

Feedback stabilization of a 3D fluid-structure model with a boundary control*

Moctar Ndiaye^{1,2}, Denis Matignon² and Jean-Pierre Raymond¹

Abstract—We study a system coupling the incompressible Navier-Stokes equations in a 3D parallelepiped type domain with a damped plate equation. The plate is located in a part of the upper boundary of the fluid domain. The fluid domain depends on the deformation of the plate, and therefore it depends on time.

We are interested in the stabilization, with a prescribed decay rate, of such a system in a neighborhood of a stationary solution, by a Dirichlet control acting at the boundary of the fluid domain.

For that, we first study the stabilizability of the corresponding linearized system and we determine a finite-dimensional feedback control able to stabilize the linearized model.

A crucial step in the analysis consists in showing that this linearized system can be rewritten thanks to an analytic semigroup, the infinitesimal generator of which has a compact resolvent.

A fixed-point argument is finally used to prove the local stabilization of the original nonlinear system. The main difficulties come from the coupling between the fluid and plate equations, and the fact that the fluid domain varies with time, giving rise to geometric nonlinearities.

The results of the paper may be adapted to other more complex geometrical configurations for the same type of system. Ongoing research concerns the numerics of the control problem.

Keywords: stabilization, boundary control, fluid-structure model, Navier-Stokes equations, plate equation

I. INTRODUCTION

In this paper we are interested in a system coupling a three dimensional fluid flow with the evolution of a structure localized at the boundary of the geometrical domain occupied by the fluid.

The fluid flow is described by the Navier-Stokes equations while the displacement of the structure satisfies a damped plate equation.

Stabilizing such systems in a neighborhood of an unstable stationary solution is a challenging problem. Depending on the applications we want to deal with, it is interesting to look for a stabilizing control acting either only in the structure equation or only in the fluid equation. The case of a control acting in the structure equation has been recently considered in [5]. Here we would like to study the local stabilization of such systems by a boundary control acting only in the fluid equation. For simplicity we do not stabilize

this coupled system around an unstable stationary solution, but we consider the simpler problem consisting of stabilizing the system around zero, with any prescribed exponential decay rate. The stabilization problem around an unstable stationary solution leads to some additional difficulties. It will be studied in a forthcoming paper. The procedure we follow in the paper is a classical method, already used in [5], consisting in looking for a feedback control stabilizing a linearized system, that we next apply to the nonlinear system.

We have to emphasize, that in the considered model, the domain occupied by the fluid depends on the displacement of the plate. Therefore the main nonlinearity in this model comes from the fact that the equations of the fluid are written in Eulerian variables (therefore in the deformed configuration), while the equation of the structure is written in Lagrangian variables (i.e. in the reference configuration).

Thus before linearizing, we have to rewrite this coupled system in the reference configuration. This is performed by using a change of variables. There are mainly two ways to do that. One way consists in writing the Navier-Stokes equations in Lagrangian variables, in that case the change of variables is associated with the flow of the fluid. The other way consists in using a change of variables associated with the displacement of the structure. The best choice usually corresponds to the change of variables associated with the most regular unknowns. In our case the displacement of the structure will be very regular. That is why it is more convenient to use a change of variables associated to the displacement of the plate.

Let us now precisely describe the equations of this coupled system, with the help of figures 1 and 2. The reference configuration of the fluid is

$$\Omega = (0, a) \times (0, b) \times (0, 1),$$

with $a > 0$ and $b > 0$.

The plate is located on $\Gamma_s = \omega_s \times \{1\}$ with $\omega_s = (0, a) \times (0, b)$.

For simplicity we choose Dirichlet boundary conditions on three sides of the parallelepiped Ω

$$\Gamma_{d,\ell} = \{0\} \times (0, b) \times (0, 1), \quad \text{left wall}$$

$$\Gamma_c = (0, a) \times (0, b) \times \{0\}, \quad \text{control region}$$

$$\Gamma_{d,r} = \{a\} \times (0, b) \times (0, 1), \quad \text{right wall}$$

and periodic boundary conditions on the two sides

$$\Gamma_{\#}^1 = (0, a) \times \{0\} \times (0, 1),$$

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¹M. Ndiaye and J.-P. Raymond are with the Institut de Mathématiques, UMR CNRS 5219, Université Paul Sabatier, 31062 Toulouse Cedex 9, France. moctar.ndiaye@math.univ-toulouse.fr and jean-pierre.raymond@math.univ-toulouse.fr

²D. Matignon is with the University of Toulouse, ISAE; 10, av. E. Belin, 31055 Toulouse Cedex 4, France. denis.matignon@isae.fr

and

$$\Gamma_{\#}^2 = (0, a) \times \{b\} \times (0, 1).$$

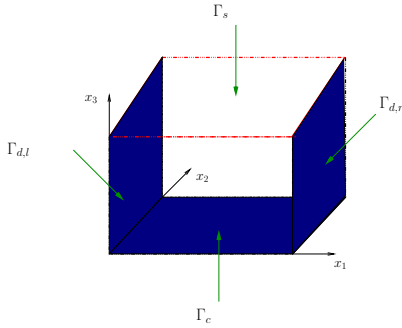


Fig. 1. Reference configuration

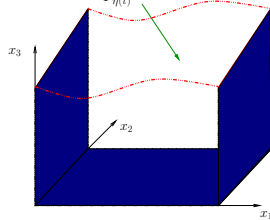


Fig. 2. Configuration at time t

More complex boundary conditions, including mixed boundary conditions, will be considered in a forthcoming paper.

We denote by \mathbf{u} and p the fluid velocity and the fluid pressure, and by η the displacement of the plate. Therefore the fluid domain at time t is

$$\Omega_{\eta(t)} = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \omega_s \text{ and } 0 \leq x_3 \leq 1 + \eta(t, x_1, x_2)\},$$

while the configuration of the plate at time t is

$$\Gamma_{\eta(t)} = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \omega_s \text{ and } x_3 = 1 + \eta(t, x_1, x_2)\}.$$

We also use the notation

$$\begin{aligned} \tilde{Q}^\infty &= \bigcup_{t \in (0, \infty)} \{t\} \times \Omega_{\eta(t)}, & \tilde{\Sigma}_s^\infty &= \bigcup_{t \in (0, \infty)} \{t\} \times \Gamma_{\eta(t)}, \\ Q^\infty &= (0, \infty) \times \Omega, & \Sigma_s^\infty &= (0, \infty) \times \Gamma_s, \\ \Sigma_{d,\ell}^\infty &= (0, \infty) \times \Gamma_{d,\ell}, & \Sigma_c^\infty &= (0, \infty) \times \Gamma_c, \\ \Sigma_{d,r}^\infty &= (0, \infty) \times \Gamma_{d,r}, & \Sigma_{s,\ell,r}^\infty &= (0, \infty) \times \Gamma_{s,\ell,r}, \\ \Gamma_{s,\ell,r} &= (\{0\} \times (0, b) \times \{1\}) \cup (\{a\} \times (0, b) \times \{1\}). \end{aligned}$$

The displacement η satisfies the equation

$$\begin{aligned} &\eta_{tt} - \beta \Delta \eta - \gamma \Delta \eta_t + \alpha \Delta^2 \eta \\ &= \sqrt{1 + |\nabla \eta|^2} \sigma(\mathbf{u}, p)|_{\Gamma_{\eta(t)}} \tilde{\mathbf{n}}(t) \cdot \mathbf{e}_3 \text{ on } \Sigma_s^\infty, \\ &\eta = 0 \text{ on } \Sigma_{s,\ell,r}^\infty, \quad \frac{\partial \eta}{\partial \mathbf{n}} = 0 \text{ on } \Sigma_{s,l,r}^\infty, \\ &\eta(0) = \eta_1^0 \text{ on } \Gamma_s, \quad \eta_t(0) = \eta_2^0 \text{ on } \Gamma_s, \end{aligned} \quad (1)$$

where

$$\sigma(\mathbf{u}, p) = \nu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) - pI,$$

$$\mathbf{e}_1 = (1, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0), \quad \mathbf{e}_3 = (0, 0, 1),$$

$\tilde{\mathbf{n}}(t) = \frac{1}{\sqrt{1 + |\nabla \eta|^2}}(-\eta_{x_1} \mathbf{e}_1 - \eta_{x_2} \mathbf{e}_2 + \mathbf{e}_3)$ is the unit normal to $\Gamma_{\eta(t)}$ exterior to $\Omega_{\eta(t)}$, ν is the fluid viscosity; $\alpha > 0$ is a mechanical constant; and $\beta \geq 0$ accounts for a wave-like component given by $\beta \Delta \eta$.

The damping term $-\gamma \Delta \eta_t$, where $\gamma > 0$, makes the semi-group associated to the equation of the structure analytic, which is essential to obtain the main result of this paper. Indeed, the semigroup corresponding to the plate equation is analytic if and only if $\gamma > 0$ (see [3]).

The fluid flow is governed by the Navier-Stokes equations

$$\begin{aligned} &\mathbf{u}_t - \operatorname{div} \sigma(\mathbf{u}, p) + (\mathbf{u} \cdot \nabla) \mathbf{u} = 0 \text{ in } \tilde{Q}^\infty, \\ &\operatorname{div} \mathbf{u} = 0 \text{ in } \tilde{Q}^\infty, \\ &\mathbf{u} = \eta_t \mathbf{e}_3 \text{ on } \tilde{\Sigma}_s^\infty, \quad \mathbf{u} = 0 \text{ on } \Sigma_c^\infty, \\ &\mathbf{u} = 0 \text{ on } \Sigma_{d,\ell}^\infty, \quad \mathbf{u} = 0 \text{ on } \Sigma_{d,r}^\infty, \\ &\mathbf{u}(0) = \mathbf{u}^0 \text{ in } \Omega_{\eta_1^0}. \end{aligned} \quad (2)$$

Let us notice that system (1) and (2) are coupled through the equations (1)₁ and (2)₂. The equation $\mathbf{u} = \eta_t \mathbf{e}_3$ corresponds to the equality of the fluid velocity and the displacement velocity of the structure, while the term in the right-hand side of (1)₁ represents the force exerted by the fluid on the structure.

Due to the incompressibility condition, we have

$$\int_{\Omega_{\eta(t)}} \operatorname{div} \mathbf{u} = \int_{\Gamma_{\eta(t)}} \mathbf{u} \cdot \tilde{\mathbf{n}}(t) = \int_{\Gamma_s} \eta_t = 0.$$

Therefore, we choose the displacement η with zero mean value.

Our goal is to use a boundary control located in Γ_c of the form

$$\mathbf{u}|_{\Gamma_c} = \sum_{i=1}^{N_c} g_i(t) \mathbf{w}_i(x_1, x_2),$$

able to stabilize with exponential decay rate $-\omega < 0$ the corresponding controlled system, provided that \mathbf{u}^0 , η_1^0 , η_2^0 are small enough in an appropriate space.

The control variable is the vector $\mathbf{g} = (g_1, \dots, g_{N_c})$. The family $(\mathbf{w}_i)_{1 \leq i \leq N_c}$ is chosen so that some extended system satisfies a stabilizability condition (see section II-C).

To achieve this goal we are going to follow a classical approach, consisting in the following steps:

- First we rewrite system (1)-(2) in the reference configuration, and we linearize it around $(0, 0, 0)$. Since the time derivative of the control appears in the state equation, we introduce an extended system.
- We study the stabilizability of the linearized extended system and we define a feedback law able to stabilize it.
- We prove that the closed loop system, obtained by applying the linear feedback law to the nonlinear system, is locally exponentially stable.

In order to rewrite system (1)-(2) in the fixed domain, we use the change of variables

$$\begin{aligned}\Omega_{\eta(t)} &\longrightarrow \Omega \\ (x_1, x_2, x_3) &\longmapsto (x_1, x_2, z) = \left(x_1, x_2, \frac{x_3}{1+\eta(t, x_1, x_2)}\right).\end{aligned}$$

We denote by $(\hat{\mathbf{u}}, \hat{p})$ the image of (\mathbf{u}, p) by this change of variables; we also denote $\hat{\mathbf{u}}^0$ the image of \mathbf{u}^0 . Thus the system satisfied by $(\hat{\mathbf{u}}, \hat{p}, \eta)$ is

$$\begin{aligned}\hat{\mathbf{u}}_t - \operatorname{div} \sigma(\hat{\mathbf{u}}, \hat{p}) &= F_f[\hat{\mathbf{u}}, \hat{p}, \eta] \text{ in } Q^\infty, \\ \operatorname{div} \hat{\mathbf{u}} &= G[\hat{\mathbf{u}}, \eta] \text{ in } Q^\infty, \\ \hat{\mathbf{u}} &= \eta_t \mathbf{e}_3 \text{ on } \Sigma_s^\infty, \hat{\mathbf{u}} = \sum_{i=1}^{N_c} g_i \mathbf{w}_i \text{ on } \Sigma_c^\infty, \\ \hat{\mathbf{u}} &= 0 \text{ on } \Sigma_{d,\ell}^\infty \cup \Sigma_{d,r}^\infty, \hat{\mathbf{u}}(0) = \hat{\mathbf{u}}^0 \text{ in } \Omega, \\ \eta_{tt} - \beta \Delta \eta - \gamma M_s \Delta \eta_t + \alpha M_s \Delta^2 \eta & \\ &= \gamma_s \hat{p} + \gamma_s F_s[\hat{\mathbf{u}}, \eta] \text{ on } \Sigma_s^\infty, \\ \eta &= 0, \frac{\partial \eta}{\partial \mathbf{n}} = 0 \text{ on } \Sigma_{s,\ell,r}^\infty, \\ \eta(0) &= \eta_1^0, \eta_t(0) = \eta_2^0 \text{ on } \Gamma_s.\end{aligned}\tag{3}$$

- The nonlinear terms F_f , G and F_s in the right-hand sides are defined by

$$\begin{aligned}F_f[\hat{\mathbf{u}}, \hat{p}, \eta] &:= -\eta \hat{\mathbf{u}}_t + z \eta_2 \hat{\mathbf{u}}_z + \nu \eta (\hat{\mathbf{u}}_{x_1 x_1} + \hat{\mathbf{u}}_{x_2 x_2}) \\ &\quad - \frac{\nu \eta}{1+\eta} \hat{\mathbf{u}}_{zz} - 2\nu z (\eta_{x_1} \hat{\mathbf{u}}_{x_1 z} + \eta_{x_2} \hat{\mathbf{u}}_{x_2 z}) \\ &\quad + \frac{\nu z^2}{1+\eta} |\nabla \eta|^2 \hat{\mathbf{u}}_{zz} - [(1+\eta) \Delta \eta - |\nabla \eta|^2] \hat{\mathbf{u}}_z \\ &\quad - \eta \hat{u}_1 \hat{\mathbf{u}}_{x_1} - \eta \hat{u}_2 \hat{\mathbf{u}}_{x_2} - (\hat{\mathbf{u}} \cdot \nabla) \hat{\mathbf{u}} \\ &\quad + z (\eta_{x_1} \hat{u}_1 + \eta_{x_2} \hat{u}_2) \hat{\mathbf{u}}_z + [z \eta_{x_1} \hat{p}_z - \eta \hat{p}_{x_1}] \mathbf{e}_1 \\ &\quad + [z \eta_{x_2} \hat{p}_z - \eta \hat{p}_{x_2}] \mathbf{e}_2, \\ G[\hat{\mathbf{u}}, \eta] &:= -\eta (\hat{u}_{1,x_1} + \hat{u}_{2,x_2}) \\ &\quad + z (\eta_{x_1} \hat{u}_{1,z} + \eta_{x_2} \hat{u}_{2,z}), \\ F_s[\hat{\mathbf{u}}, \eta] &:= \nu \frac{2\eta - |\nabla \eta|^2}{1+\eta} \hat{u}_{3,z} + \frac{\nu \eta_{x_1}}{1+\eta} \hat{u}_{1,z} + \frac{\nu \eta_{x_2}}{1+\eta} \hat{u}_{2,z} \\ &\quad + \nu (\eta_{x_1} \hat{u}_{3,x_1} + \eta_{x_2} \hat{u}_{3,x_2}) - 2\nu \gamma_s \hat{u}_{3,z}.\end{aligned}\tag{4}$$

- The operator M_s is the orthogonal projection from $L^2_\#(\Gamma_s)$ onto

$$L^2_{0,\#}(\Gamma_s) = \{k \in L^2_\#(\Gamma_s) \text{ such that } \int_{\Gamma_s} k = 0\},$$

defined by

$$M_s k := k - \frac{1}{|\Gamma_s|} \int_{\Gamma_s} k.$$

The operator $\gamma_s \in \mathcal{L}(H^\sigma_\#(\Omega), H^{\sigma-\frac{1}{2}}_\#(\Omega))$, with $\sigma > \frac{1}{2}$, is defined by

$$\gamma_s p := M_s(p|_{\Gamma_s}) = p|_{\Gamma_s} - \frac{1}{|\Gamma_s|} \int_{\Gamma_s} p|_{\Gamma_s}.$$

The main theorem of the paper is a local stabilization result for initial data in the space

$$\begin{aligned}\mathbf{H}_{cc} &= \left\{ (\mathbf{v}, \eta_1, \eta_2) \in \mathbf{H}^1_\#(\Omega) \times H^3_\#(\Gamma_s) \times H^1_\#(\Gamma_s) \right. \\ &\quad \text{such that } \mathbf{v} = \eta_2 \mathbf{e}_3 \text{ on } \Gamma_s, \mathbf{v} = 0 \text{ on } \Gamma_{d,\ell} \cup \Gamma_c \cup \Gamma_{d,r}, \\ &\quad \left. \eta_1 = \eta_2 = \frac{\partial \eta_1}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_{s,\ell,r}, \int_{\Gamma_s} \eta_1 = \int_{\Gamma_s} \eta_2 = 0 \right\}.\end{aligned}$$

Let us also introduce the functional spaces

$$\begin{aligned}\mathbf{H}^{2,1}_\#(Q^\infty) &= L^2(0, \infty; \mathbf{H}^2_\#(\Omega)) \cap H^1(0, \infty; \mathbf{L}^2_\#(\Omega)), \\ H^{4,2}(\Sigma_s^\infty) &= L^2(0, \infty; H^4_\#(\Gamma_s)) \cap H^2(0, \infty; L^2_\#(\Gamma_s)), \\ H^{2,1}(\Sigma_s^\infty) &= L^2(0, \infty; H^2_\#(\Gamma_s)) \cap H^1(0, \infty; L^2_\#(\Gamma_s)), \\ \mathcal{H}^1_\#(\Omega) &= \{p \in H^1_\#(\Omega) \text{ such that } \int_\Omega p = 0\}.\end{aligned}$$

Theorem 1. *Let $\omega > 0$ be given. There exists $r > 0$ such that for all $(\hat{\mathbf{u}}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}_{cc}$ satisfying*

$$\|(\hat{\mathbf{u}}^0, \eta_1^0, \eta_2^0)\|_{\mathbf{H}_{cc}} \leq r,$$

there exists a control $(g_i)_{i=1, \dots, N_c} \in H^1(0, \infty; \mathbb{R}^{N_c})$ in feedback form, such that system (3) admits a unique solution $(\hat{\mathbf{u}}, \hat{p}, \eta) \in \mathbf{H}^{2,1}_\#(Q^\infty) \times L^2(0, \infty; \mathcal{H}^1_\#(\Omega)) \times H^{4,2}(\Sigma_s^\infty)$ satisfying

$$\|(\hat{\mathbf{u}} - \mathbf{u}_s, \eta)\|_{\mathbf{H}^2_\#(\Omega) \times H^4_\#(\Gamma_s)} \leq C_1 e^{-\omega t},$$

with $C_1 > 0$.

II. STABILIZABILITY OF THE LINEARIZED SYSTEM

• Let $(\hat{\mathbf{u}}, \hat{p}, \eta)$ be a solution of system (3). The linearization around $(\mathbf{0}, 0, 0, 0)$ of the system satisfied by $(e^{\omega t} \hat{\mathbf{u}}, e^{\omega t} \hat{p}, e^{\omega t} \eta, e^{\omega t} \eta_t)$ is

$$\begin{aligned}\mathbf{v}_t - \operatorname{div} \sigma(\mathbf{v}, p) - \omega \mathbf{v} &= 0, \operatorname{div} \mathbf{v} = 0 \text{ in } Q^\infty, \\ \mathbf{v} &= \eta_2 \mathbf{e}_3 \text{ on } \Sigma_s^\infty, \mathbf{v} = \sum_{i=1}^{N_c} g_i \mathbf{w}_i \text{ on } \Sigma_c^\infty, \\ \mathbf{v} &= 0 \text{ on } \Sigma_{d,\ell}^\infty \cup \Sigma_{d,r}^\infty, \mathbf{v}(0) = \mathbf{v}^0 \text{ in } \Omega, \\ \eta_{1,t} - \omega \eta_1 - \eta_2 &= 0 \text{ on } \Sigma_s^\infty, \\ \eta_{2,t} - \omega \eta_2 - \beta \Delta \eta_1 - \gamma M_s \Delta \eta_2 + \alpha M_s \Delta^2 \eta_1 & \\ &= \gamma_s p \text{ on } \Sigma_s^\infty, \\ \eta_1 &= 0, \frac{\partial \eta_1}{\partial \mathbf{n}} = 0 \text{ on } \Sigma_{s,\ell,r}^\infty, \\ \eta_1(0) &= \eta_1^0, \eta_2(0) = \eta_2^0 \text{ on } \Gamma_s.\end{aligned}\tag{5}$$

We are going to rewrite system (5) as a controlled evolution equation. First, we have to eliminate the pressure from the equations. The space $\mathbf{L}^2_\#(\Omega)$ admits the orthogonal decomposition

$$\mathbf{L}^2_\#(\Omega) = \mathbf{V}^0_{n,\#}(\Omega) \oplus \nabla H^1_\#(\Omega),$$

with

$$\mathbf{V}^0_{n,\#}(\Omega) = \{\mathbf{y} \in \mathbf{L}^2_\#(\Omega) \mid \operatorname{div} \mathbf{y} = 0, \mathbf{y} \cdot \mathbf{n} = 0 \text{ on } \partial_\# \Omega\},$$

where $\partial_{\#}\Omega = \Gamma_s \cup \Gamma_{d,\ell} \cup \Gamma_{d,r} \cup \Gamma_c$.

We denote by P the orthogonal projector in $L^2_{\#}(\Omega)$ onto $\mathbf{V}^0_{n,\#}(\Omega)$. If we apply projector P to the Stokes equations of system (5), we obtain an equation satisfied by $P\mathbf{v}$. For that, we first introduce the unbounded operator $(A_0, D(A_0))$ on $\mathbf{V}^0_{n,\#}(\Omega)$ by

$$D(A_0) = \mathbf{H}^2_{\#}(\Omega) \cap \mathbf{H}^1_{0,\#}(\Omega) \cap \mathbf{V}^0_{n,\#}(\Omega),$$

and

$$A_0 = \nu P\Delta.$$

We also introduce $D \in \mathcal{L}(\mathbf{V}^0_{\#}(\Gamma), \mathbf{V}^0_{\#}(\Omega))$, the Dirichlet operator associated to A_0 and which is defined by $D\mathbf{h} = \mathbf{w}$ where (\mathbf{w}, q) is the solution of the equation

$$\begin{aligned} -\operatorname{div} \sigma(\mathbf{w}, q) &= 0 \text{ in } \Omega, \\ \operatorname{div} \mathbf{w} &= 0 \text{ in } \Omega, \quad \mathbf{w} = \mathbf{h} \text{ on } \partial_{\#}\Omega. \end{aligned}$$

The spaces $\mathbf{V}^0_{\#}(\Gamma)$ and $\mathbf{V}^0_{\#}(\Omega)$ are defined by

$$\begin{aligned} \mathbf{V}^0_{\#}(\Omega) &= \{\mathbf{y} \in \mathbf{L}^2_{\#}(\Omega) \text{ such that } \operatorname{div} \mathbf{y} = 0\}, \\ \mathbf{V}^0(\Gamma) &= \{\mathbf{y} \in \mathbf{L}^2_{\#}(\Gamma) \text{ such that } \int_{\Gamma} \mathbf{y} \cdot \mathbf{n} = 0\}. \end{aligned}$$

We set

$$D_s \eta_2 := D(\eta_2 \mathbf{e}_3 \chi_{\Gamma_s}) \text{ and } D_c \mathbf{h} := D(\mathbf{h} \chi_{\Gamma_c}).$$

The projection $P\mathbf{v}$ of \mathbf{v} satisfies the equation

$$\frac{d}{dt} P\mathbf{v} = (A_0 + \omega I)P\mathbf{v} + (-A_0)PD_s \eta_2 + \sum_{i=1}^{N_c} g_i (-A_0)PD_c \mathbf{w}_i.$$

• In order to deal with the plate equation we have to express the pressure p in terms of \mathbf{v} , η_1 and \mathbf{g} . For that, we introduce the Neumann operator N defined by $Nh = q$, where q is the solution of the equation

$$\Delta q = 0 \text{ in } \Omega, \quad \frac{\partial q}{\partial \mathbf{n}} = h \text{ on } \partial_{\#}\Omega.$$

We set

$$N_s \eta_2 := N(\eta_2 \mathbf{e}_3 \chi_{\Gamma_s}) \text{ and } N_c h := N(h \chi_{\Gamma_c}).$$

The pressure p is defined by

$$p = N(\nu \Delta \mathbf{v} \cdot \mathbf{n}) + \omega N_s \eta_2 - N_s \eta_{2,t} - \sum_{i=1}^{N_c} (g_{i,t} - \omega g_i) N_c(\mathbf{w}_i \cdot \mathbf{n}).$$

For the structure, we define the unbounded operator $(A_{\alpha,\beta}, D(A_{\alpha,\beta}))$ on $L^2_{\#}(\Gamma_s)$ by

$$D(A_{\alpha,\beta}) = H^4_{\#}(\Gamma_s) \cap H^2_{0,\#}(\Gamma_s) \cap L^2_{0,\#}(\Gamma_s),$$

and

$$A_{\alpha,\beta} = \beta \Delta - \alpha M_s \Delta^2.$$

A. First Abstract Reformulation of the Control Problem

We equip the Hilbert space $\mathbf{H} = \mathbf{V}^0_{n,\#}(\Omega) \times \{H^2_{0,\#}(\Gamma_s) \cap L^2_{0,\#}(\Gamma_s)\} \times L^2_{0,\#}(\Gamma_s)$ with the inner product

$$\begin{aligned} ((\mathbf{v}, \eta_1, \eta_2), (\mathbf{u}, \xi_1, \xi_2))_{\mathbf{H}} &:= (\mathbf{v}, \mathbf{u})_{\mathbf{L}^2_{\#}(\Omega)} \\ &+ ((-A_{\alpha,\beta})\eta_1, \xi_1)_{L^2_{\#}(\Gamma_s)} + (K_s \eta_2, \xi_2)_{L^2_{\#}(\Gamma_s)}, \end{aligned}$$

with $K_s := (1 + \gamma_s N_s)^{-1}$.

We define the unbounded operator $(\mathcal{A}, D(\mathcal{A}))$ on \mathbf{H} by

$$\begin{aligned} D(\mathcal{A}) &= \{(P\mathbf{v}, \eta_1, \eta_2) \in \{\mathbf{H}^2_{\#}(\Omega) \cap \mathbf{V}^0_{\#}(\Omega)\} \times D(A_{\alpha,\beta}) \\ &\times \{H^2_{0,\#}(\Omega) \cap L^2_{0,\#}(\Gamma_s)\} | P\mathbf{v} = PD_s \eta_2 \text{ on } \Gamma_s, \\ &P\mathbf{v} = 0 \text{ on } \Gamma_{d,\ell} \cup \Gamma_c \cup \Gamma_{d,r}\}, \end{aligned}$$

and

$$\mathcal{A} = \begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & I \\ K_s \gamma_s N(\nu \Delta(\cdot) \cdot \mathbf{n}) & K_s A_{\alpha,\beta} & \gamma K_s \Delta \end{pmatrix}.$$

Proposition 2. *The quadruplet $(\mathbf{v}, p, \eta_1, \eta_2) \in \mathbf{H}^{2,1}_{\#}(Q_{\infty}) \times L^2(0, \infty; \mathcal{H}^1_{\#}(\Omega)) \times H^{4,2}_{\#}(\Sigma^{\infty}_s) \times H^{2,1}_{\#}(\Sigma^{\infty}_s)$ is a solution of system (5) if and only if*

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} &= (\mathcal{A} + \omega I) \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} + \mathcal{B}_c \mathbf{g} + \mathcal{B}_d \mathbf{g}_t, \\ (P\mathbf{v} \quad \eta_1 \quad \eta_2)(0) &= (P\mathbf{v}^0 \quad \eta_1^0 \quad \eta_2^0), \end{aligned} \tag{6}$$

$$(I - P)\mathbf{v} = (I - P)D_s \eta_2 + \sum_{i=1}^{N_c} g_i (I - P)D_c \mathbf{w}_i,$$

$$\begin{aligned} p &= N(\nu \Delta \mathbf{v} \cdot \mathbf{n}) + \omega N_s \eta_2 - N_s \eta_{2,t} \\ &+ \omega \sum_{i=1}^{N_c} g_i N_c(\mathbf{w}_i \cdot \mathbf{n}) - \sum_{i=1}^{N_c} g_{i,t} N_c(\mathbf{w}_i \cdot \mathbf{n}), \end{aligned}$$

where

$$\mathcal{B}_c \mathbf{g} = \begin{pmatrix} \sum_{i=1}^{N_c} g_i (-A_0)PD_c \mathbf{w}_i \\ 0 \\ \omega \sum_{i=1}^{N_c} g_i K_s \gamma_s N_c \mathbf{w}_i \end{pmatrix},$$

and

$$\mathcal{B}_d \mathbf{g}_t = \begin{pmatrix} 0 \\ 0 \\ -\sum_{i=1}^{N_c} g_{i,t} K_s \gamma_s N_c(\mathbf{w}_i \cdot \mathbf{n}) \end{pmatrix}.$$

B. The Extended System

We can see that the time derivative of \mathbf{g} appears in system (6). In order to have an evolution equation in the form of a controlled system, we write an extended system by considering \mathbf{g} as a new state variable and by introducing $\mathbf{f} := \mathbf{g}_t - \Lambda \mathbf{g}$ as a new control variable, where Λ is a diagonal matrix satisfying conditions (C1) and (C2) in section II-C. Thus the first equation of system (6) may be rewritten as

$$\frac{d}{dt} \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \\ \mathbf{g} \end{pmatrix} = \mathcal{A}_e \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \\ \mathbf{g} \end{pmatrix} + \mathcal{B}_e \mathbf{f}, \quad (7)$$

$$(P\mathbf{v} \ \eta_1 \ \eta_2 \ \mathbf{g})(0) = (P\mathbf{v}^0 \ \eta_1^0 \ \eta_2^0 \ 0),$$

where $(\mathcal{A}_e, D(\mathcal{A}_e))$ is an unbounded operator on $\mathbf{H}_e = \mathbf{H} \times \mathbb{R}^{N_c}$ defined by

$$\mathcal{A}_e = \begin{pmatrix} \mathcal{A} + \omega I & \mathcal{B}_c + \mathcal{B}_d \Lambda \\ 0 & \Lambda \end{pmatrix},$$

and

$$\begin{aligned} D(\mathcal{A}_e) = & \{(\mathbf{v}, \eta_1, \eta_2, \mathbf{g}) \in \{\mathbf{H}_{\#}^2(\Omega) \cap \mathbf{V}_{\#}^0(\Omega)\} \times D(\mathcal{A}_{\alpha, \beta}) \\ & \times \{H_{0, \#}^2(\Omega) \cap L_{0, \#}^2(\Gamma_s)\} \times \mathbb{R}^{N_c} | \mathbf{v} = 0 \text{ on } \Gamma_{d, \ell} \cup \Gamma_{d, r}, \\ & \mathbf{v} = PD_s \eta_2 \text{ on } \Gamma_s \text{ and } \mathbf{v} = \sum_{i=1}^{N_c} g_i PD \mathbf{w}_i \text{ on } \Gamma_c\}. \end{aligned}$$

We have

$$\mathcal{B}_e \mathbf{f} = \begin{pmatrix} 0 \\ 0 \\ -\sum_{i=1}^{N_c} f_i K_s \gamma_s N_c (\mathbf{w}_i \cdot \mathbf{n}) \\ \mathbf{f} \end{pmatrix}.$$

Now that we have written the problem of control in the classical form (7), we will focus on its stabilization by feedback. We have the following result.

Theorem 3. *Let $\omega > 0$ be given. Then for all $(\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}_{cc}$ there exists a control $\mathbf{f} \in L^2(0, \infty; \mathbb{R}^{N_c})$ defined in feedback form, such that the solution $(P\mathbf{v}, \eta_1, \eta_2, \mathbf{g})$ to (7) obeys:*

$$\begin{aligned} & \|(P\mathbf{v}, \eta_1, \eta_2, \mathbf{g})\|_{\mathbf{H}_{\#}^2(\Omega) \times H_{\#}^4(\Gamma_s) \times H_{\#}^2(\Gamma_s) \times \mathbb{R}^{N_c}} \\ & \leq C_2 \|(P\mathbf{v}^0, \eta_1^0, \eta_2^0)\|_{\mathbf{H}_{cc}}, \end{aligned}$$

with $C_2 > 0$.

To prove this theorem, we need the stabilizability of the pair $(\mathcal{A}_e, \mathcal{B}_e)$ established in the next section.

C. Stabilizability of the Pair $(\mathcal{A}_e, \mathcal{B}_e)$

We will show how to construct the family $(\mathbf{w}_i)_{1 \leq i \leq N_c}$ and the diagonal matrix Λ in order to obtain the stabilizability of the pair $(\mathcal{A}_e, \mathcal{B}_e)$. The criterion that will be used here is that of Hautus, namely

$$\text{Ker}(\lambda I - \mathcal{A}_e^*) \cap \text{Ker}(\mathcal{B}_e^*) = \{0\}, \quad (8)$$

for all $\lambda \in \mathbb{C}$ such that $\Re e(\lambda) \geq 0$.

The adjoint $(\mathcal{A}_e^*, D(\mathcal{A}_e^*))$ of $(\mathcal{A}_e, D(\mathcal{A}_e))$ in \mathbf{H}_e is

$$\mathcal{A}_e^* = \begin{pmatrix} \mathcal{A}^* + \omega I & 0 \\ \mathcal{J} & \Lambda \end{pmatrix},$$

where $(\mathcal{A}^*, D(\mathcal{A}^*))$ is the adjoint of $(\mathcal{A}, D(\mathcal{A}))$ in \mathbf{H} . The operator \mathcal{J} belonging to $\mathcal{L}(D(\mathcal{A}^*), \mathbb{R}^{N_c})$ and \mathcal{B}_e^* the adjoint of \mathcal{B}_e in \mathbf{H}_e are defined by

$$\mathcal{J} \begin{pmatrix} \mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = - \left(\int_{\Gamma_c} \sigma(\mathbf{v}, p) \mathbf{n} \cdot \mathbf{w}_i \right)_{1 \leq i \leq N_c},$$

and

$$\mathcal{B}_e^* \begin{pmatrix} \mathbf{v} \\ \eta_1 \\ \eta_2 \\ \mathbf{g} \end{pmatrix} = \left(\mathbf{g}_i - \int_{\Gamma_c} (N_s \eta_2) \mathbf{n} \cdot \mathbf{w}_i \right)_{1 \leq i \leq N_c},$$

for all $(\mathbf{v}, \eta_1, \eta_2)$ in $D(\mathcal{A}^*)$ and $\mathbf{g} \in \mathbb{R}^{N_c}$. If λ belongs to $\text{spec}(\mathcal{A}_e^*)$, we have

$$(\mathcal{A}^* + \omega I) \begin{pmatrix} \mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix},$$

and

$$\mathcal{J} \begin{pmatrix} \mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} + \Lambda \mathbf{g} = \lambda \mathbf{g}.$$

We choose $\Lambda = \text{diag}(\Lambda_1, \dots, \Lambda_{N_c})$ in such a way that the following condition is satisfied:

$$\text{spec}(\mathcal{A}^* + \omega I) \cap \text{spec}(\Lambda) = \{0\}. \quad (\text{C1})$$

Thanks to (C1), if λ belongs to $\text{Ker}(\lambda I - \mathcal{A}_e^*)$ and if $(\mathbf{v}, \eta_1, \eta_2, \mathbf{g})$ is an eigenfunction then we have

- either λ belongs to $\text{spec}(\mathcal{A}^*)$ and

$$\mathbf{g} = (\lambda I - \Lambda)^{-1} \mathcal{J} \begin{pmatrix} \mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix},$$

- or λ belongs to $\text{spec}(\Lambda)$ and $(\mathbf{v}, \eta_1, \eta_2) = (\mathbf{0}, 0, 0)$.

We can easily check (8) in the latter case. Indeed, if $(\mathbf{v}, \eta_1, \eta_2, \mathbf{g})$ belongs to $\text{Ker}(\lambda I - \mathcal{A}_e^*) \cap \text{Ker}(\mathcal{B}_e^*)$ and if λ belongs to $\text{spec}(\Lambda)$, then we have $(\mathbf{v}, \eta_1, \eta_2) = (\mathbf{0}, 0, 0)$ and

$$\mathcal{B}_e^* \begin{pmatrix} \mathbf{v} \\ \eta_1 \\ \eta_2 \\ \mathbf{g} \end{pmatrix} = \left(\mathbf{g}_i - \int_{\Gamma_c} (N_s \eta_2) \mathbf{n} \cdot \mathbf{w}_i \right)_{1 \leq i \leq N_c} = 0,$$

which gives $\mathbf{g} = 0$ because $N_s \eta_2$ is equal to zero.

To finish, we have to verify criterion (8) for λ belonging to $\text{spec}(\mathcal{A}^*)$ such that $\Re e(\lambda) \geq -\omega$.

We begin by proving that the unbounded operator $(\mathcal{A}, D(\mathcal{A}))$ has a compact resolvent. Thus the spectrum of \mathcal{A} consists of isolated eigenvalues of finite multiplicity.

Next, we prove that $(\mathcal{A}, D(\mathcal{A}))$ is the generator of an analytic semigroup on \mathbf{H} . Thus the number of eigenvalues

of \mathcal{A} with real part equal or greater than $-\omega$ is finite. More precisely, we have

$$\dots < \Re \lambda_{N_\omega+1} < -\omega \leq \Re \lambda_{N_\omega} \leq \dots \leq \Re \lambda_1,$$

where $(\lambda_j)_j$ are the eigenvalues of $(\mathcal{A}^*, D(\mathcal{A}^*))$, repeated according to their multiplicity.

We are going to see that, for a suitable choice of the family $(\mathbf{w}_i)_{1 \leq i \leq N_c}$, the stabilizability criterion for $(\mathcal{A}_e, \mathcal{B}_e)$ is verified. For that, we choose Λ satisfying the following additional condition:

For all $1 \leq i \leq N_\omega$, we have

$$(\lambda_i - \Lambda_i)(N_s \eta_{2,i}) \mathbf{n} + \sigma(\mathbf{v}_i, p_i) \mathbf{n} \neq 0 \text{ on } \Gamma_c, \quad (\text{C2})$$

where $(\mathbf{v}_i, \eta_{1,i}, \eta_{2,i}, \mathbf{g}_i)$ is a non zero eigenfunction of \mathcal{A}_e^* .

We choose the family $(\mathbf{w}_i)_{1 \leq i \leq N_c}$ as the union of

$$((\lambda_i - \Lambda_i)(N_s \eta_{2,i}) \mathbf{n} + \sigma(\mathbf{v}_i, p_i) \mathbf{n})_i \text{ if } \lambda_i \in \mathbb{R},$$

$$(\Re[(\lambda_i - \Lambda_i)(N_s \eta_{2,i}) \mathbf{n} + \sigma(\mathbf{v}_i, p_i) \mathbf{n}])_i,$$

and

$$(\Im[(\lambda_i - \Lambda_i)(N_s \eta_{2,i}) \mathbf{n} + \sigma(\mathbf{v}_i, p_i) \mathbf{n}])_i \text{ if } \lambda_i \notin \mathbb{R}.$$

Now let us verify (8) for λ belonging to $\text{spec}(\mathcal{A}^*)$ such that $\Re(\lambda) \geq -\omega$. Let i_0 belong to $\{1, \dots, N_\omega\}$ and $(\mathbf{v}, \eta_1, \eta_2, \mathbf{g})$ belong to

$$\text{Ker}(\lambda_{i_0} I - \mathcal{A}_e^*) \cap \text{Ker}(\mathcal{B}_e^*).$$

Then, we have

$$\begin{aligned} \lambda_{i_0} \mathbf{v} - \text{div } \sigma(\mathbf{v}, p) &= 0, \text{ div } \mathbf{v} = 0 \text{ in } \Omega, \\ \mathbf{v} &= \eta_2 \mathbf{e}_3 \text{ on } \Gamma_s, \mathbf{v} = 0 \text{ on } \Gamma_{d,\ell} \cup \Gamma_c \cup \Gamma_{d,r}, \\ \lambda_{i_0} \mathbf{g} - \Lambda \mathbf{g} &= - \left(\int_{\Gamma_c} \sigma(\mathbf{v}, p) \mathbf{n} \cdot \mathbf{w}_i \right)_{1 \leq i \leq N_c} \\ \lambda \eta_1 + \eta_2 &= 0 \text{ on } \Sigma_s^\infty, \\ \lambda_{i_0} \eta_2 + \beta \Delta \eta_1 - \gamma M_s \Delta \eta_2 - \alpha M_s \Delta^2 \eta_1 &= \gamma_s p \text{ on } \Sigma_s^\infty, \\ \eta_1 &= 0, \frac{\partial \eta_1}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_{s,\ell,r}, \\ \left(\mathbf{g}_i - \int_{\Gamma_c} (N_s \eta_2) \mathbf{n} \cdot \mathbf{w}_i \right)_{1 \leq i \leq N_c} &= 0. \end{aligned} \quad (9)$$

Since λ_{i_0} does not belong to $\text{spec}(\Lambda)$, equation (9)₃ can be solved and we obtain

$$\mathbf{g} = -(\lambda_{i_0} I - \Lambda)^{-1} \left(\int_{\Gamma_c} \sigma(\mathbf{v}, p) \mathbf{n} \cdot \mathbf{w}_i \right)_{1 \leq i \leq N_c}. \quad (10)$$

Thus, the last equation of system (9) becomes

$$\left(\int_{\Gamma_c} [(\lambda_{i_0} - \Lambda_i)(N_s \eta_2) \mathbf{n} + \sigma(\mathbf{v}, p) \mathbf{n}] \cdot \mathbf{w}_i \right)_{1 \leq i \leq N_c} = 0.$$

Thanks to the definition of the family $(\mathbf{w}_i)_{1 \leq i \leq N_c}$, the vector $((\lambda_{i_0} - \Lambda_i)(N_s \eta_2) \mathbf{n} + \sigma(\mathbf{v}, p) \mathbf{n})_{1 \leq i \leq N_c}$ belongs to $\text{Vect}(\mathbf{w}_1, \dots, \mathbf{w}_{N_c})$. Thus we have

$$\int_{\Gamma_d} \left| (\lambda_{i_0} - \Lambda_i)(N_s \eta_2) \mathbf{n} + \sigma(\mathbf{v}, p) \mathbf{n} \right|^2 = 0.$$

Therefore, thanks to (C2), we have $\mathbf{v} = 0$, $\eta_1 = 0$ and $\eta_2 = 0$. Finally, we use equation (10) to obtain $\mathbf{g} = 0$. Thus the stabilizability of the pair $(\mathcal{A}_e, \mathcal{B}_e)$ is proved.

III. STABILIZATION OF THE NONLINEAR SYSTEM

The last part of the work consists in the following steps:

- Derive estimates for the nonlinear terms F_f , G and F_s in (4).
- Study the nonhomogeneous version of the linearized system (5), with right-hand sides f , g , h playing the roles of the nonlinear terms in place of F_f , G , F_s respectively.
- Use a fixed point method to prove the local stabilization of the closed loop nonlinear system.

IV. CONCLUSION AND PERSPECTIVES

The stabilization of a coupled 3D fluid-structure system has been studied thanks to a finite-dimensional boundary control for the fluid. For a suitable choice of the finite dimensional control space, we state the existence of a feedback control able to locally stabilize around 0 the solution of the full nonlinear system with a prescribed exponential decay rate.

The study of the stabilization of an unstable equilibrium can be performed with the same strategy, but with more involved technicalities. This will be the subject of a future paper.

Moreover, numerical experiments for the boundary control of such coupled systems are part of an ongoing research project.

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