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Synchronization Conditions for Diffusively Coupled Linear Systems

Tian Xia and Luca Scardovi

***Abstract*— The paper addresses the synchronization problem for a network of identical, linear time-invariant state-space models. The notion of synchronizability is investigated and a set of sufficient and necessary conditions relating synchronizability to the dynamical properties of the subunits are provided. The paper also extends recent results about synchronization of passive linear systems by proving that networks of linear, detectable, passive systems can be synchronized by any (possibly directed) connected interconnection topology. The theory is illustrated with several examples.**

I. INTRODUCTION

Synchronization has been recently a popular subject in systems and control motivated by many applications in physics, biology, and engineering [1], [2].

Coordination problems, typical in the engineering world, can often be addressed as synchronization problems in which both the dynamical aspects of the subunits and the limited communication aspects governing their interaction play a role. Designing control laws and investigating interconnection structures that can ensure synchronization is therefore a problem that has attracted quite some attention in the recent years [3]–[7].

The present paper focuses on the synchronization problem in the linear case. Assuming N identical dynamical systems, each described by the linear, time-invariant (LTI), state-space model (A, B, C), we investigate conditions on the individual dynamics for the existence of a static distributed controller that synchronizes the network.

Several papers have been published on synchronization of linear systems [8]–[15]. Synchronization using static controllers is addressed in [14]–[17], and partially in [9]. These results focused mainly on neutrally stable systems (such as harmonic oscillators), and double integrators. Some of these limitations can be overcome by using dynamic controllers. In [9] a condition for synchronizability using a class of dynamic controllers is presented while [8] presents a fairly general solution to the synchronization problem when the interconnection topology is time-varying. Finally [10] provides synchronization conditions when the systems are not assumed to be identical.

After presenting a preliminary result that relates synchronization to the eigen-structure of the interconnection matrix, in the first part of the paper we provide sufficient and necessary conditions for the synchronizability of a collection of LTI systems. In short a collection of LTI systems is synchronizable if it is possible to find a distributed controller (including the communication topology) that synchronizes the network. The results draw a connection between synchronizability and output-feedback stabilizability of the subunits and provide the minimum requirement for synchronization by a static controller. The results are derived for general interconnection structures, where the weights are not constrained to be nonnegative, and for general LTI systems (the dynamics are not supposed to be stable). The results are then specialized to graphs with nonnegative weights and undirected graphs. By leveraging on these results, we then prove that networks of passive and detectable systems can be synchronized by any connected communication topology with nonnegative weights, thus extending recent results where the communication topology was assumed to satisfy certain symmetry conditions (Theorem 4 in [8]).

This paper is organized as follows. In Section II the notation used throughout the paper is summarized. Section III introduces the synchronization problem. In Section IV and V the main results are presented. The theory is illustrated with several examples.

II. BACKGROUND AND NOTATIONS

Throughout the paper we will use the following notation. Given a set of vectors $x_i \in \mathbb{R}^n, i = 1, 2, \ldots, N$ we indicate the stacked vector as $x = [x_1^T, x_2^T, \ldots, x_N^T]^T$. Given a complex matrix P we indicate with P^* its conjugate transpose. We use $\mathbf{1}_N$ and $\mathbf{0}_N$ to denote the column vectors in $\mathbb{R}^N$ containing a 1 in every entry and a zero in every entry respectively. We use I_N to indicate the identity matrix with N elements.

Let Q_N be a $(N-1) \times N$ real matrix satisfying the following properties:

$$Q_N \mathbf{1}_N = 0,\ Q_N^T Q_N = \Pi_N,\ Q_N Q_N^T = I_{N-1}, \qquad (1)$$

where $\Pi_N := I_N - \frac{1}{N}\mathbf{1}_N \mathbf{1}_N^T$. Q_N exists but it is non-unique. Q_N will consistently denote one of such matrices. It can be easily verified that Q_N is a matrix whose rows form an orthonormal basis for $\mathbf{1}_N^{\perp}$ (the vector space orthogonal to the linear space spanned by the vector $\mathbf{1}_N$). Given the stacked vector $x = [x_1^T, x_2^T, \ldots, x_N^T]^T$ we define

$$\begin{aligned} x_s &:= \frac{1}{N}(\mathbf{1}_N^T \otimes I_n)x, \\ x_n &:= (Q_N \otimes I_n)x, \end{aligned} \qquad (2)$$

and express x as

$$x = (Q_N^T \otimes I_n)x_n + (\mathbf{1}_N \otimes I_n)x_s. \qquad (3)$$

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Tian Xia and Luca Scardovi are with the Department of Electrical and Computer Engineering, University of Toronto, Toronto, ON, M5S 3G4, Canada; e-mail: t.xia@mail.utoronto.ca; scardovi@scg.utoronto.ca.

The vector x_s is the average of the vectors $x_i, i = 1, \dots, N$, while x_n vanishes whenever $x_i = x_j$ for every $i, j = 1, \dots, N$.

A. Graph theory

A *directed graph* G consists of the triple $(\mathcal{V}, \mathcal{E}, \Sigma)$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges and $\Sigma \in \mathbb{R}^{N \times N}$ is a weighted adjacency matrix. Each $\sigma_{i,j}$ (an element of Σ) will be nonzero if and only if $(i, j) \in \mathcal{E}$, otherwise $\sigma_{i,j} = 0$. When $(k, j) \in \mathcal{E}$, node j is called a *neighbor* of node k. We assume that there are no self-loops, i.e. $\sigma_{i,i} = 0$ for $i = 1 \dots N$. Unless differently stated, we allow for negative weights $\sigma_{i,j}$. The set of graphs with the properties above is denoted with the symbol $\mathcal{G}$. Two subsets of $\mathcal{G}$ are given special notations: $\mathcal{G}_+$ is the subset of all graphs with non-negative weights ($\sigma_{i,j} \geq 0$); and $\mathcal{G}_u$ is the subset of graphs with symmetric matrices Σ.

Given a graph $G \in \mathcal{G}$ we define the *interconnection matrix* L as the $N \times N$ matrix with elements

$$l_{i,j} := \begin{cases} \sum_{k=1}^{N} \sigma_{i,k}, & i = j, \\ -\sigma_{i,j}, & i \neq j. \end{cases} \tag{4}$$

The matrix L always contains 0 and $\mathbf{1}_N$ as an eigenvalue-eigenvector pair (since L has zero row sum). L has special properties when restricted to $\mathcal{G}_+$ and $\mathcal{G}_u$.

For graphs contained in $\mathcal{G}_u$, L is a symmetric matrix, hence has real eigenvalues. For graphs contained in $\mathcal{G}_+$, the associated interconnection matrix L is called *Laplacian matrix*. All the eigenvalues of a Laplacian matrix have non-negative real part and the zero eigenvalue has multiplicity one if and only if the graph is connected[1]. According to Proposition 1 in [18], the spectrum of an N-dimensional Laplacian matrix is localized to a subset of

$$\Gamma := \left\{ \delta + i\omega : \|\omega\| \leq \|\delta\| \cot\frac{\pi}{N},\ \delta \geq 0 \right\}. \tag{5}$$

Given the matrix Q satisfying properties (1) and an interconnection matrix L, we define the *reduced interconnection matrix* by

$$\tilde{L} := QLQ^T. \tag{6}$$

When L is a Laplacian matrix, $\tilde{L}$ is called the *reduced Laplacian matrix*.

The spectrum of the $(N-1) \times (N-1)$ matrix $\tilde{L}$ will be denoted by Λ. Λ is the spectrum of L with one zero eigenvalue removed [19]. Furthermore, for a reduced Laplacian matrix, Λ is contained in the closed right-half plane.

[1] A path between two nodes n_1, n_l is a sequence of nodes $\{n_1, n_2, \dots, n_l\}$ such that n_i, n_{i+1} is an edge for $i = 1, \dots l-1$. A graph is *connected* if it has one node to which there exists a path from every other node

III. Problem Statement

Consider a collection of N identical LTI systems

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i, \\ y_i &= Cx_i, \end{aligned} \tag{7}$$

where $x_i \in \mathbb{R}^n$, $y_i \in \mathbb{R}^l$ and $u_i \in \mathbb{R}^m$, $i = 1, \dots, N$, $N \geq 2$.

Given a graph G with N nodes we associate each system (7) to a node. We assume that each system can receive information only from its neighbors according to the following coupling structure

$$u_i = K \sum_{j=1}^{N} \sigma_{i,j}(y_j - y_i), \quad i = 1, \dots, N. \tag{8}$$

where K is a suitable matrix to be designed.

The next Definition formalizes the notion of network synchronization.

Definition 1. *The network defined by (7), where u_i are defined by (8) is said to synchronize if*

$$\lim_{t \to \infty} (x_i(t) - x_j(t)) = 0$$

for any initial conditions and for all $i, j = 1, 2, \dots, N$.

One of the main contributions of this paper is to investigate under what conditions the collection of systems (7) is *synchronizable* in the sense specified by the next definition.

Definition 2. *The collection of LTI systems defined in (7) is statically-synchronizable with respect to a set of graphs $\mathcal{S}$ if there exist a static controller (8) and a graph $G \in \mathcal{S}$ such that the network synchronizes.*

IV. Conditions for Static Synchronizability

In order to study the synchronization properties of (7), (8), we will make use of the notion of *synchronization region* [20], [21].

Definition 3. *Given a LTI system (A, B, C) and a $m \times l$ real matix K, the synchronization region S_K is the subset of the complex plane defined by*

$$S_K := \{s \in \mathbb{C} : A - sBKC \text{ is Hurwitz}\}. \tag{9}$$

The term *synchronization region* is justified by the synchronization criterion presented below.

Theorem 1. *The network (7), (8) synchronizes if and only if $\Lambda \subset S_K$, where Λ is the spectrum of $\tilde{L}$.*

Proof. The closed loop system (7) with (8) reads

$$\dot{x} = (I_N \otimes A - L \otimes BKC)x. \tag{10}$$

Notice that the network synchronizes if and only if for any initial conditions of (10), $\lim_{t \to \infty} x_n(t) = 0$, where x_n has been defined in (2).

After some algebraic manipulations and by using (3), the dynamics of x_n can be written as

$$\dot{x}_n = (I_{N-1} \otimes A - \tilde{L} \otimes BKC)x_n. \tag{11}$$

Let $\tilde{L} = P\Omega P^{-1}$ where Ω is the Jordan normal form of $\tilde{L}$. Because $(I_{N-1}\otimes A-\tilde{L}\otimes BKC) = (P\otimes I_n)(I_{N-1}\otimes A-\Omega\otimes BKC)(P\otimes I_n)^{-1}$, the matrix $(I_{N-1}\otimes A-\tilde{L}\otimes BKC)$ is Hurwitz if and only if $(I_{N-1}\otimes A-\Omega\otimes BKC)$ is Hurwitz. Notice that the matrix $(I_{N-1}\otimes A-\Omega\otimes BKC)$ is block upper-triangular and therefore it is Hurwitz if and only if all diagonal blocks $A-\lambda_i BKC$ are Hurwitz, where $\{\lambda_i\}_{i=1}^{N-1} = \Lambda$. This implies that system (7), (8) synchronizes if and only if $\Lambda \subset S_K$. □

Some comments on the synchronization region S_K are now in place. It is clear from (9) that it depends only on the structural properties of (7) and the choice of the feedback matrix K, it is therefore independent of the interconnection topology. The region S_K is an open set and it is symmetric with respect to the real axis because the eigenvalues of $A - sBKC$ and $A - s^*BKC$ are complex conjugated. In general, S_K has a non-trivial topology. Section 2 of [21] provided examples of systems with bounded, unbounded, or disconnected synchronization regions.

Due to Theorem 1, once the matrix K is chosen, the synchronization region S_K provides information on the set of interconnection topologies required for synchronization, as the following example illustrates.

Example 1. *In certain situations it turns out that, once the matrix K is fixed, for the network to synchronize the adjacency matrix Σ must contain some negative entries. In other words the network will not synchronize for any graph belonging to the set $\mathcal{G}_+$ but synchronize for some graphs in $\mathcal{G}$. As an example consider a network of three LTI systems each with dynamics*

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -8 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ C = I_2, \tag{12}$$

and choose $K = I_2$. From elementary computations the synchronization region is

$$S_{I_2} = \left\{ \delta + i\omega : \omega^2 > \frac{\delta^2+32}{1-\frac{\delta^2}{36}},\ \|\delta\| < 6 \right\},$$

(shaded region in Fig. 1). Suppose that $G \in \mathcal{G}_+$ and let L be the associated Laplacian matrix. From (5) the eigenvalues of L are contained in the set $R = \{\delta + i\omega : \|\omega\| \leq \|\delta\| cot\frac{\pi}{3},\ \delta \geq 0\}$ (dashed region in Fig. 1). Since the sets are disjoint, any graph $G \in \mathcal{G}_+$ will not synchronize the network.

Given a set of interconnection topologies, the spectrum of the associated interconnection matrices can discriminate the set of feedback matrices K that guarantee synchronization. For example let $G \in \mathcal{G}_u$, the set of undirected graphs. Undirected graphs are associated to symmetric interconnection matrices which possess all real eigenvalues. From Theorem 1, if the graph is undirected, synchronization requires the existence of a matrix K such that S_K intersects the real line. If $S_K \cap \mathbb{R} = \emptyset$ for every K, any synchronizing graph is associated to an asymmetric interconnection matrix. The next result provides a first characterization of static

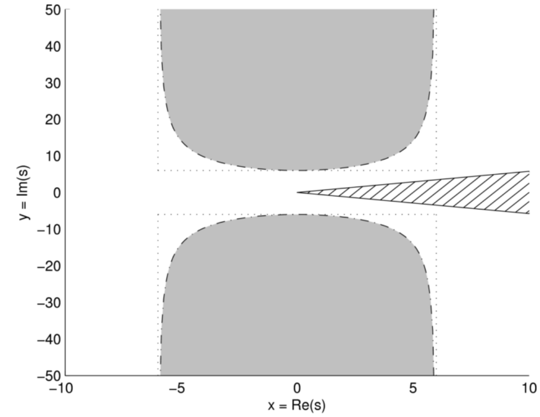

Fig. 1: Synchronization region and eigenvalues region for any graph in $\mathcal{G}_+$ (Example 1). The synchronization region is denoted by the shaded area. It is bounded inside the region $\{\delta + i\omega : \|\delta\| < 6,\ \|\omega\| > 6\}$ indicated by dotted lines. The eigenvalues of any Laplacian matrix are contained in the dashed region. Since the two regions are not intersecting synchronization is not possible with any graph in $\mathcal{G}_+$

synchronizability with respect to different sets of graphs by making use of the notion of synchronization region.

Lemma 1. *Consider the collection of systems (7), the following facts hold true:*

i) The collection is statically synchronizable with respect to the set of graphs $\mathcal{G}$ if and only if there exists a matrix K such that

$$\begin{cases} S_K \cap \mathbb{R} \neq \emptyset, & if\ N\ is\ even; \\ S_K \neq \emptyset, & if\ N\ is\ odd. \end{cases} \tag{13}$$

ii) The collection is statically synchronizable with respect to the set of graphs $\mathcal{G}_u$ if and only if there exists a matrix K such that

$$S_K \cap \mathbb{R}. \neq \emptyset \tag{14}$$

Proof. Proof of i):
($\Rightarrow$) When N is even, the reduced interconnection matrix $\tilde{L}$ has an odd number of eigenvalues. Therefore, Λ contains at least one real number. By Theorem 1, there exists a matrix K such that S_K contains Λ. We conclude that S_K must intersect the real axis. When N is odd, $\Lambda \subset S_K$. Thus, $S_K \neq \emptyset$ since $N \geq 2$.
($\Leftarrow$) If N is even and $S_K \cap \mathbb{R} \neq \emptyset$, there exists a real $p \in S_K \cap \mathbb{R}$. Consider the complete graph $\sigma_{i,j} = \frac{p}{N}$ for all $i \neq j$. It can be readily checked that the eigenvalues of L are $\{0, p, \ldots, p\}$ and therefore all eigenvalues of $\tilde{L}$ are contained in S_K. If N is odd and $S_K \neq \emptyset$, there exist $p \in S_K$ and $p^* \in S_K$ (where * denotes complex conjugation). Define L as:

$$L = P \begin{bmatrix} 0 & & & 0 \\ & R & & \\ & & \ddots & \\ 0 & & & R \end{bmatrix} P^{-1}, \qquad R = \begin{bmatrix} \text{Re}\{p\} & -\text{Im}\{p\} \\ \text{Im}\{p\} & \text{Re}\{p\} \end{bmatrix}, \tag{15}$$

where P is any invertible real matrix with first column $\mathbf{1}_N$. The eigenvalues of every R block are $\mathrm{Re}\{p\} \pm i\mathrm{Im}\{p\} \in \{p, p^*\}$. Therefore, all eigenvalues of the reduced interconnection matrix are contained in S_K. It remains to ensure that L is a interconnection matrix. This is true as long as L is a real matrix satisfying $L\mathbf{1}_N = 0$. Clearly, L is a real matrix. By construction of P, $P[1, 0, \ldots, 0]^T = \mathbf{1}_N$ or equivalently $P^{-1}\mathbf{1}_N = [1, 0, \ldots, 0]^T$. Therefore, L is a valid interconnection matrix with all the non-zero eigenvalues contained in S_K.

Proof of ii): any Laplacian matrix in $\mathcal{G}_u$ has real eigenvalues. Therefore, from Theorem 1, it is necessary that the synchronization region S_K intersects the real axis for some K for synchronization to take place. The sufficiency part follows from the same construction in the proof of fact i). □

Lemma 1 indicates that static synchronizability mainly depends on the synchronization region of the constituent system, which is exclusively a property of the systems dynamics. The next results, based on Lemma 1, relate static synchronizability to output-feedback stabilizability.

Theorem 2. *Consider the collection of systems (7). Then following facts hold true*

i) Assume that N is even. The collection (7) is statically synchronizable with respect to the set of graphs $\mathcal{G}$ if and only if the system triple (A, B, C) is output-feedback-stabilizable.

ii) The collection (7) is statically synchronizable with respect to the set of graphs $\mathcal{G}_u$ if and only if the system triple (A, B, C) is output-feedback-stabilizable.

Proof. Proof of i):
($\Rightarrow$) output-feedback-stabilizability guaranties that there exists some real matrix K such that $A - BKC$ is Hurwitz. This in turn implies that S_K intersects the real axis at $s = 1$. By Lemma 1, the collection of systems (7) is therefore statically synchronizable with respect to the set of graphs $\mathcal{G}_u$ and therefore statically synchronizable with respect to the set of graphs $\mathcal{G}$.
($\Leftarrow$) Because the collection is statically synchronizable with respect to the set of graphs $\mathcal{G}$ and N is even, Lemma 1 guarantees the existence of a real matrix K and $s \in \mathbb{R}$ such that $A - sBKC$ is Hurwitz. Hence, the system is output-feedback stabilizable.

Proof of ii): from Lemma 1 static synchronizability with respect to the set of graphs $\mathcal{G}_u$ is equivalent to static synchronizability with respect to the set of graphs $\mathcal{G}$ when N is even. This proves fact ii). □

The previous theorem does not characterize static synchronizability with respect to $\mathcal{G}$ when N is odd. The next theorem completes the picture.

Theorem 3. *Assume N is odd. The collection (7) is statically synchronizable with respect to $\mathcal{G}$ if and only if the following conditions are satisfied:*

i) (A, B) is stabilizable;

ii) (A, C) is detectable;

iii) There exists real matrix K, complex number s, and Hermitian positive-semidefinite P such that

$$(A + sBKC)^* P + P(A + sBKC) + C^*C + C^*K^*KC = 0. \tag{16}$$

Proof. ($\Rightarrow$) Because the collection is statically synchronizable for some odd N, Lemma 1 guarantees the existence of a real matrix K and of a complex number s such that $A - sBKC$ is Hurwitz (S_K is not empty). Let (A, B, C) be expressed in the canonical Kalman form

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ 0 & A_{2,2} & 0 & A_{2,4} \\ 0 & 0 & A_{3,3} & A_{3,4} \\ 0 & 0 & 0 & A_{4,4} \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, \tag{17}$$
$$C = \begin{bmatrix} 0 & C_2 & 0 & C_4 \end{bmatrix},$$

where $A_{1,1}$ is the controllable/unobservable mode, $A_{2,2}$ the controllable/observable mode, $A_{3,3}$ the uncontrollable/unobservable mode, and $A_{4,4}$ the uncontrollable/observable mode. The matrix $A - sBKC$ can be rewritten in the block matrix form

$$\begin{bmatrix} A_{1,1} & A_{1,2} - sB_1KC_2 & A_{1,3} & A_{1,4} - sB_1KC_4 \\ 0 & A_{2,2} - sB_2KC_2 & A_{2,3} & A_{2,4} - sB_2KC_4 \\ 0 & 0 & A_{3,3} & A_{3,4} \\ 0 & 0 & 0 & A_{4,4} \end{bmatrix}. \tag{18}$$

Since $A - sBKC$ is Hurwitz the matrices $A_{1,1}$, $A_{3,3}$, and $A_{4,4}$ are Hurwitz. This guarantees that the uncontrollable mode and the unobservable modes of the system are asymptotically stable and therefore the system is stabilizable and detectable.

It remains to prove statement iii). Define matrices $G = A - sBKC$ and $H = C^*C + C^*K^*KC$. G is Hurwitz and H is Hermitian and positive-semidefinite because C^*C and C^*K^*KC both are. The matrix $e^{G^*t}He^{Gt}$ satisfies the following property:

$$\frac{d}{dt}e^{G^*t}He^{Gt} = G^*\left(e^{G^*t}He^{Gt}\right) + \left(e^{G^*t}He^{Gt}\right)G, \tag{19}$$

$\forall t \in \mathbb{R}$. Integrating both sides from zero to infinity and using the fact that $\lim_{t\to\infty} e^{Gt} \to 0$ we get

$$-H = G^*P + PG, \tag{20}$$

where $P = \int_0^\infty e^{G^*t}He^{Gt}dt$. The existence of P is guaranteed by the observation that $\|e^{Gt}\|$ converges exponentially to zero. Moreover, P is Hermitian and positive-semidefinite because $e^{G^*t}He^{Gt}$ is. We therefore conclude that property iii) holds.

($\Leftarrow$) Given K, P, and s satisfying (16), let $M = A + sBKC$ and $J = \begin{bmatrix} C \\ KC \end{bmatrix}$. Note that (16) is equivalent to $M^*P + PM + J^*J = 0$. We want to show that M is Hurwitz. The equation $M^*P + PM + J^*J = 0$ implies that:

$$\int_0^t e^{M^*\tau}J^*Je^{M\tau}d\tau = -\int_0^t e^{M^*\tau}\left(M^*P + PM\right)e^{M\tau}d\tau = P - e^{M^*t}Pe^{Mt}. \tag{21}$$

Since $P \geq 0$ the matrix $e^{M^* t} P e^{Mt}$ is also positive-semidefinite. Therefore for any complex vector x,

$$\int_0^t x^* e^{M^* \tau} J^* J e^{M\tau} x d\tau = x^* P x - x^* e^{M^* t} P e^{Mt} x \leq x^* P x. \tag{22}$$

Assume by contradiction that M is not Hurwitz. Then M has an eigenvalue λ satisfying $\text{Re}\{\lambda\} \geq 0$. Let v be a corresponding eigenvector, then $e^{Mt} v = e^{\lambda t} v$. Substituting v into (22) we obtain

$$v^* J^* J v \int_0^t e^{\lambda^* t} e^{\lambda t} d\tau \leq v^* P v. \tag{23}$$

The only way for (23) to hold is that $Jv = 0$.

Detectability of (A, C) implies the existence of a real matrix D such that $A + DC$ is Hurwitz. Notice that $= A + DC = M + \begin{bmatrix} D & -sB \end{bmatrix} J$. Post-multiplying by the eigenvector v we obtain that $\left(M + \begin{bmatrix} D & -sB \end{bmatrix} J\right) v = \lambda v$ and therefore $A + DC$ cannot be Hurwitz, contradicting the hypothesis. We conclude that M is Hurwitz and by Lemma 1, that the collection is static synchronizable when N is odd. □

The main theorem in [22] provides necessary and sufficient conditions for output-feedback-stabilizability in LTI systems. The conditions are similar to the conditions in Theorem 3 except the additional constraint that s is real in (16). This implies that the conditions in Theorem 3 are weaker than the conditions in Theorem 2. We conclude that the conditions of Theorem 3 are the minimum requirement for synchronization by a static controller.

Remark 1. *If a collection is statically synchronizable with respect to the set of graphs $\mathcal{G}$ for even N then satisfies the same property for odd N. In fact, by Lemma 1, static synchronizability with respect to the set of graphs $\mathcal{G}$ for even N implies the existence of controller K such that $S_K \cap \mathbb{R} \neq \emptyset$. Therefore, $S_K \neq \emptyset$ and the collection is statically synchronizable for any odd N. It is natural to ask whether the converse holds. The answer is no, counterexamples exist and one is presented in the next example.*

Example 2. *We provide an example of a collection of systems that is statically synchronizable with respect to the set of graphs $\mathcal{G}_+$ if and only if N is odd. Consider the system*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 6.4 & 6.4 & 4.4 \end{bmatrix}. \tag{24}$$

To prove that the system is not synchronizable when N is even, it suffices to show (A, B, C) is not output-feedback-stabilizable (Theorem 2). Notice the special structure of $A - BKC$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 - 6.4K & -1 - 6.4K & -2 - 4.4K \end{bmatrix}. \tag{25}$$

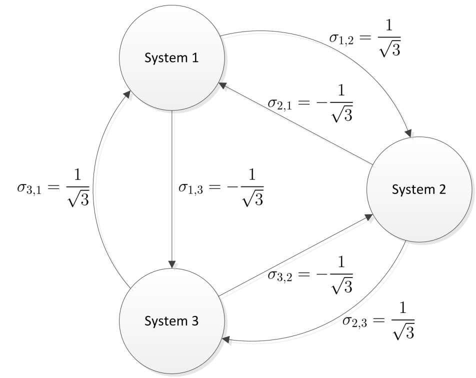

Fig. 2: Circulant interconnection topology used in Example 2. Each system has two neighbors, it is attracted by one and repelled by the other.

The characteristic polynomial of $A - BKC$ is $\chi_{A-BKC}(\lambda) = \lambda^4 + (4.4K + 2)\lambda^3 + (6.4K + 1)\lambda^2 + (6.4K - 1)\lambda - 1$. By the Routh-Hurwitz criterion, it can be easily shown that $A - BKC$ is non Hurwitz regardless the choice of K. Static synchronizability of odd collections is guaranteed by Theorem 1 by observing that $A - sBKC$ is Hurwitz for $K = 1$ and $s = i$ and by choosing an interconnection matrix with purely imaginary eigenvalues. As an example, suppose that $N = 3$. By choosing $K = 1$ and the interconnection shown in Fig. 2, the eigenvalues of the reduced interconnection matrix are $\pm i \in S_K$. By Theorem 1 the network synchronizes.

V. Synchronization of Passive LTI Systems

In the previous Section we focused on conditions for the existence of a distributed feedback (8) (involving the feedback matrix K and a graph G) that yields synchronization. The next result deals with a different question: under what conditions on (7) is it possible to find a feedback matrix K such that the network synchronizes for any connected graph in $\mathcal{G}_+$? We prove that passivity and detectability are such conditions. Synchronization of passive systems have been studied in literature (see e.g. Theorem 4 in [8]). Since detectable passive systems are output-feedback-stabilizable, from Theorem 2 we can conclude that detectable passive linear systems are synchronizable with respect to the set of graphs $\mathcal{G}_u$. Moreover, since for detectable passive LTI systems $A - kBC$ is a Hurwitz matrix for any positive real scalar k, the synchronization region S_I contains the positive real line. In conjunction with the properties of Laplacian matrices, this implies that any connected-undirected graph with non-negative weights, $G \in \mathcal{G}_+ \cap \mathcal{G}_u$, is sufficient to synchronize a collection of passive systems (in agreement with e.g. [8]). With the next result we show that this result can be generalized and synchronization of passive-detectable systems can be achieved by choosing any connected graph in $\mathcal{G}_+$.

Theorem 4. *If (A, B, C) is passive and detectable then*
i) The collection is statically synchronizable with respect to the set of graphs $\mathcal{G}_u$.

ii) The network synchronizes for any connected graph in $\mathcal{G}_+$ and any symmetric and positive definite feedback matrix K.

Proof. Proof of i): since detectable-passive systems are output-feedback-stabilizable the result follows directly from Theorem 2.

Proof of ii): by Theorem 1 we need to prove that $A - \lambda_i BKC$ is Hurwitz for any $K = K^T > 0$ and any graph $G \in \mathcal{G}_+$, where $\{\lambda_i\}_{i=1}^{N-1} = \Lambda$. Since Λ is contained in the open right-half plane for Laplacian matrices associated to connected graphs in $\mathcal{G}_+$, it suffices to show that $A - sBKC$ is Hurwitz for any s satisfying $\text{Re}\{s\} > 0$, and any $K = K^T > 0$. We will proceed by contradiction. Assume that there exists a complex number $\bar{s}$, with $\text{Re}\{\bar{s}\} > 0$, and a symmetric real matrix $\bar{K} > 0$ such that $A - \bar{s}B\bar{K}C$ is not Hurwitz. Let λ be an eigenvalue of $A - \bar{s}B\bar{K}C$ such that $\text{Re}\{\lambda\} \geq 0$. Let v be the corresponding eigenvector, then $(A - \bar{s}B\bar{K}C)v = \lambda v$ and

$$\begin{aligned} &v^* \left(P\left(A - \bar{s}B\bar{K}C\right) + \left(A - \bar{s}B\bar{K}C\right)^* P\right) v \\ &= 2\text{Re}\{\lambda\} v^* P v \geq 0 \end{aligned} \tag{26}$$

for any complex $P = P^* > 0$.

Since (A, B, C) is passive, there exists a real symmetric matrix $\bar{P} > 0$ such that

$$\bar{P}A - A^T\bar{P} \leq 0\,, \tag{27}$$
$$B^T\bar{P} = C. \tag{28}$$

By substituting $\bar{P}$ into (26) and by using (28) we obtain

$$\begin{aligned} &v^* \left(\bar{P}\left(A - \bar{s}B\bar{K}C\right) + \left(A - \bar{s}B\bar{K}C\right)^* \bar{P}\right) v \\ &= v^* \left(\bar{P}A + A^T\bar{P} - 2\text{Re}\{\bar{s}\}C^T\bar{K}C\right) v \\ &= 2\text{Re}\{\lambda\} v^* \bar{P} v \geq 0. \end{aligned}$$

Since $K > 0$, $v^*(\bar{P}A + A^T\bar{P})v \leq 0$ and $-\text{Re}\{\bar{s}\}v^*C^T\bar{K}Cv \leq 0$, it follows that $Cv = 0$. But this means that v belongs to both an unstable eigenspace of A and to the kernel of C implying that the (A, B, C) is not detectable thus leading to a contradiction.

□

VI. Conclusion

In this paper we have related the synchronizability property of a collection of LTI systems with the dynamical properties of the subunits. We defined the notion of synchronizability as the existence of a distributed feedback that synchronizes the network. We showed that synchronizability is strictly related to the property of output feedback stabilizability of each subunit. We showed next that a network of passive and detectable LTI systems can be synchronized with any communication topology, assuming non negative weights.

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