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McMillan Degree of Impedance, Admittance Functions of RLC Networks

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Abstract— **We examine RLC networks described by the impedance, or admittance matrices, which have a natural topology associated with the *R*, *L*, *C* structural matrices of the network. The McMillan degree of the integral- differential description introduced by the impedance and admittance matrices is related to the rank properties of the *R*, *L*, *C* structural matrices. We prove that the maximum possible value of the McMillan degree is given as the sum of the ranks of the capacitance and inductance matrices. We also provide necessary and sufficient conditions, of determinantal or of rank type, for this degree to be achieved.**

I. INTRODUCTION

Classical network theory [1], [2], [3], [4] introduces for a large family of systems an integral- differential description, in terms of the impedance and admittance models, which in turn provide an implicit system description. In this description the natural topology of the network introduced by the *R*, *L*, and *C* structural matrices is explicitly described. These type of networks may be considered within the systems theory setting [5], modelled as a set of integro-differential equations relating the basic variables of the network i.e. the vectors of currents and voltage sources [3], [4]. In the frequency domain these equations are transformed to the so called loop or impedance model where a rational matrix of the form $W(s) = sL + R + (1/s)C$ plays the role of a generalised transfer function [6]. The properties of $W(s)$ are central to the study of the network from the system theory point of view [6]. In this work we address the problem of associating the McMillan degree of $W(s)^{-1}$ with the matrices R,L,C of the network elements. The McMillan degree defines the minimum number of dynamic elements needed to describe the network fully. A result which is intuitively known but not rigorously proven in the circuit literature [1], [7], [3] is that this degree has to be equal to the minimum independent number of capacitors and inductors in the circuit. The main theoretic tools which were used for the derivation of the following results are given in the framework of compound matrices and exterior algebra [8]. Here we examine rigorously this question proving that the maximum possible McMillan degree of such networks is given by $rank(L) + rank(C)$ and this value is attained provided some regularity (or independence) conditions are valid for the network. These conditions are necessary and sufficient, i.e. optimal, and they are expressed in various forms that are all testable [6]. The first set of conditions are of determinantal type and relate the highest and lowest order coefficients of s in the expansion of the determinant $det(s^2L + sR + C)$ to the matrices L,R,C. The second set of conditions relates the property of these coefficients to be non-zero with some rank properties of matrices related to the three fundamental matrices R,L,C. These conditions imply some regularity properties for the network similar to the ones considered intuitively in the literature. This result and analysis may be used as a starting point for a more general study of this type of networks in terms of its algebraic properties such as the study of the McMillan form of $W(s)$ the nature of zero elementary divisors or the structure of the pole divisors. Such considerations will make possible the use of system and control theoretic tools in RLC network theory and will facilitate the definition and solution of new analysis and design problems. The central results in this paper are given with proof. A complete treatment may be found in the technical report[6].

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II. PROBLEM STATEMENT

In this paper we consider the dynamic properties of an RLC network, as described by its impedance model [1], [2]. In this model the variables are selected such that the vertex law is automatically satisfied. The process of working out the equations involves the selection of internal independent loops, the definition of loop currents and the transformation of current sources to equivalent voltage sources.

If we denote by $(f_1, f_2, ..., f_q)$ the set of the Laplace transforms of the loop currents and by $(u_{s1}, ..., u_{sq})$ the set of Laplace transforms of equivalent voltage sources, then the loop or impedance model [2] is defined by: $Z(s)$, where $z_{ii}(s)$ is the sum of impedances in loop and $z_{ij}(s)$ is the sum of impedances common between loops i and j. These equations can be written in short as:

$$Z(s)f(s) = u_s(s). \tag{1}$$

This is referred to as the loop or impedance model and the symmetric matrix Z(s) is referred to as the network impedance matrix.

Similarly, the node or admittance model is described in short as:

$$Y(s)v(s) = f_s(s). \tag{2}$$

and the symmetric matrix Y(s) is referred to as the network admittance matrix. The general modelling for passive network provides a description of networks in terms of

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symmetric integral-differential operators, the impedance and admittance models which are described in a general way by the unified operator [9]:

$$W(s) = sL + s^{-1}C + R \tag{3}$$

where for the case of impedance we have that L is the matrix of inductances, C is the matrix of capacitances and R is the matrix of resistances. The operator W(s) is thus a common description of the Y(s) and Z(s) matrices and its properties will be investigated next. Clearly, the W(s) matrix is symmetric and the structure of L, C, R matrices characterizes the topology associated with the network [2]. Such matrices have a structure and properties that underpin the development of system theoretic framework based on network models. The operator W(s) describes the dynamics of the network and of special interest are the properties of its zeros [6], [10]. Furthermore, this integral - differential operator, defined by $W(s)$, introduces a *new implicit system description*:

$$(p\mathbf{L} + {}^{1}\!/_{p}\mathbf{C} + \mathbf{R}) \cdot \xi = 0$$

where ξ can be seen as an internal vector. Clearly, the above description has no inputs and no outputs but as a rational matrix, $W(s)$, has a McMillan degree which is linked to a notion of *minimality* of the implicit description.

The McMillan degree [11] of the system may be computed via various methods, i.e. by determining the Smith-McMillan form or via exterior algebra and a determinantal treatment of the problem. The main purpose of the paper is, for a network that is described through the implicit description defined by the integral- differential operator $W(s)$ ($W(s) = sL + s^{-1}C + R$), to relate its McMillan degree to the properties of the structured matrices R, L, C, to derive testable conditions and to interpret the results.

III. The McMillan degree of Impedance / Admittance functions of an RLC network. The main results.

In this section the main results for the McMillan degree calculation are presented and necessary and sufficient conditions are given in terms of composite matrices for this are derived. From this results an interpretation about the network equations.

A. McMillan Degree Calculation

The McMillan degree in an *RLC* network can be easily calculated. In this paper we propose an alternative method which can be easily implemented. In this section two important theorems are stated. In the first, the McMillan degree of a network is computed in terms of the maximum and minimum coefficients of s in the polynomial matrix $det(W(s))$.

Theorem 3.1: Let $W(s)^{-1}$ be the transfer function of an *RLC* network, where $W(s) = sL + R + \frac{1}{s}C$ and $W(s)$ non-singular. Then the *McMillan degree* of $W(s)^{-1}$ is given by:

$$\delta_m = n_{\max} - n_{\min}$$

where $n_{\max}$ and $n_{\min}$ are the maximum and minimum degrees of s in the expansion of the determinant:

$$\det(s^2L + sR + C).$$

□

Remark 3.1: : It may be easily proven that $W(s)$ has always full rank, i.e. is non-singular (for proof see [6]).

The second theorem deals with the values of these coefficients and provides necessary conditions in terms of the rank of the matrices of dynamical elements in the network.

Theorem 3.2: Let $A(s) = s^2L + sR + C$ with $rank(L) = p$, $rank(C) = q$ and let the polynomial $\det[A(s)] = \alpha s^{n_2} + \cdots + \beta s^{n_1}$ with the powers in descending order. Then: $n_2 - n_1 \leq p + q$ with the equality iff $n_2 = n + p$ and $n_1 = n - q$.

□

B. Necessary and sufficient conditions

This subsection deals with the necessary and sufficient conditions that must hold in order to compute the McMillan degree of an RLC network. Because each network has a unique representation, it was considered necessary to benefit from the framework of exterior algebra, and in particular to use compound matrices [8].

Theorem 3.3: Let $A(s) = s^2L + sR + C$ the matrix representation of a RLC circuit. Let $k_{\max}$, $k_{\min}$, $n_{\max}$, $n_{\min}$ be the maximum and minimum coefficients and degrees of the determinant $\det[A(s)]$ respectively. Assume also that $rank(L) = p$, $rank(C) = q$ which implies that

$$C_p(L) = \alpha_1 \cdot \alpha_2^t \ , \ \alpha_1, \alpha_2 \in R^{\binom{n}{p} \times 1}$$

and that

$$C_q(C) = \beta_1 \cdot \beta_2^t \ , \ \beta_1, \beta_2 \in R^{\binom{n}{q} \times 1}$$

Then the following hold true:

(i) When $p < n$ then: $n_{\max} \leqslant n + p$ and $n_{\max}$ takes the maximum possible value $n + p$ iff

$$k_{\max} = tr(C_p(L) \cdot Adj_p(R)) = \alpha_2^t \cdot Adj_p(R) \cdot \alpha_1 \neq 0.$$

In the case where $n = p$ then $n_{\max}$ takes the maximum possible value, $2n$, iff:

$$k_{\max} = \det(L) \neq 0.$$

(ii) When $q < n$ then: $n_{\min} \geqslant n - q$ and $n_{\min}$ takes the minimum possible value $n - q$ iff

$$k_{\min} = tr(C_q(C) \cdot Adj_q(R)) = \beta_2^t \cdot Adj_q(R) \cdot \beta_1 \neq 0.$$

In the case, where $n = q$ then $n_{\min}$ takes the minimum possible value, 0, iff:

$$k_{\min} = \det(C) \neq 0.$$

Proof: Denote l_i,r_i,c_i the columns of the matrices $\mathbf{L}$,$\mathbf{R}$,$\mathbf{C}$ respectively. The $\det(A(s))$ is a sum of terms:

$$(-1)^{\sigma} \cdot \underbrace{l_{i_1} \wedge l_{i_2} \wedge \cdots \wedge l_{i_{f_1}}}_{f_1 \text{from} L} \wedge \underbrace{r_{j_1} \wedge r_{j_2} \wedge \cdots \wedge r_{j_{f_2}}}_{f_2 \text{from} R} \wedge$$

$$\underbrace{\wedge c_{m_1} \wedge c_{m_2} \wedge \cdots \wedge c_{m_{n-f_1-f_2}}}_{n-f_1-f_2 \text{from} C} \cdot s^{2f_1+f_2} \tag{4}$$

(i) To find the maximum possible degree of the polynomial $\det(A(s))$ we have to solve the *integer-programming problem:*

$$\begin{gathered}\max n = 2f_1 + f_2 \\ \text{s.t.} \\ f_1, f_2 \geq 0, f_1 + f_2 \leq n, f_1 \leq p, n - f_1 - f_2 \leq q\end{gathered}$$

This has the obvious solution: $f_1 = p, f_2 = n-p$ and $n_{\max} = 2p + n - p = n + p$ i.e. take p columns from $\mathbf{L}$ and $n-p$ columns from $\mathbf{R}$. In this case:

$$k_{\max} = \sum_{\omega \in Q_n^p} A_\omega$$

where A_ω are all $n \times n$ determinants of matrices formed by p rows of $\mathbf{L}$ and $n-p$ complementary rows of $\mathbf{R}$. For a given selection of columns of $\mathbf{L}$: $\omega = (i_1, i_2, \cdots, i_p) \in Q_n^p$ the *Laplace Expansion Theorem* gives:

$$A_\omega = \begin{vmatrix} r_{j_1} \\ l_{i_1} \\ r_{j_2} \\ l_{i_2} \\ \vdots \\ l_{i_p} \\ r_{j_{n-p}} \end{vmatrix} = \sum_{\beta \in Q_n^p} C_p(\mathbf{L})_{\omega,\beta} \cdot Adj_p(\mathbf{R})_{\beta,\omega}$$

Therefore,

$$\sum A_\omega = \sum_{\omega \in Q_n^p} \sum_{\beta \in Q_n^p} C_p(\mathbf{L})_{\omega,\beta} \cdot Adj_p(\mathbf{R})_{\beta,\omega} =$$

$$= tr(C_p(\mathbf{L}) \cdot Adj_p(\mathbf{R}))$$

Since, $\mathbf{L}$ has rank p we have: $C_p(\mathbf{L}) = \alpha_1 \cdot \alpha_2^t$. Thus

$$k_{\max} = tr(C_p(\mathbf{L}) \cdot Adj_p(\mathbf{R})) = \alpha_2^t \cdot Adj_p(\mathbf{R})) \cdot a_1$$

When $p = n$ then: $n_{max} = 2n$, i.e. take all the columns from $\mathbf{L}$ and 0 columns from $\mathbf{R}$. In this case:

$$k_{\max} = \det(L) \neq 0.$$

(ii) As $\det(A(s))$ is a sum of terms as in (4). To find the minimum degree we have to solve the *integer-programming problem:*

$$\begin{gathered}\min n = 2f_1 + f_2 \\ \text{s.t.} \\ f_1, f_2 \geq 0, f_1 + f_2 \leq n, n - f_1 - f_2 \leq q, f_1 \leq p\end{gathered}$$

which has the obvious solution: $f_1 = 0$ and $f_2 = n-q$. In this case: $n_{\min} = 2 \cdot 0 + n - q = n - q$. Then

$$k_{\min} = \sum_{\omega \in Q_n^q} B_\omega$$

where B_ω are all $\mathrm{n} \times \mathrm{n}$ determinants of matrices formed by q rows of C and $n-q$ complementary rows of R. For $\omega = (i_1, i_2, ..., i_q) \in Q_n^q$ using the *Laplace Expansion theorem* we have:

$$\mathrm{B}_\omega = \begin{vmatrix} r_{j_1} \\ c_{i_1} \\ r_{j_2} \\ c_{i_2} \\ \vdots \\ c_{i_q} \\ r_{j_{n-q}} \end{vmatrix} = \sum_{\beta \in Q_n^p} C_q(\mathbf{C})_{\omega,\beta} \cdot Adj_q(\mathbf{R})_{\beta,\omega}$$

Therefore,

$$k_{\min} = \sum B_\omega = \sum_{\omega \in Q_n^p} \sum_{\beta \in Q_n^p} C_q(\mathbf{C})_{\omega,\beta} \cdot Adj_q(\mathbf{R})_{\beta,\omega} =$$

$$= tr(C_q(\mathbf{C}) \cdot Adj_q(\mathbf{R}))$$

Since, $\mathbf{C}$ has rank q we have: $C_q(\mathbf{C}) = \beta_1 \cdot \beta_2^t$ proving that:

$$k_{\min} = tr(C_q(\mathbf{C}) \cdot Adj_q(\mathbf{R})) = \beta_2^t \cdot Adj_q(\mathbf{R})) \cdot \beta_1$$

When $n = q$ then: $n_{min} = 0$, i.e. take all the columns from $\mathbf{C}$ and 0 columns from $\mathbf{R}$. In this case:

$$k_{\min} = \det(C) \neq 0.$$

□

Next, the necessary conditions are settled for the minimum and maximum coefficients of $det(W(s))$ to be non-zero.

Proposition 3.1: (1) A *necessary condition* for $k_{n+p} \neq 0$, is that the matrices $\begin{bmatrix} L & R \end{bmatrix}$ have *full rank.*

(2) A *necessary condition* for $k_{n-q} \neq 0$, is that the matrices $\begin{bmatrix} R & C \end{bmatrix}$ have *full rank.*

□

The following proposition provides an alternative description for the computation of the maximum coefficient in the expression of the determinant of $W(s)$. This expression is given with respect to the composite matrices that are formed from the matrices R,L,C.

Proposition 3.2: Let $L = L' \cdot L''$, $L' \in R^{n \times p}$, $L'' \in R^{p \times n}$, $p < n$. Then:

$$C_p(L'') \cdot Adj_p(R) \cdot C_p(L') = (-1)^p \cdot \begin{vmatrix} R & L' \\ L'' & 0 \end{vmatrix}$$

□

Similarly, a different expression is derived for the minimum coefficient.

Corollary 3.1: Let $C = C' \cdot C''$, $\mathbf{C}' \in R^{n \times q}$, $\mathbf{C}'' \in R^{q \times n}$, $q < n$. Then:

$$C_q(C'') \cdot Adj_q(R) \cdot C_q(C') = (-1)^q \cdot \begin{vmatrix} R & C' \\ C'' & 0 \end{vmatrix}$$

□

The next theorem is crucial. It provides the necessary and sufficient conditions in order to determine the maximum coefficient. This is given in terms of the properties of the R,L,C structural matrices.

Theorem 3.4: (i) If $p < n$ then:

$$k_{n+p} = C_p(L'') \cdot Adj_p(R) \cdot C_p(L') \neq 0$$
$$iff\ rank\left(\begin{bmatrix} R & L \\ L & 0 \end{bmatrix}\right) = n + rank(L)$$

(ii) If $p = n$ then: $k_{2n} \neq 0$ iff $\det(L) \neq 0$, which is equivalent to: $rank\left(\begin{bmatrix} R & L \\ L & 0 \end{bmatrix}\right) = n + rank(L)$

Proof: Let $p = rank(\mathbf{L})$. Moreover,

$$rank\left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix}\right) \leq rank(\mathbf{L}) + rank\left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \end{bmatrix}\right) = n + p$$

Therefore, for $rank\left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix}\right) = n + p$, there must be $C_{n+p}\left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix}\right) \neq 0$.

Taking into account the identity:

$$\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{I_n} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}' \end{bmatrix} \cdot \begin{bmatrix} \mathbf{R} & \mathbf{L}' \\ \mathbf{L}'' & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}'' \end{bmatrix}$$

By the *Binet-Cauchy* theorem we have:

$$C_{n+p}\left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix}\right) = \det\left(\begin{bmatrix} \mathbf{R} & \mathbf{L}' \\ \mathbf{L}'' & \mathbf{0} \end{bmatrix}\right) \cdot$$
$$\cdot C_p\left(\begin{bmatrix} \mathbf{I_n} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}' \end{bmatrix}\right) \cdot C_{n+p}\left(\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}'' \end{bmatrix}\right)$$

Hence,

$$C_{n+p} = \left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix}\right) \neq 0$$

iff

$$\det\left(\begin{bmatrix} \mathbf{R} & \mathbf{L}' \\ \mathbf{L}'' & \mathbf{0} \end{bmatrix}\right) \neq 0$$

Since, $k_{n+p} = (-1)^p \cdot \det\left(\begin{bmatrix} \mathbf{R} & \mathbf{L}' \\ \mathbf{L}'' & \mathbf{0} \end{bmatrix}\right)$ (see proposition), we have that: $k_{n+p} \neq 0$ iff $rank\left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix}\right) = n + p$.

□

Correspondingly, for the minimum coefficient:

Corollary 3.2: (i) If $q < n$ then: $k_{n-q} \neq 0$ iff $rank\left(\begin{bmatrix} R & C \\ C & 0 \end{bmatrix}\right) = n + rank(C)$.

(ii) If $q = n$ then: $k_0 \neq 0$ iff $\det(C) \neq 0$, which is equivalent to: $rank\left(\begin{bmatrix} R & C \\ C & 0 \end{bmatrix}\right) = n + rank(C)$

□

Subsequently, summarising all the above:

Corollary 3.3: Let δ_m be the McMillan degree of $W^{-1}(s) = \left(sL + R + {}^1/_s C\right)^{-1}$. Then the following are equivalent:

(a) $\delta_m = rank(L) + rank(C)$.

(b) $rank\left(\begin{bmatrix} R & L \\ L & 0 \end{bmatrix}\right) = n + rank(L)$ and $rank\left(\begin{bmatrix} R & C \\ C & 0 \end{bmatrix}\right) = n + rank(C)$.

□

Remark 3.2: The link between the McMillan degree and the structural matrices of the network dynamical elements is summarised below.

□

Corollary 3.4: The necessary conditions for $\delta_m = rank(L) + rank(C)$ are:

(a) $rank\left(\begin{bmatrix} R & L \end{bmatrix}\right) = n$.

(b) $rank\left(\begin{bmatrix} R & C \end{bmatrix}\right) = n$.

(c) $rank(R) \geqslant n - \min(rank(L), \text{rank}(C))$.

□

C. Interpretation of results

This subsection provides an interpretation of the results derived before. This is linked to the equations of each RLC network.

Proposition 3.3: $rank\begin{bmatrix} R & L \\ L & 0 \end{bmatrix} < n + rank(L)$ *iff* there exist vectors $x_1, x_2 \in \mathbb{R}^n$ with $x_1 \neq 0$ such that:

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} R & L \\ L & 0 \end{bmatrix} = 0.$$

□

Corollary 3.5: Let δ_m be the McMillan degree of $W^{-1}(s) = \left(sL + R + {}^1/_s C\right)^{-1}$. Then the following are equivalent:

(a) $\delta_m < rank(L) + rank(C)$.

(b) There is a network equation of the form of one of the following types:

- $\left(-x_2 L + {}^1/_s x_1 C\right) \underline{i}(s) = x_1 \underline{v}(s)$ where $x_1 \neq 0$, $x_1 L = 0$

or

- $(s x_1 L - x_2 C)\, \underline{i}(s) = x_1 \underline{v}(s)$ where $x_1 \neq 0$, $x_1 C = 0$.

□

IV. Examples

In this section we make use of all the above by demonstrating two examples.

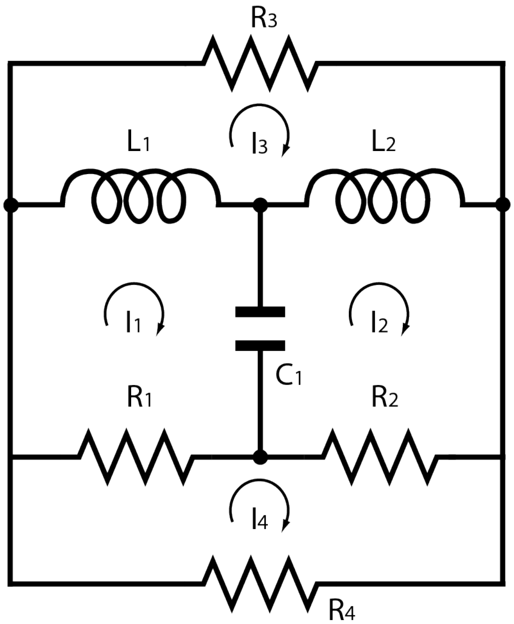
Example 1: First, let us investigate an RLC network with $n = 4$ loops, 2 inductors and 1 capacitor arranged as shown in the figure below. The operator $W(s) = s^2 L + sR + C$ is given by the following matrices:

$$L = \begin{bmatrix} L_1 & 0 & -L_1 & 0 \\ 0 & L_2 & -L_2 & 0 \\ -L_1 & -L_2 & L_1 + L_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (5)$$

$$R=\begin{bmatrix} R_1 & 0 & 0 & -R_1 \\ 0 & R_2 & 0 & -R_2 \\ 0 & 0 & R_3 & 0 \\ -R_1 & -R_2 & 0 & R_1+R_2+R_4 \end{bmatrix} \tag{6}$$

$$C=\begin{bmatrix} C_1 & -C_1 & 0 & 0 \\ -C_1 & C_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

By inspection: $rank(L)=p=2$ and $rank(C)=q=1$. Using



RLC network with n=4, p=2, q=1

the formulas derived in this paper we may find the minimum and maximum coefficients of the determinant of the $W(s)$ operator. For these coefficients we need to compute:

(I) $C_p(L)=C_2(L)$, because $p=2$.
(II) $C_q(C)=C_1(C)=C$, because $q=1$.
(III) $Adj_q(R)=Adj_1(R)$ and $Adj_p(R)=Adj_2(R)$.

Thus, we have:

$$C_2(L)=\begin{bmatrix} L_1L_2 & -L_1L_2 & 0 & L_1L_2 & 0 & 0 \\ -L_1L_2 & L_1L_2 & 0 & -L_1L_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ L_1L_2 & -L_1L_2 & 0 & L_1L_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}=$$
$$=\underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\alpha_1}\cdot\underbrace{\begin{bmatrix} L_1L_2 & -L_1L_2 & 0 & L_1L_2 & 0 & 0 \end{bmatrix}}_{\alpha_2^t} \tag{8}$$

$$C_1(C)=C=\begin{bmatrix} C_1 & -C_1 & 0 & 0 \\ -C_1 & C_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}=$$
$$=\underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}}_{\beta_1}\cdot\underbrace{\begin{bmatrix} C_1 & -C_1 & 0 & 0 \end{bmatrix}}_{\beta_2^t} \tag{9}$$

Finally, for the compound adjoints of R we have that:

$$Adj_1(R)=$$
$$=\begin{bmatrix} R_2R_3(R_1+R_4) & R_1R_2R_3 & 0 & R_1R_2R_3 \\ R_1R_2R_3 & R_1R_3(R_2+R_4) & 0 & R_1R_2R_3 \\ 0 & 0 & R_1R_2R_4 & 0 \\ R_1R_2R_3 & R_1R_2R_3 & 0 & R_1R_2R_3 \end{bmatrix} \tag{10}$$

$$Adj_2(R)=$$
$$=\begin{bmatrix} a_{11} & 0 & R_2R_3 & 0 & -R_1R_3 & 0 \\ 0 & a_{22} & 0 & R_1R_2 & 0 & -R_1R_2 \\ R_2R_3 & 0 & R_2R_3 & 0 & 0 & 0 \\ 0 & R_1R_2 & 0 & a_{44} & 0 & -R_1R_2 \\ -R_1R_3 & 0 & 0 & 0 & R_1R_3 & 0 \\ 0 & -R_1R_2 & 0 & -R_1R_2 & 0 & R_1R_2 \end{bmatrix} \tag{11}$$

where $a_{11}=R_3(R_1+R_2+R_4)$, $a_{22}=R_2(R_1+R_4)$, $a_{44}=R_1(R_2+R_4)$.
Hence, for the maximum and minimum coefficients using the following formulas:

$$k_{\max}=\alpha_2^t\cdot Adj_p(R)\cdot\alpha_1 \text{ and } k_{\min}=\beta_2^t\cdot Adj_q(R)\cdot\beta_1$$

we finally find that:

$$K_{\min}=C_1(R_1+R_2)R_3R_4$$

$$K_{\max}=L_1L_2(R_3R_4+R_1(R_3+R_4)+R_2(R_3+R_4))$$

$$=L_1L_2R_3R_4+L_1L_2R_1R_3+L_1L_2R_1R_4+L_1L_2R_2R_3+L_1L_2R_2R_4$$

and by substracting their corresponding degrees n_{max}, n_{min} we get the McMillan degree: $\delta_\mu=1$.
Alternatively, we may use the composite matrices as denoted in Proposition (2) and Corollary (1):

$$(-1)^q\begin{vmatrix} R & C' \\ C'' & 0 \end{vmatrix} \tag{12}$$

$$(-1)^p\begin{vmatrix} R & L' \\ L'' & 0 \end{vmatrix} \tag{13}$$

Firstly, we need to decompose matrix C from Corollary 1 to its corresponding dyads, $C=C'\cdot C''$, as indicated below, where $C'\in\mathbb{R}^{4\times 1}$ and $C''\in\mathbb{R}^{1\times 4}$. Then, C can be written as:

$$C=\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} C_1 & -C_1 & 0 & 0 \end{bmatrix}$$

Hence, the composite matrix which used to calculate the minimum coefficient of the $det(W(s))$ operator, $Kmin$, is expressed as:

$$(-1)^q \begin{vmatrix} R & C' \\ C'' & 0 \end{vmatrix} =$$

$$= (-1) \cdot \begin{vmatrix} R_1 & 0 & 0 & -R_1 & 1 \\ 0 & R_2 & 0 & -R_2 & -1 \\ 0 & 0 & R_3 & 0 & 0 \\ -R_1 & -R_2 & 0 & R_1+R_2+R_4 & 0 \\ C_1 & -C_1 & 0 & 0 & 0 \end{vmatrix}$$

Similarly, we need to decompose matrix L from Proposition 2 to its corresponding dyads, $L = L' \cdot L''$, where $L' \in \mathbb{R}^{4\times 2}$ and $L'' \in \mathbb{R}^{2\times 4}$. Then, L can be written as:

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} L_1 & 0 & -L_1 & 0 \\ 0 & L_2 & -L_2 & 0 \end{bmatrix}$$

and the composite matrix which used to calculate the highest coefficient $Kmax$ is expressed as:

$$(-1)^p \begin{vmatrix} R & L' \\ L'' & 0 \end{vmatrix} =$$

$$= (-1)^2 \cdot \begin{vmatrix} R_1 & 0 & 0 & -R_1 & 1 & 0 \\ 0 & R_2 & 0 & -R_2 & 0 & 1 \\ 0 & 0 & R_3 & 0 & -1 & -1 \\ -R_1 & -R_2 & 0 & R_1+R_2+R_4 & 0 & 0 \\ L_1 & 0 & -L_1 & 0 & 0 & 0 \\ 0 & L_2 & -L_2 & 0 & 0 & 0 \end{vmatrix}$$

Therefore, by computing the determinants of the composite matrices above we derive the minimum coeffiecient as:

$$K_{\min} = C_1(R_1+R_2)R_3R_4$$

and the maximum coefficient:

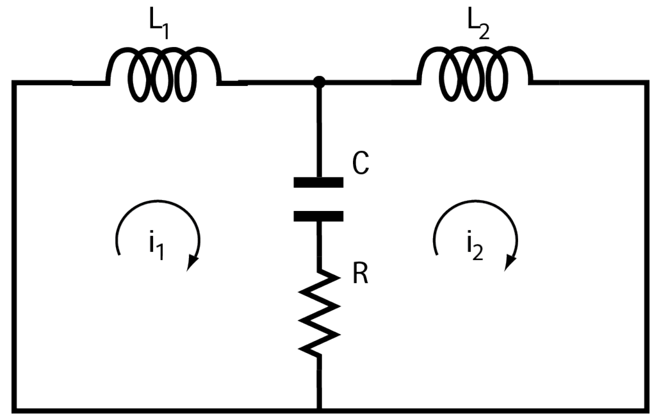
$$K_{\max} = L_1L_2R_3R_4 + L_1L_2R_1R_3 + L_1L_2R_1R_4 +$$
$$+ L_1L_2R_2R_3 + L_1L_2R_2R_4$$

exactly same as before. Thus, it is verified that both computational methods produces the same results, i.e. McMillan degree $\delta_\mu = 1$. Subsequently, if the composite matrices are formed (Theorem 4, Corrolary 2) it will be proven that the necessary and suficient conditions are met. Hence, the minimun and maximum coefficient of the McMillan degree are non-zero.

Example 2: Now, lets examine a simple RLC network with $n = 2$ loops, 2 inductors and 1 capacitor arranged as shown in the next Figure: The operator $W(s) = s^2L + sR + C$ for the RLC network is:

$$W(s) = s^2 \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} + s \begin{bmatrix} R_1 & -R_1 \\ -R_1 & R_1 \end{bmatrix} + \begin{bmatrix} C & -C \\ -C & C \end{bmatrix}$$

In this example, the McMillan degree of the system cannot exceed the total number of the network's dynamical elements i.e capacitors and inductors. Hence, the McMillan degree



RLC with n=2, p=2, q=1

should be $\delta_M = 3$. Let's calculate the actual McMillan degree of the network by computing the maximum and minimum coefficients and their corresponding degrees. In this case we have: $K_{\max} = L_1L_2 \cdot s^4$ and $K_{\min} = C(L_1+L_2) \cdot s^2$. As we can see, $\delta = K_{max} - K_{min} = 4 - 2 = 2$. This shows that the actual McMillan degree (i.e. $\delta = 2$) of the system is not the maximum achievable degree (i.e. $\delta_M = 3$), and this is due to the fact that the necessary and sufficient conditions of Corollary 3.3 are not met, i.e.

i. $rank \begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} R_1 & -R_1 & L_1 & 0 \\ -R_1 & R_1 & 0 & L_2 \\ L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \end{bmatrix} =$
$= n + rank(\mathbf{L}) = 4$
but

ii. $rank \begin{bmatrix} \mathbf{R} & \mathbf{C} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} R_1 & -R_1 & C & -C \\ -R_1 & R_1 & -C & C \\ C & -C & 0 & 0 \\ -C & C & 0 & 0 \end{bmatrix} = 2 \neq$
$\neq n + rank(\mathbf{C}) = 4.$

V. CONCLUSIONS

We have consider the implicit system descriptions introduced by the impedance and admittance functions of a network which provide a natural description of the distribution of the R, L, C elements in terms of the structure of the associated symmetric matrices R, L, C. Such implicit descriptions (in terms of the loop currents, or nodal voltages) have a McMillan degree. We have examined the problem of determination of maximum McMillan degree of such a network and a necessary and sufficient condition was given in terms of the properties of the matrices R, L, C. This condition expresses in a rigorous manner what was intuitively known for these types of networks, i.e. that this number is the sum of ranks of the capacitor and inductor matrices. The analysis also provides a new insight on the conditions for which an *RLC* network achieves or fails to achieve this maximum McMillan degree. The results provide some basic properties required to develop a systems theory based on integral differential descriptions. This framework is essential for defining a appropriate representations for studying the problem of network redesign.

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