

Synthesis of Microwave filters: a novel approach based on computer algebra

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Abstract — This paper presents an algebraic setting of the synthesis problem concerning coupled-resonators filters. The latter yields a rigorous understanding of the relationship between the coupling geometry and the corresponding set of realizable filtering characteristics as well as an algorithm, based on Groebner bases computations, which solves exhaustively the coupling matrix synthesis problem. Numerical results are presented for a 10^{th} order filter.

Index Terms — synthesis, coupling matrix, Groebner basis, microwave filters

I. INTRODUCTION

The synthesis procedures for multiple coupled resonators filter are usually composed of two related steps. The first one takes place in the frequency domain where a low pass characteristic is designed (for example using elliptic filtering functions). The latter is chosen so as to be in accordance with the frequency domain specifications (return loss, group delay) as well as with the anticipated coupling structure of the filter. In particular this presupposes a precise description of the set of filtering characteristics one can expect to realize with a given coupling geometry.

The second step called the realization step consists in the computation of a or all coupling matrices with the proper coupling geometry that synthesize the previous filtering function. A wide class of algorithms have been proposed to solve this problem based for example on the repetitive extraction of elementary sections [1] or on the application of a series of similarity transforms to the coupling matrix [2]. To our knowledge none of this methods is guaranteed to find all admissible coupling sets.

The purpose of this paper is to develop a formalism which allows a rigorous understanding of the relationship between the coupling geometry and the corresponding set of realizable filtering characteristics. In particular we will introduce the notion of non-redundant geometry which ensures in some sense the well posedness of the coupling matrix synthesis problem. Finally we will present an algorithm based on computer algebra which solves the latter in an exhaustive manner. Its use in practice have already been advocated in [3] for the synthesis of asymmetric filtering functions.

We first present a general framework concerning parameterized linear dynamical systems which we will adapt to our

filter synthesis situation in section III. Due to the format of the paper the proofs of the results are not detailed.

II. PARAMETERIZED LINEAR DYNAMICAL SYSTEMS

We introduce the following parameterized state space equations for linear dynamical systems,

$$\begin{aligned} \dot{x}(t) &= A(p)x(t) + B(p)u(t) \\ y(t) &= C(p)x(t) \end{aligned} \quad (1)$$

where $p = \{p_1, \dots, p_r\}$ is a finite set of r parameters and $(A(p), B(p), C(p))$ are matrices whose entries are polynomials (on the field \mathbb{C}) of the variables $p_1 \dots p_r$. Using standard notations (those of [4]) we have $A \in \mathbb{C}[p]^{n \times n}$, $B \in \mathbb{C}[p]^{n \times m}$, $C \in \mathbb{C}[p]^{k \times n}$ where k is the dimension of the input, n the dimension of the state space, m is the dimension of the output. For fixed integers r, k, n, m we call $\Sigma_{r,k,n,m}$ the class of all such systems. A typical example is given by the relationship between voltages and currents of the low pass equivalent circuit of a microwave filter: in this case n is the number of resonators, $k = m = 2$, and the parameters are the couplings and the i/o load impedances.

The transfer function of the system defined by (1) is given by

$$H(s) = C(p)(sId - A(p))^{-1}B(p) = \sum_{i=1}^{\infty} \frac{C(p)A^i(p)B(p)}{s^i} \quad (2)$$

The latter equality is meant between formal series of the variable $1/s$. The matrices $C(p)B(p), \dots, C(p)A^i(p)B(p) \dots$ are called the Markov parameters of the system. A classical result in system theory states that $H(s)$ is entirely characterized by the first $2n$ Markov parameters. For each system $\sigma \in \Sigma_{r,k,n,m}$ we therefore define the following map:

$$\begin{aligned} \pi_{\sigma} : \mathbb{C}^r &\rightarrow (\mathbb{C}^{k \times m})^{2n} \\ p &\rightarrow (C(p)B(p), \dots, C(p)A^{2n-1}(p)B(p)) \end{aligned} \quad (3)$$

Thanks to the preceding remark the set $\pi_{\sigma}(\mathbb{C}^r)$ is isomorphic to the set of all the possible transfer functions of our parameterized system obtained when the parameters vary in all of \mathbb{C}^r . In the case of our microwave filter the latter is the set

of all admittance matrices one can realize with a given coupling geometry (supposing the couplings can take complex values!). For a system σ we define $V(\sigma) = \overline{\pi_\sigma(\mathbb{C}^r)}$ where the bar in the latter means “taking” the Zariski closure (see [4]). The reader who might not be familiar with this algebraic material can think of $V(\sigma)$ as the set $\pi_\sigma(\mathbb{C}^r)$ where only a “few” elements have been added so as it becomes an algebraic variety (a zero set of some polynomials is system).

Given two systems $\sigma_1 \in \Sigma_{r_1, n, m, k}$ and $\sigma_2 \in \Sigma_{r_2, n, m, k}$ we define the following parameterized sets:

$$\begin{aligned} p \in \mathbb{C}^{r_1}, E_{\sigma_1}(p) &= \{q \in \mathbb{C}^{r_1}, \pi_{\sigma_1}(q) = \pi_{\sigma_1}(p)\} \quad (4) \\ p \in \mathbb{C}^{r_2}, E_{\sigma_1, \sigma_2}(p) &= \{q \in \mathbb{C}^{r_1}, \pi_{\sigma_1}(q) = \pi_{\sigma_2}(p)\} \quad (5) \end{aligned}$$

In other words the set $E_{\sigma_1}(p)$ contains all the parameter vectors giving rise to the same transfer function as p does. In what follows we say that a property $\mathcal{P}(p)$ depending on a vector $p \in \mathbb{C}^r$ is “generic” if it is true for all $p \in \mathbb{C}^r$ unless may be on a set included in a proper algebraic variety (for example in \mathbb{C}^2 an algebraic curve is proper algebraic, in \mathbb{C}^3 it is the case for an algebraic surface).

Proposition 1

- i) *The cardinalities of $E_{\sigma_1}(p)$ and $E_{\sigma_1, \sigma_2}(p)$ are generically constant with respect to p . We denote by $\Theta(\sigma_1)$ (resp. $\Theta(\sigma_1, \sigma_2)$) the corresponding constant and call it the parameterization’s order of σ_1 (resp. of σ_1 over σ_2). If the parameterization’s order of a system is finite we call the latter “non-redundant”.*
- ii) *The system σ_1 is non-redundant iff the Jacobian matrix associated to the mapping π_{σ_1} is generically of rank r_1 on \mathbb{C}^{r_1} .*
- iii) *If the system σ_1 is non-redundant then the dimension of $V(\sigma_1)$ (as a variety) is r_1 .*
- iv) *Suppose $r_1 \geq r_2$, $V(\sigma_1) \subset V(\sigma_2)$ and σ_1 is non-redundant then $r_2 = r_1$ and $V(\sigma_1) = V(\sigma_2)$.*
- v) *If $V(\sigma_1) = V(\sigma_2)$ then $\Theta(\sigma_1) = \Theta(\sigma_1, \sigma_2)$*
- vi) *If $r_1 < r_2$ and σ_2 is non-redundant then $\Theta(\sigma_1, \sigma_2) = 0$.*

Remarks. The justification of (i) is purely algebraic: the set $E_{\sigma_1}(p)$ is characterized by a system of polynomial equations parameterized by the entries of p . More precisely we can think of it as polynomial system on the field of fractions $\mathbb{C}(p_1, p_2 \dots p_{r_1})$ [4]. It is now well known that algebraic properties like the cardinality of the zero set of a polynomial system remains constant for all specializations (i.e when p

is given a value in \mathbb{C}^r) of p , except may be on a zero set of some polynomial system in the unknowns $(p_1 \dots p_{r_1})$.

The points (ii),(iii) and (iv) follow from the application of the implicit function theorem and the fact that the $V(\sigma)$ ’s are irreducible varieties. Finally (v) and (vi) are straightforward. ■

III. THE LOW PASS PROTOTYPE

The input/output system relating the voltages and currents of the classical low pass prototype of a coupled-resonators filter [2] leads to the following linear system:

$$\begin{cases} A_y = jM = jM^t \\ C_y = B_y^t = \begin{bmatrix} \sqrt{Z_1} & 0 & \dots & 0 \\ 0 & 0 & \dots & \sqrt{Z_2} \end{bmatrix} \end{cases} \quad (6)$$

where M is the matrix of normalized couplings and (Z_1, Z_2) are the corresponding normalized load impedances. The transfer matrix Y of this system is the so-called reduced admittance matrix ($Y = (Id - S)/(Id + S)$). In order to fit into the framework of parameterized systems of the last section our vector of parameters will be formed of $\sqrt{Z_1}, \sqrt{Z_2}$ (or only $\sqrt{Z_1}$ if $Z_1 = Z_2$ is imposed) and of all the independent non-zero couplings of the corresponding filter geometry. We call $\mathcal{F}_{r, n}$, the class of all parameterized systems of the form (6) with r independent parameters and n resonators. In the following we call “loss-less” a filter which admittance matrix is positive real and purely imaginary on the imaginary axes.

Proposition 2

- i) *Let $M = [m_{i, j}]$ be the coupling matrix of a filter, then the entry (l, m) of the matrix M^k is a sum of monomials in the variables $m_{i, j}$, and there is a one to one canonical correspondence between each of this monomials and all the paths from l to m of length k in the associated coupling graph. In particular for a filter of order n , the lowest integer k' for which the Markov parameter $j^{k'} C_y M^{k'} B_y$ has a non zero $(1, 2)$ diagonal entry corresponds to the length of the shortest path between 1 and n in the associated coupling graph.*
- ii) *Let $\sigma_1 \in \mathcal{F}_{r_1, n}, \sigma_2 \in \mathcal{F}_{r_2, n}$ be two parameterized systems associated to some filter geometries. Suppose that σ_1 and σ_2 are generically of MacMillan degree n , then the integers $\theta(\sigma_1), \theta(\sigma_1, \sigma_2)$ are multiples of 2^n . We therefore define $\theta^*(\sigma_1) = \theta(\sigma_1)/2^n$ (resp. $\theta^*(\sigma_1, \sigma_2) = \theta(\sigma_1, \sigma_2)/2^n$) and call it the reduced order of the coupling geometry of the filter σ_1 (resp. of σ_1 over σ_2).*
- iii) *If all the parameters of a filter are given real values then the corresponding filter is loss-less.*

Remarks. (i) is also known as the “minimum path rule” (see. [5], [6]). Point (ii) is due to the natural symmetry of the parameterization in the case of a filter, i.e: if (A_y, B_y, C_y) is a filter realization then for any sign matrix Δ , $(\Delta A_y \Delta, \Delta B, C \Delta)$ is a realization with the same coupling geometry and same admittance matrix. Point (iii) is not an iff equivalence: for some coupling geometries there may exist complex valued parameter sets that leads to loss-less filters. ■

An element $\sigma \in \mathcal{F}_{r,n}$ is entirely characterized by its set of independent parameters $\mathcal{M}(\sigma)$. We now define two types of filters $\phi_{n,k}$ and $\hat{\phi}_{n,k}$ with n resonators,

$$\begin{aligned} \mathcal{M}(\phi_{n,k}) = & \{\sqrt{Z_1}, \sqrt{Z_2}\} \cup \{m_{i,i}, i = 1 \dots n\} \\ & \cup \{m_{i,i+1}, i = 1 \dots n-1\} \cup \{m_{i,n}, i = k \dots n-2\} \end{aligned} \quad (7)$$

$$\begin{aligned} \mathcal{M}(\hat{\phi}_{n,k}) = & \{\sqrt{Z_1}, \sqrt{Z_2}\} \cup \{m_{i,i+1}, i = 1 \dots n-1\} \\ & \cup \begin{cases} \text{if } n+k \text{ odd} & \{m_{i,n}, i = k, k+2 \dots, n-3\} \\ \text{if } n+k \text{ even} & \{m_{i,n}, i = k+1, k+3 \dots, n-1\} \end{cases} \end{aligned} \quad (8)$$

For example the coupling matrix of $\phi_{4,1}$ is,

$$\begin{bmatrix} m_{1,1} & m_{1,2} & 0 & m_{1,4} \\ m_{1,2} & m_{2,2} & m_{1,3} & m_{2,4} \\ 0 & m_{1,3} & m_{3,3} & m_{3,4} \\ m_{1,4} & m_{2,4} & m_{3,4} & m_{4,4} \end{bmatrix} \quad (9)$$

and the one of $\hat{\phi}_{4,1}$ is (9) where all the diagonal couplings $m_{1,1} \dots m_{4,4}$ as well as $m_{2,4}$ have been set to 0. The following proposition shows that this “arrow form” of the coupling matrix (visually one can see an arrow by looking at the matrix!) of the filters $\phi_{n,k}$ and $\hat{\phi}_{n,k}$ is in some sense canonical.

Proposition 3 *For all $n \geq 2$ and $1 \leq k \leq n-1$ the following holds,*

- i) $\theta^*(\phi_{n,k}) = \theta^*(\hat{\phi}_{n,k}) = 1$
- ii) $V(\phi_{n,k})$ is equal to the set of all (2×2) reciprocal ($Y_{1,2} = Y_{2,1}$), strictly proper transfer functions of MacMillan degree less or equal to n for which the k first Markov parameters are diagonal (i.e their $(1, 2)$ and $(2, 1)$ terms are zero)
- iii) Suppose n is odd and k is even then $V(\hat{\phi}_{n,k})$ is equal to the set of all (2×2) odd ($Y(-s) = -Y(s)$), reciprocal, strictly proper transfer functions of MacMillan degree less or equal to n for which the k first Markov parameters are diagonal (by definition, for k odd we have in this case that $\hat{\phi}_{n,k} = \hat{\phi}_{n,k+1}$). If n is even and k odd then $V(\hat{\phi}_{n,k})$ is equal to the set of all (2×2) symmetric,

strictly proper transfer functions of MacMillan degree less or equal to n where $Y_{1,1}$ and $Y_{2,2}$ are odd functions and $Y_{1,2} = Y_{2,1}$ is an even function and the k first Markov parameters are diagonal.

- iv) For any generic transfer function in $V(\phi_{n,k})$ or $V(\hat{\phi}_{n,k})$ the corresponding synthesis with a coupling matrix in the “arrow form” can be computed by a simple orthonormalization algorithm. If in addition the transfer is loss-less then all the parameters of the “arrow form” are real.

Remarks. (i) states that up to sign changes the “arrow” form is generically unique. The assertion concerning the Markov parameters in (iii) is a direct consequence of the “minimum path rule”. We will not detail (iv) but a constructive algorithm to obtain an arrow form have already been studied in [7]. ■

We will now tackle the problem of computing the reduced order $\theta^*(\sigma)$ and the set $E_\sigma(p)$ for any coupling geometry. Given $p \in \mathbb{C}^r$ the equations characterizing $E_\sigma(p)$ in (4) form a non-linear algebraic system whose zero set can be computed using methods of computer algebra.

IV. COMPUTING THE ZEROS OF A ZERO DIMENSIONAL ALGEBRAIC SYSTEM

We will give a summary of the two step approach proposed in [8]. The first step of the latter consist in computing a Groebner basis of the system. Given a polynomial system Γ , we say that an equation is a polynomial consequence of Γ if it is obtained by algebraic manipulation of equations of the latter. For example $x(x-y^2) - (x^2 - y^3 - 1) = -xy^2 + y^3 + 1 = 0$ is a polynomial consequence of the system $x-y^2 = 0$, $x^2 - y^3 - 1 = 0$. We call $I(\Gamma)$ the set of all polynomial consequences of Γ . For short a Groebner basis of Γ is a finite subset of $I(\Gamma)$ that allows to decide by means of a simple division algorithm involving its elements whether a given algebraic equation is in $I(\Gamma)$ or not [4]. Computing a Groebner basis, using for example Buchberger’s algorithm, is theoretically straight forward but can be extremely time and memory consuming. In practice, the use of specialized algorithms [9] and their effective software implementation [10] is strongly recommended.

Once a Groebner basis of a system have been computed informations about the geometry of the zero set are easily deduced. In particular one can decide if it consists of a finite number of points; in the latter case the system is called zero dimensional and its zero set can be expressed in terms of the roots of a single univariate polynomial. This is the so called rational univariate representation (RUR for short) which can be computed in a direct manner from a Groebner basis [8]. Finally computing numerically the roots of the corresponding univariate polynomial yields a numerical estimation of the zero set of the latter system.

V. EXAMPLE OF A 10^{th} ORDER FILTER

We will study the case of a filter geometry $\sigma_0 \in \mathcal{F}_{15,10}$ defined by,

$$\mathcal{M}(\sigma_0) = \{\sqrt{Z_1}, \sqrt{Z_2}\} \cup \{m_{i,i+1}, i = 1 \dots 9\} \cup \{m_{1,10}, m_{1,4}, m_{6,9}, m_{5,10}\} \quad (10)$$

Using a computer algebra system like Maple and (ii) of proposition 1 we check that σ_0 is non-redundant. Now invoking the minimum path rule and (iv) of proposition 1 we conclude that

$$V(\sigma_0) = V(\hat{\phi}_{10,1}). \quad (11)$$

Practically this means that generically all the reciprocal transfer functions of MacMillan degree 10 with 8 transmission zeros (having the parity properties of elements of $V(\hat{\phi}_{10,1})$ described in proposition 3) can be synthesized under the geometry of σ_0 (when accepting complex couplings). Now solving the algebraic system defining $E_{\sigma_1}(p)$ in (4) for a generic value of p leads to

$$\theta^*(\sigma_0) = \theta^*(\sigma_0, \hat{\phi}_{10,1}) = 3. \quad (12)$$

In other words, to every coupling set of σ_0 there corresponds 2 others which realize the same transfer function. We now follow a classical synthesis method: we compute a 10^{th} degree elliptical filter characteristic with 8 transmission zeros and build with its 20 first Markov parameters a system similar to the one defined in (4). As expected from (12), solving the latter yields 3 equivalent coupling sets (there are 3×2^{10} sets if we consider all possible sign changes).

$$\sqrt{Z_1} = 1.0022 \quad \sqrt{Z_2} = 1.0022$$

$$\text{Set 1: } [1, 2] = .653 \quad [2, 3] = .890 \quad [3, 4] = .359 \quad [4, 5] = .530 \\ [5, 6] = -.535 \quad [6, 7] = .493 \quad [7, 8] = .287 \quad [8, 9] = .516 \quad [9, 10] = -.822 \\ [1, 4] = -.501 \quad [6, 9] = .236 \quad [5, 10] = .065 \quad [1, 10] = .024$$

$$\text{Set 2: } [1, 2] = -.800 + .027j \quad [2, 3] = .437 - .067j \\ [3, 4] = -.552 + .011j \quad [4, 5] = .548 + .007j \quad [5, 6] = .518 + .010j \\ [6, 7] = .453 - .015j \quad [7, 8] = -.751 - .043j \quad [8, 9] = .540 - .022j \\ [9, 10] = -.818 - .008j \quad [1, 4] = .219 + .097j \quad [6, 9] = .159 + .083j \\ [5, 10] = -.119 + .055j \quad [1, 10] = .024$$

$$\text{Set 3: } [1, 2] = -.800 - .027j \quad [2, 3] = .437 + .067j \\ [3, 4] = -.552 - .011j \quad [4, 5] = .548 - .007j \quad [5, 6] = .518 - .010j \\ [6, 7] = .453 + .015j \quad [7, 8] = -.751 + .043j \quad [8, 9] = .540 + .022j \\ [9, 10] = -.818 + .008j \quad [1, 4] = .219 - .097j \quad [6, 9] = .159 - .083j \\ [5, 10] = -.119 - .055j \quad [1, 10] = .024$$

In this example, only the first set is real and the two others are complex conjugated. It can be proven that this conjugation property of complex solutions is true whenever starting from a loss-less prototype and yields, thanks to the odd parity of the reduced order, the generic existence of a real solution to the synthesis problem associated to σ_0 (this for loss-less characteristics in $V(\hat{\phi}_{10,1})$).

Equating the Markov parameters is not the only way of determining equivalent coupling sets: as in classical synthesis methods [2] one can use a formulation based on orthogonal similarity transforms. In a recent work which we will detail in a future publication we developed an algebraic version of this ideas which allows to break the natural sign symmetry of the synthesis problem. The latter in turn simplifies tremendously the computation of the corresponding Groebner basis and allows the computation of both the reduced order and equivalent coupling sets for all usual coupling geometries. With this formulation the latter numerical results were obtained in less than 10 seconds on a 1 Ghz PC.

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