then

$$\Gamma_1 - K_1C = \Gamma_2 \tag{45}$$

Hence the feedback with gain K_1 on the numerator matrix Γ_1 flips the zeros of $W_1(z)$ to those of $W_2(z)$.

The following lemma establishes that for $P \in \mathcal{P}$, the difference $\Sigma = P - P_0$ satisfies a quadratic equation which can be seen as a generalization of the algebraic Riccati equation which enters in the analysis of the steady-state Kalman filter.

Lemma 20: Let $Q \in \mathbb{S}_+^{n\times n}$ and $P = R^{-1}(Q) \in \mathcal{P}$ be an arbitrary solution of the ARI (35) and let P_0 be a solution of the ARE (37). Then $\Sigma = P - P_0$ satisfies the Riccati equation

$$\Sigma = \Gamma\Sigma\Gamma' - L\Delta(P_0)L' + Q\,. \tag{46}$$

where Γ and L are defined in (40) and (42). For an arbitrary fixed P_0, all solutions of the ARI (35) are in one-to-one correspondence with solutions of (46) via the relation $P = \Sigma + P_0$.

Dually, $\bar{\Sigma} = P_0 - P = -\Sigma$, satisfies

$$\bar{\Sigma} = \Gamma_0\bar{\Sigma}\Gamma_0' - L_0\Delta(P)L_0' - Q \tag{47}$$

where $L_0 := \Gamma_0\bar{\Sigma}C'\Delta(P)^{-1}$.

Hence we have a parametrization of $\mathcal{P}$ as the set of translates by an arbitrary element $P_0 \in \mathcal{P}_0$, of the solution set of the Riccati equation (46), when Q ranges on $\mathbb{S}_+^{n\times n}$. Letting $Q = 0$ in (46) and (47) we obtain an analogous description of $\mathcal{P}_0$.

Corollary 21: Let P_1 and P_2 be two solutions of the ARE; then the difference $\Sigma := P_1 - P_2$ satisfies the following homogeneous quadratic equation

$$\Sigma = \Gamma_1\Sigma\Gamma_1' - L_1\Delta_2L_1' \tag{48}$$

while the difference $\bar{\Sigma} = P_2 - P_1 = -\Sigma$, satisfies

$$\bar{\Sigma} = \Gamma_2\bar{\Sigma}\Gamma_2' - L_2\Delta_1L_2' \tag{49}$$

where $\Delta_i = \Delta(P_i)$, $i = 1, 2$ and $L_i := \Gamma_i\bar{\Sigma}C'\Delta_j^{-1}$. Conversely, for any Σ satisfying (48) and fixed $P_2 \in \mathcal{P}_0$, the matrix $P_1 := \Sigma + P_2$ satisfies the ARE (37); i.e. $P_1 \in \mathcal{P}_0$. The alternative expressions

$$\Sigma = (\Gamma_1 - L_1C)\Sigma(\Gamma_1 - L_1C)' + L_1\Delta_2L_1' \tag{50a}$$
$$\bar{\Sigma} = (\Gamma_2 - L_2C)\bar{\Sigma}(\Gamma_2 - L_2C)' + L_2\Delta_2L_1' \tag{50b}$$

are equivalent to (48) and (49).

Note that (48) can be written also as

$$\Sigma = \Gamma_1\Sigma[\Gamma_1 - \Gamma_1\Sigma C'\Delta_2^{-1}C]' = \Gamma_1\Sigma[\Gamma_1 - L_1C]' = \Gamma_1\Sigma\Gamma_2' \tag{51}$$

and since by regularity, all numerator matrices cannot have eigenvalues at zero as these would be zeros of the spectral density, this implies

$$\Gamma_1^{-1}\Sigma = \Sigma\Gamma_2' \tag{52}$$

which leads to the following statement.

Theorem 22: Let Σ *be a solution of the Riccati equation* (48)*; then in the orthogonal decomposition*

$$\mathbb{R}^n = \ker\Sigma \oplus \mathrm{Im}\ \Sigma \tag{53}$$

$\ker\Sigma$ *is an invariant subspace for both* Γ_2' *and* Γ_1' *while* $\mathrm{Im}\ \Sigma$ *is an invariant subspace for* Γ_2 *and* Γ_1. *In a basis adapted to the orthogonal decomposition* (53) Σ *transforms into a block diagonal* $\hat\Sigma = \mathrm{diag}\{0,\ \hat\Sigma_2\}$*, the restrictions of* Γ_2 *and* Γ_1 *to* $\ker\Sigma$ *coincide, while the restrictions to* $\mathrm{Im}\ \Sigma$ *satisfy the relation*

$$\hat\Sigma_2 = \hat\Gamma_1\hat\Sigma_2\hat\Gamma_2' \tag{54}$$

In other words, $\hat\Gamma_2'$ *is similar to* $\hat\Gamma_1^{-1}$ *so that the eigenvalues of* $\hat\Gamma_2$ *are flipped to the reciprocals of those of* $\hat\Gamma_1$.

If $P_2 = P_-$ is the solution of the ARE yielding the numerator matrix Γ_- with spectrum in $\bar{\mathcal{D}}_-$, one may show by a Lyapunov argument, that the solution of (51) are nonnegative definite; i.e. $\Sigma \geq 0$, so that $P_1 \geq P_-$ for all $P_1 \in \mathcal{P}_0$.

For a square matrix M, let us denote:

$$\mathcal{L}_<(M)\,, \quad \mathcal{L}_1(M)\,, \quad \mathcal{L}_>(M)\,,$$

the (invariant) eigenspaces corresponding to eigenvalues, respectively inside the open unit disc, on the unit circle and outside of the closed unit disc.

Lemma 23 (See Lemma 3.1 and 3.2 of [10]): Let P *be a solution of the ARE and let* $X := P - P_-$ *and* $Y := P_+ - P$*; then* X *and* Y *satisfy the equations*

$$X = \Gamma_-^{-1}X\Gamma_-^{-\prime} - XC'\Delta(P_-)^{-1}CX \tag{55}$$

$$Y = \Gamma_+^{-1}Y\Gamma_+^{-\prime} + YC'\Delta(P)^{-1}CY \tag{56}$$

and for every P *solving the ARE, the subspace* $\mathrm{Ker}\ X$ *is* Γ_-'*-invariant and* $\mathrm{Ker}\ Y$ *is* Γ_+'*-invariant. Moreover*

$$\mathcal{L}_1(\Gamma_-') \subset \mathrm{Ker}\ X \qquad \mathcal{L}_1(\Gamma_+') \subset \mathrm{Ker}\ Y \tag{57}$$

for every $P \in \mathcal{P}_0$.

The main result of the theory of the discrete algebraic Riccati equation then follows.

Theorem 24: The numerator matrices Γ_+ *and* Γ_- *correspond to the extreme solutions* P_+ *and* P_- *of the ARE, that is*

$$\Gamma_+ = \Gamma_{P_+} = A - B_+D_+^{-1}C\,, \qquad \Gamma_- = \Gamma_{P_-} = A - B_-D_-^{-1}C\,. \tag{58}$$

where $P_- \leq P \leq P_+$ *for all* $P \in \mathcal{P}_0$ *and*

$$B_+D_+' = \bar C' - AP_+C', \qquad D_+D_+' = \Lambda_0 - CP_+C'$$
$$B_-D_-' = \bar C' - AP_-C', \qquad D_-D_-' = \Lambda_0 - CP_-C'\,.$$

If and only if the spectral density $\Phi(z)$ *has no zeros on the unit circle,*

- *All eigenvalues of* Γ_+ *have modulus strictly greater than one; i.e.* $\sigma(\Gamma_+) \subset \mathcal{D}_+ = \{z; |z| > 1\}$
- *All eigenvalues of* Γ_- *have modulus strictly less than one; i.e.* $\sigma(\Gamma_-) \subset \mathcal{D}_- = \{z; |z| < 1\}$
- *The* gap $P_+ - P_-$ *is positive definite, in particular non singular.*

Hence there are as many P's (or as many Γ_P) in $\mathcal{P}_0$ as there are *zero flipping* feedback transformations (45) to reciprocal positions of some of the zeros of W_+ or W_-, keeping in mind that the zeros on the unit circle remain fixed in the flipping process. Since on $\mathrm{Ker}\ X$ we have $\Gamma_P = \Gamma_-$ while on $\mathrm{Ker}\ Y$ we have $\Gamma_P = \Gamma_+$, all numerator matrices and the solutions $P \in \mathcal{P}_0$ can be parametrized also in terms of projections onto these invariant subspaces

$$P = \Pi_-P_- + (I - \Pi_-)P_+\,, \qquad P \in \mathcal{P}_0$$

Since this is well-known from the literature we shall not report on these refinements.

References

[1] Brian D. O. Anderson. The inverse problem of stationary covariance generation. *J. Statist. Phys.*, 1:133–147, 1969.

[2] P. Faurre. Realisations markoviennes de processus stationnaires. Technical Report 13, INRIA (LABORIA), Le Chesnay, France, March 1973.

[3] A. Ferrante, G. Picci, and S. Pinzoni. Silverman algorithm and the structure of discrete time stochatic systems. *Linear Algebra and its Applications*, 351-352:219–242, 2002.

[4] K. Glover. All optimal hankel norm approximations of linear multivariable systems and their l^∞ error bounds. *Intern. J. Control*, 39:1115–1193, 1984.

[5] R. E. Kalman. Lyapunov functions for the problem of Lur′e in automatic control. *Proc. Nat. Acad. Sci. U.S.A.*, 49:201–205, 1963.

[6] A. Lindquist and G. Picci. *Linear stochastic systems: a geometric approach*. In preparation, 2014.

[7] Anders Lindquist and Giorgio Picci. On the stochastic realization problem. *SIAM J. Control Optim.*, 17(3):365–389, 1979.

[8] M. Pavon. Stochastic realization and invariant directions of the matrix Riccati equation. *SIAM J. Control and Optimization*, 28:155–180, 1980.

[9] V. M. Popov. Hyperstability and optimality of automatic systems with several control functions. *Rev. Roumaine Sci. Tech. Sér. Électrotech. Énergét.*, 9:629–690, 1964.

[10] ACM Ran and HL Trentelman. Linear quadratic problems with indefinite cost for discrete time systems. *SIAM journal on matrix analysis and applications*, 14(3):776–797, 1993.

[11] Jan C. Willems. Least squares stationary optimal control and the algebraic Riccati equation. *IEEE Trans. Automatic Control*, AC-16:621–634, 1971.

[12] Harald K. Wimmer. A parametrization of solutions of the discrete-time algebraic riccati equation based on pairs of opposite unmixed solutions. *SIAM J. Control and Optimization*, 44(6):19922005, 2006.

[13] Harald K. Wimmer and Allen D. Ziebur. Remarks on inertia theorems for matrices. *Czechoslovak Mathematical Journal*, 25(4):556–561, 1975.

[14] V. A. Yakubovich. The solution of some matrix inequalities encountered in automatic control theory. *Dokl. Akad. Nauk SSSR*, 143:1304–1307, 1962.