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Lifetime of transient dynamics on complex networks *

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***Abstract*— We study individual agents with identical linear dynamics interconnected in a network, and in particular the lifetime of the transient behaviour, before the asymptotically dominant behaviour is reached. In application and main contribution, we give explicit expressions for the mixing time of non-Poissonian random walkers on a network, and show that it is typically either determined by the network topology alone, or by the waiting time distribution alone.**

I. Networks of dynamical systems: dominant and transient behaviour

When many agents interact together given a certain interconnection network, the dynamics of the global many-agent system results from both the individual dynamics and the way they are interconnected. Finding in which ways has been a central topic in the field of Complex Systems, Complex Networks, Networked Control, Multi-Agent Systems. In this first section we sketch the general formalism in which we study the lifetime of the transient behaviour of a networked dynamics. In the next section we illustrate with specific results regarding the mixing times of random walks and convergence time of the corresponding consensus dynamics.

In this extended abstract we consider linear dynamics for the agents. Linearity may result for example from linearisation of the dynamics around a fixed point. Simple examples of this include continuous-time first-order consensus

$$\dot{x} = Lx, \tag{1}$$

where x is a list of scalar positions (e.g. representing an opinion) of the agents, and every agent moves in the direction of (the weighted average position of) its neighbours in the interconnection network. Provided that the graph is strongly connected, this will result in the agents' positions converging to a single value called the consensus value. In this situation, L is the (weighted) Laplacian of the network, whose every row sums to zero and the eigenvalues are in the negative half-plane, except for the single dominant eigenvalue zero.

Variants include among many others the second-order damped consensus

$$\ddot{x} - \alpha\dot{x} = Lx, \tag{2}$$

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the discrete-time consensus

$$x(t+1) - x(t) = Lx. \tag{3}$$

More generally one can consider arbitrary linear dynamics for every agent, and one can consider weighted matrices L that, although compatible with the structure of the interconnection network (i.e. contains non-zero entries only between agents neighbours in the network), do not have the properties of the Laplacian, and in particular have an arbitrary spectrum.

Therefore we consider the systems of the form

$$Dx = Lx \tag{4}$$

where D is a linear operator applied to every agent's trajectory $(x_i(t))_{t\in[0,+\infty]}$, e.g. a combination of derivatives or delays. This implies in particular that every agent is supposed to have the same dynamics. We keep this restrictive assumption in the present abstract so as to obtain simple statements regarding the interaction between individual dynamics (operator D) and interconnection network (matrix L). As a further simplification, we assume in this abstract that the matrix L is real diagonalisable.

Perhaps the most natural question when faced with a linear system is to determine the characteristic time of growth or decay, i.e. the number τ_{char} such that $\|x(t)\| = \Theta(e^{t/\tau_{char}})$. Here a negative time indicates an exponential decay to zero. This characteristic time can also be read in the Laplace-domain expression of the dynamics. We denote variable and operators in the Laplace domain in boldface, e.g. the Laplace transform of $x(t)$ is $\mathbf{x}(s)$, and for example the linear operator $D = d/dt$ creates a corresponding operator $\mathbf{D} = s - \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}$ (for any γ in the domain of definition of $\mathbf{x}(s)$), since the Laplace transform of dx/dt is $s\mathbf{x} - x(0)$. We may therefore write

$$\mathbf{D}\mathbf{x} = L\mathbf{x}. \tag{5}$$

Expressing the problem in the variables $y = T^{-1}x$, where T is a transformation that diagonalises L, i.e. $T^{-1}LT$ is the diagonal matrix of eigenvalues λ_k, the system decouples to the individual study of modes

$$\mathbf{D}\mathbf{y}_k = \lambda_k\mathbf{y}_k. \tag{6}$$

The solution space for $\mathbf{y}_k$ is therefore the eigenspace of the operator $\mathbf{D}$ for the eigenvalue λ_k. A solution $\mathbf{y}_k(s)$ has a domain of definition consisting of a right half-plane $\{s \in \mathbb{C}|\mathrm{Re}(s) > \sigma\}$ (possibly with all or part of the border), corresponding in the time domain to a characteristic time σ^{-1}, since $y_k(t)$ behaves asymptotically as $e^{\sigma t}$. When

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the eigenspace has more than one dimension, one can have solutions with different characteristic times. We order them in the following way. Consider the subspace of solutions with the largest σ, which we denote $\sigma_{\lambda_k,0}$. In the complement of this subspace, we take the largest σ, which we denote $\sigma_{\lambda_k,1}$, and the corresponding subspace. Removing this subspace again, we find $\sigma_{\lambda_k,3}$, etc. Overall, an eigentrajectory $y_k(t)$ can be expressed as the linear combination of elementary trajectories behaving asymptotically as $e^{\sigma_{\lambda_k,0}t}$, $e^{\sigma_{\lambda_k,1}t}$, etc.

We are however interested in the behaviour of x_i rather than y_k. Every agent's behaviour is a linear combination $\sum_j \alpha_j y_k(t)$, where $y_k(t)$ is itself a combination of eigentrajectories of characteristic times $\sigma^{-1}_{k,0}, \sigma^{-1}_{\lambda_k,1}, \ldots$ The characteristic time of $x_i(t)$ and $\|x(t)\|$ is therefore typically $\sigma^{-1}_{\lambda_0,0}$, if we suppose that the eigenvalues have been ordered so that $\sigma_{\lambda_0,0} > \sigma_{\lambda_1,0} > \sigma_{\lambda_2,0} > \ldots$ Incidentally, remark that this ordering of the eigenvalues depends on the individual dynamics of the agent. This is not surprising however, if one thinks for example that in discrete time ($Dx(t) = x(t+1)-x(t)$) dominant eigenvalues λ are those with largest modulus $|1+\lambda|$, while in continuous time ($Dx(t) = \dot{x}(t)$) dominant eigenvalues are those with the largest real part. Stability for the global dynamics occur when $\sigma_{\lambda_0,0} < 0$, which is elegantly characterised in [FM04] by a Nyquist-like criterion.

Although the characteristic time of the trajectories is a precious indication on the behaviour of system, it is important to complement it with the characteristic time τ_{trans} of the transient, i.e. the trajectory from which is removed the asymptotically dominant component. It is important to identify the transient characteristic time as it gives an indication of when the asymptotic analysis becomes correct and the trajectories can effectively be reduced to their dominant component, while the transient has effectively vanished. This is especially relevant for processes where the dominant eigenvalue $\sigma_{\lambda_0,1}$ is zero, such as consensus and random walks. In the latter case, the transient characteristic time is called the *mixing time*, and is a fundamental quantity in random walk theory.

II. Mixing time of non-Poissonian random walks

Random walks on a network is a particular situation where we can illustrate the respective impact of the network and the dynamics on the transient behaviour.

A Markovian random walk is defined by the jumps of a random walker from a node to a neighbouring node with certain transition probabilities encoded in a row-stochastic matrix P. In discrete-time, the jumps occur at times $t = 0, 1, 2, \ldots$ The probability π_i of presence on node i can be arranged into a row vector π checking the master equation $D\pi = \pi(P-I)$, where $D\pi(t) = \pi(t+1) - \pi(t)$.

In continuous time, one may assume that the waiting time of the walker between two consecutive jumps is itself a random distribution, possibly depending on the node. In the simplest case, the distribution is exponential with same expected waiting time τ for each node, in which case the jumping times form a Poisson process. The probability of presence therefore satisfy the equation $D\pi = \pi(P-I)$, where now $D = \tau d/dt$. It is clear from these equations that those random walks are dual in their dynamics to corresponding consensus dynamics on the same set of agents, and the Laplacian $L = P - I$.

For a general waiting time probability density $\rho(t)$ (identical on every node), one may also derive an equation for the probability of presence $\pi(t)$ of the form $D\pi = \pi L$, where D depends on ρ, see details in [HPL12], [DLR13]. The characteristic time of the transient in this case is called mixing time in the random walk or Markov chain methodology. It is clear every such random walk is dual to a corresponding consensus dynamics $Dx = Lx$. The dominant eigenvalue of L is trivial $\lambda_0 = 0$, while the second eigenvalue $\lambda_1 < 0$ determines the spectral gap $|\lambda_1|$.

It has been shown in the last decade that human interactions show strongly non-Poissonian patterns, displaying in particular heavier tails (i.e. converging more slowly to zero) and larger variance than a Poisson walker with same expected waiting time τ [Bar], [Bar10]. A large variance implies that the walker typically performs several fast jumps in bursts, followed by long waiting times. Therefore the modelling of diffusion processes, such as epidemics, random walks or opinion consensus occurring on social networks is expected to take into account those characteristics. So far the studies in that direction have been largely numerical, with a lack of analytical tools, see e.g. [IM09], [RLH11], [SWP+13], [LTD13], [MGV11]. The present results constitute a step towards a deeper theoretical understanding of non-Poissonian random walks. We consider in particular

We apply the general framework to derive exact and approximate formulas for the mixing time of the random walk, the main results of this extended abstract.

We focus on a reversible random walk with a waiting time probability density $\rho(t) = \rho_{heavy} e^{-\tau_{tail}}$, where ρ_{heavy} decreases with a heavy tail, i.e. at a subexponential rate (e.g. a power law), and $e^{-\tau_{tail}}$ is a so-called 'soft cut-off' often used as model indicating that the heavy tail is only observed for a range of times, followed by an eventual fast decay. We can see τ_{tail} as a quantitative measure for the tail heaviness, being infinite for a pure heavy tail.

This kind of probability law is common to model bursty human activities, as mentioned above. In this case, we find that the mixing time is of the form

$$\tau_{mix} = \max(\tau_0, \tau_{tail}). \tag{7}$$

Here τ_0 is the largest solution to the the Laplace-domain equation $\boldsymbol{\rho}(-1/\tau_0) = \frac{1}{1-|\lambda_1|}$. We can show that this can be approximated by

$$\tau_0 \approx \frac{\tau}{|\lambda_1|}(1 + \beta|\lambda_1|). \tag{8}$$

in the limit of large τ_0. Here $\beta = (\sigma^2 - \tau^2)/2\tau^2$, with τ being the expected waiting time as we have seen and the σ^2 the variance. It captures the burstiness of waiting time as $\beta = 0$ for an exponential distribution, i.e. a Poisson random walker, and $\beta > 0$ for a bursty random walker. A further simplification, valid up to a factor two at most, is

$$\tau_{mix} \approx \max(\tau_{tail}, \tau/|\lambda_1|, \tau\beta). \tag{9}$$

This approximate formula—the main contribution of this abstract—has the merit to show how the mixing time is influenced by the individual dynamics through expected waiting time τ, tail heaviness τ_{tail} and burstiness β, and by the network through λ_1. In particular we see that the mixing can be obstructed either by dynamical causes (heavy tail or burstiness) or by network topology causes (small spectral gap $|\lambda_1|$). See proof details in [DLR13].

III. Conclusion

To conclude this extended abstract, we observe that an analytical study of the interaction between dynamics of the agents and interconnection topology is possible under some simplifying assumptions, and sheds light on fundamental phenomena underlying the emergence of a global dynamical behaviour from interconnected individual dynamics, through the example of non-Poissonian random walks.

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