21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

Computing (Reverse) Dichotomic Solutions of Non-symmetric Algebraic Riccati Equations via the Matrix Sign Function

Marc Jungers

Abstract— **The Non-symmetric Algebraic Riccati Equations are an extension of Algebraic Riccati Equations: they are equations which are quadratic in a rectangular matrix variable. Such equations may exhibit distinct solutions. Several among them have particular properties in terms of robustness and are useful in control system theory and in game theory. One can cite, if they exist: the strongly stabilizing solution, dichotomic solutions and reverse dichotomic solutions. This paper provides numerical efficient algorithms to compute them by using the Matrix Sign Function of appropriated characteristic matrices. The suitability and efficiency of these algorithms are illustrated by several academic examples.**

I. Introduction

Standard algebraic Riccati equations (ARE) are quadratic equations in a square matrix variable. They appear in a great number of problems in control system theory: optimal control problem for linear systems with quadratic costs and optimal filtering problem via Kalman filter design (See [1] and [2] for example). Their numerical and theoretic solvings have generated an important literature (see for instance [3]–[6] and the therein mentioned references).

In other frameworks like game theory [6]–[8], or transport theory [9]–[11], a more tricky class of equations occurs: the rectangular or non-symmetric algebraic Riccati equations (NARE), that is quadratic equations in a rectangular matrix variable [6]. For example, when considering a linear differential game with quadratic cost functions, the Nash [12] or Stackelberg [6] strategies with open-loop information structure may be solved via coupled algebraic Riccati equations (CARE), which can be rewritten as a NARE.

As the ARE, the NARE may offer several solutions, among which we can cite the stabilizing ones or strongly stabilizing one (See [12, Definition 7.2]). The strongly stabilizing solution, when it exists, is the single solution stabilizing the state closed-loop matrix and anti-stabilizing the co-state closed-loop matrix. This property is of great importance in control system theory in terms of robustness, due to the unicity of the stabilizing solution. Other particular solutions are, when they exist, dichotomic and reverse dichotomic solutions [13]. They are also of interest in control system theory and game theory and their numerical computation will be the focus of this paper.

The lack of symmetry of NARE introduces on one hand increased difficulties in their solving and on the other hand a greater number of solutions. A large literature on ARE provides several techniques which have been extended to cope with NARE. The technique based on invariant subspaces of an Hamiltonian matrix (see [14]) has been extended to the one related to a characteristic matrix (without Hamiltonian symmetry) [6] and to the deflating subspace of a matrix pencil in continuous-time [15] and discrete-time [16]. The Newton's method [17] has been extended for the NARE (see for instance [10]).

Concerning the method using invariant subspaces of a characteristic method, several numerical procedures are available. The two main numerical procedures for ARE involve a Schur decomposition [18] or the Matrix Sign Function (details may be found in [19], [20] and refinements in [21]). Even if the Schur decomposition method has been extended naturally for solving the NARE, this is not the case in what concerns the Matrix Sign Function. As noted in [22], the literature offers only an extension for solving the strongly stabilizing solution. This is due to the important fact that the stable invariant subspace should have exactly the number of columns of the matrix variable. As far as we know, this extension has not been provided in the literature for other types of solutions, like dichotomic and reverse dichotomic solutions of the NARE. This constitutes the aim of this paper.

The paper is organized as follows. In Section II, the definition of Non-symmetric Algebraic Riccati Equations, the definitions of the particular solutions and their main properties are presented. Section III provides the definition, the algorithm and the properties of the Matrix Sign Function. The procedures to solve the strongly stabilizing solution, reverse dichotomic and dichotomic solutions of a NARE are given in Section IV and illustrated by numerical examples in Section V, before concluding remarks in Section VI.

Notation. By $\mathbb{R}$ and $\mathbb{C}$ we denote the real axis and the complex plane, respectively. The open left–half plane (the stability domain in continuous–time) is denoted by $\mathbb{C}^-$ and the open right–half plane by $\mathbb{C}^+$. $\mathbb{N}$ and $\mathbb{N}^*$ are respectively the set of non-negative integers and the set of positive integers. For a constant matrix A we denote by A' its transpose and if A is invertible by A^{-1} its inverse. $\Lambda(A)$ is the spectrum of a square matrix A. An eigenvalue of A is called *stable* provided it is inside $\mathbb{C}^-$, and *antistable* otherwise. I_{n_1} and $0_{n_1 \times n_2}$ denote the identity matrix of size $n_1 \times n_1$ and the null matrix of size $n_1 \times n_2$, respectively. $\mathrm{diag}(\alpha_1, \cdots, \alpha_n)$ is the diagonal matrix with the scalars $\alpha_1, \cdots, \alpha_n$ on its diagonal.

M. Jungers is with Université de Lorraine, CRAN, UMR 7039, 2 avenue de la Forêt de Haye, Vandœuvre-lès-Nancy, 54516, France and CNRS, CRAN, UMR 7039, France. (marc.jungers@univ-lorraine.fr).

ISBN: 978-90-367-6321-9

II. Non-symmetric Algebraic Riccati Equations

A. Definition

The algebraic equation, with the variable $K \in \mathbb{R}^{p\times n}$ $(n, p \in \mathbb{N}^*)$

$$M_{21} + M_{22}K - KM_{11} - KM_{12}K = 0_{p\times n}, \quad (1)$$

where $M_{11} \in \mathbb{R}^{n\times n}$, $M_{12} \in \mathbb{R}^{n\times p}$, $M_{21} \in \mathbb{R}^{p\times n}$ and $M_{22} \in \mathbb{R}^{p\times p}$, is called Non-symmetric Algebraic Riccati Equation (NARE) in continuous time. The term *Non-symmetric* is applied in order to specify that p and n are not necessary equal, that is the variable K is a rectangular matrix.

This equation is associated with the characteristic matrix

$$M = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right] \in \mathbb{R}^{(n+p)\times(n+p)}. \quad (2)$$

More precisely, the solution of the NARE (1) are closely related to the invariant subspaces of the matrix M, as emphasized in the following section.

B. Invariant subspaces solving method

The following theorems are inspired by the Radon's Lemma [23], [24]. They are also discussed in [6], [12], [25].

Theorem 1: Let $K \in \mathbb{R}^{p\times n}$ be a solution of equation (1), then there exist three matrices $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{p\times n}$ and $J \in \mathbb{R}^{n\times n}$, such that X is invertible, J is a Jordan canonical form verifying

$$M \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} J, \quad (3)$$

and

$$Y = KX. \quad (4)$$

Proof: If K is a solution of the NARE (1), then

$$M \begin{bmatrix} I_n \\ K \end{bmatrix} = \begin{bmatrix} I_n \\ K \end{bmatrix} (M_{11} + M_{12}K). \quad (5)$$

The proof is obtained by choosing J and X via the Jordan canonical form of the matrix $(M_{11} + M_{12}K)$, that is

$$(M_{11} + M_{12}K) = XJX^{-1}, \quad (6)$$

and finally $Y = KX$. ∎

Theorem 2: Let $\mathcal{V} \subset \mathbb{R}^{(n+p)}$ an invariant subspace of the characteristic matrix M. Let also $X \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{p\times n}$ be such that $\mathcal{V} = \mathrm{Im}\begin{bmatrix} X \\ Y \end{bmatrix}$. Then there exists a Jordan canonical form matrix $J \in \mathbb{R}^{n\times n}$ verifying

$$M \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} J, \quad (7)$$

If X is invertible, then $K = YX^{-1}$ is a solution of the equation (1). Due to the latter relation, it is clear that K is independent of the specific choice of the basis of $\mathcal{V}$.

Proof: See [6, Theorem 3.1.1] for instance. ∎

Remark 1: It should be observed that the reduction matrix J has its spectrum included in the spectrum of M. The solution $K = YX^{-1}$ is thereby associated with the n eigenvalues of J, which are also eigenvalues of M.

Among the possible solutions of the NARE (1), several have specific properties and should be distinguished. They are defined as follows.

Definition 1: A matrix K, solution of the equation (1), is said to be *stabilizing* if $(M_{11} + M_{12}K)$ is Hurwitz:

$$\Lambda(M_{11} + M_{12}K) \subset \mathbb{C}^-. \quad (8)$$

Definition 2 (Definition 7.2 in [12]): A matrix K, solution of the equation (1), is said to be *strongly stabilizing* (or (n, p) c-splitting see [22]) if it is stabilizing and $(M_{22} - KM_{12})$ is anti-Hurwitz, that is

$$\Lambda(M_{22} - KM_{12}) \subset \mathbb{C}^+. \quad (9)$$

In order to define more particular solutions, let us denote the eigenvalues of the matrix M ordered by nondecreasing real part by

$$\mathrm{Re}(\lambda_1(M)) \le \mathrm{Re}(\lambda_2(M)) \le \cdots \le \mathrm{Re}(\lambda_{n+p}(M)). \quad (10)$$

Definition 3 (Definition 2 in [13]): If the characteristic matrix M verifies

$$\mathrm{Re}(\lambda_p(M)) < \mathrm{Re}(\lambda_{p+1}(M)), \quad (11)$$

M is called *dichotomically separable*. In this case, the solution of the NARE (1) associated with the n eigenvalues $\{\lambda_{p+1}(M); \cdots; \lambda_{p+n}(M)\}$ is called the *dichotomic solution*. If M verifies

$$\mathrm{Re}(\lambda_n(M)) < \mathrm{Re}(\lambda_{n+1}(M)), \quad (12)$$

M is called *reverse dichotomically separable*. In this case, the solution of the NARE (1) associated with the n eigenvalues $\{\lambda_1(M); \cdots; \lambda_n(M)\}$ is called the *reverse dichotomic solution*.

Thanks to Theorem 2, the existence of the stabilizing and the strongly stabilizing solution of the NARE is related to the spectrum of the matrix M; like the existence of the (reverse) dichotomic solutions. These conditions are gathered in Propositions 1 and 2.

Proposition 1: The NARE admits a stabilizing solution if and only if the matrix M has at least n stable eigenvalues. That is

$$\mathrm{Re}(\lambda_n(M)) < 0. \quad (13)$$

Proposition 2: The NARE admits a strongly stabilizing solution if and only if the matrix M exactly has n stable eigenvalues and p antistable eigenvalues, that is

$$\mathrm{Re}(\lambda_n(M)) < 0 < \mathrm{Re}(\lambda_{n+1}(M)). \quad (14)$$

Furthermore if a strongly stabilizing solution exists, it is unique.

Proof: A sketch of proof is given by a change of basis of matrix M (see [6], [26] for more details). Assume that K is a strongly stabilizing solution of the NARE, let us introduce

$$T = \begin{bmatrix} I_n & 0_{n\times p} \\ -K & I_p \end{bmatrix}; \; T^{-1} = \begin{bmatrix} I_n & 0_{n\times p} \\ K & I_p \end{bmatrix}. \quad (15)$$

That implies

$$TMT^{-1} = \begin{bmatrix} M_{11} + M_{12}K & M_{12} \\ 0_{p\times n} & M_{22} - KM_{12} \end{bmatrix}. \quad (16)$$

The matrix M has then exactly n stable and p antistable eigenvalues. The converse is obtained by selecting the unique stable invariant subspace of M of dimension n and checking with Equation (16) that this stabilizing solution is a strongly stabilizing one. ■

Remark 2: It is noteworthy that if the strongly stabilizing, dichotomic and reverse dichotomic solutions exist, they are real. This fact is induced by the splitting of the eigenvalues imposing to select the both eigenvalues of a non-real complex conjugate pair of eigenvalues. See for more details [13] and [6].

This is noteworthy that the strongly stabilizing solution is a particular dichotomic one. This property will be judiciously considered to solve a dichotomic solution by transforming it into a strongly stabilizing solution of an other characteristic matrix. The research of solutions of Equation (1) is very close to the design of invariant subspaces of the characteristic matrix M. As explained in the next section, the Matrix Sign Function preserves the invariant subspaces of a matrix. This main property will be the heart of the provided algorithm.

III. Matrix Sign Function

A. Definition

The sign function for complex scalars is given by

$$\forall \alpha \in \mathbb{C},\ \operatorname{sign}(\alpha) = \begin{cases} +1 \text{ if } \operatorname{Re}(\alpha) > 0, \\ -1 \text{ if } \operatorname{Re}(\alpha) < 0, \\ \text{undefined otherwise.} \end{cases} \tag{17}$$

The Matrix Sign Function is defined as an extension of the sign function for matrices as follows. Other definitions and properties may be founded in [27]. Let be $A \in \mathbb{R}^{(n+p)\times(n+p)}$, without purely imaginary eigenvalue. A Jordan canonical form of A is given by

$$A = V(D+N)V^{-1}, \tag{18}$$

where $D = \operatorname{diag}(d_1, \cdots, d_{n+p})$ and N is nilpotent and commutes with D. Then

$$\operatorname{sign}(A) = V \operatorname{diag}(\operatorname{sign}(d_1), \cdots, \operatorname{sign}(d_{n+p}))V^{-1}. \tag{19}$$

The main properties of the Matrix Sign Function are

- $\operatorname{sign}(A)$ is a diagonalizable matrix;
- $\operatorname{sign}(A)$ is a square root of the identity matrix;
- $\operatorname{sign}(A)$ commutes with A;
- the rank of $(\operatorname{sign}(A) - I_{n+p})$ is equal to the number of anti-stable eigenvalues (multiplicity counted) of A and the rank of $(\operatorname{sign}(A) + I_{n+p})$ is equal to the number of stable eigenvalues of A;
- If A is Hurwitz, $\operatorname{sign}(A) = -I_{n+p}$;
- for any real scalar $\alpha \neq 0$, $\operatorname{sign}(\alpha A) = \operatorname{sign}(\alpha)\operatorname{sign}(A)$. (called *scaling property*).
- The invariant subspaces of the matrix A are also invariant for the matrix $\operatorname{sign}(A)$. If there exists Z and Q with appropriated dimensions such that $AZ = ZQ$, then $\operatorname{sign}(A)Z = Z\operatorname{sign}(Q)$, see [22, Lemma 1.7].

These properties are useful to compute the Matrix Sign Function.

B. Algorithms computing Matrix Sign Function

In [28], an algorithm is provided to obtain the Matrix Sign Function, for any matrix M, without purely imaginary eigenvalues. It is based on a Newton–Kantorovich procedure and on the observation that $(\operatorname{sign}(M))^2 = I_{n+p}$. The sequence of matrices defined by

$$M_{k+1} = \left(M_k + M_k^{-1}\right), \quad M_0 = M \tag{20}$$

satisfies

$$\lim_{k\to+\infty} M_k = \operatorname{sign}(M). \tag{21}$$

Thanks to the scaling property of the Matrix Sign Function, a refinement improving the convergence rate is given by the modified sequence [29]:

$$M_{k+1} = \frac{1}{2c_k}\left(M_k + c_k^2 M_k^{-1}\right), \quad M_0 = M, \tag{22}$$

where

$$c_k = |\det(M_k)|^{\frac{1}{n+p}}. \tag{23}$$

Several improved algorithms and their cost per iteration are proposed in the literature (See [30], [29], [31]). A discussion about the breaking condition of the algorithm is available in [32] or in [28]. Here we will select as breaking condition, the condition $\|M_{k+1} - M_k\|$ less than a fixed threshold.

The Matrix Sign Function has been used in the literature to compute invariant subspaces of matrices and the positive definite solution of Hermitian Riccati equations [32], [33], but also [19], [28]. The key idea is to transform the NARE into an overdetermined linear system of equations. This method is presented in the following section.

IV. Computing the (reverse) dichotomic solution of NARE with the Matrix Sign Function

As explained in the previous section, a strongly stabilizing solution is a dichotomic one. We will first show how to obtain a strongly stabilizing solution via the Matrix Sign Function.

Theorem 3 (Theorem 3.9 in [22]): If the NARE (1), characterized by the matrix M admits a strongly stabilizing solution, then it is the unique solution of the overdetermined system

$$\begin{bmatrix} W_{12} \\ W_{22} + I_p \end{bmatrix} K = -\begin{bmatrix} W_{11} + I_n \\ W_{21} \end{bmatrix}, \tag{24}$$

where

$$\operatorname{sign}(M) = \left[\begin{array}{c:c} W_{11} & W_{12} \\ \hdashline W_{21} & W_{22} \end{array}\right], \tag{25}$$

with $W_{11} \in \mathbb{R}^{n\times n}$, $W_{12} \in \mathbb{R}^{n\times p}$, $W_{21} \in \mathbb{R}^{p\times n}$ and $W_{22} \in \mathbb{R}^{p\times p}$.

Proof: If the NARE admits a solution K, we have the relation (5). Thanks to the last property of the Matrix Sign Function, this implies

$$\operatorname{sign}(M)\begin{bmatrix} I_n \\ K \end{bmatrix} = \begin{bmatrix} I_n \\ K \end{bmatrix}\operatorname{sign}(M_{11} + M_{12}K). \tag{26}$$

If the solution K is stabilizing, we have $\mathrm{sign}\,(M_{11} + M_{12}K) = -I_n$ and we have finally

$$\mathrm{sign}\,(M) \begin{bmatrix} I_n \\ K \end{bmatrix} = - \begin{bmatrix} I_n \\ K \end{bmatrix}, \tag{27}$$

which rewrites as Equation (24). The unicity is due to the strongly stabilizability of the solution and concludes the proof. ■

It should be underlined that the method is based on the fact that the stable invariant subspace is exactly of dimension n. The method cannot apply straightforwardly for reverse dichotomic solution which is not strongly stabilizing. Nevertheless by noticing that for any scalar δ:

$$\delta K = (\delta I_n)K = K(\delta I_p), \tag{28}$$

we infer

$$\begin{aligned} M_{21} + (M_{22} - \delta I_p)K - K(M_{11} - \delta I_n) - KM_{12}K \\ = M_{21} + M_{22}K - KM_{11} - KM_{12}K. \end{aligned} \tag{29}$$

The idea for solving the reverse dichotomic solution of the NARE is to shift the characteristic matrix as stated in Theorem 4, reformulating the reverse dichotomic solution as a strongly stabilizing one.

Theorem 4: Consider a reverse dichotomically separable matrix M and K the associated reverse dichotomic solution. Then K is also the strongly stabilizing solution associated with the characteristic matrix $M - \delta_{\mathrm{rds}} I_{n+p}$, where the scalar δ_{rds} verifies

$$\mathrm{Re}\,(\lambda_n(M)) < \delta_{\mathrm{rds}} < \mathrm{Re}\,(\lambda_{n+1}(M))\,. \tag{30}$$

Proof: The proof is based on Equation (29), emphasizing two writings of the same NARE. The distinction between them is related to the characteristic matrices, namely M and $M - \delta I_{n+p}$. The shift of the characteristic matrix M modifies the eigenvalues of M and thus allows to modify the size of the invariant subspaces of $\mathrm{sign}\,(M - \delta I_{n+p})$ related respectively to the eigenvalues -1 and 1. By selecting $\delta = \delta_{\mathrm{rds}}$ verifying the relation (30), $M - \delta I_{n+p}$ has exactly n stable eigenvalues and p anti-stable eigevalues. K is then the strongly stabilizing solution associated with this characteristic matrix. ■

Remark 3: The shift provided in Theorem 4 is applied on the whole matrix M. This is a different shifting technique from the ones presented in [34] or in [35], which act only locally on the partitions of M.

The same technique can be used to compute the dichotomic solution of the NARE, as presented in Theorem 5.

Theorem 5: Consider a dichotomically separable matrix M and K the associated dichotomic solution. Then K is also the strongly stabilizing solution associated with the characteristic matrix $-(M - \delta_{\mathrm{ds}} I_{n+p})$, where the scalar δ_{ds} verifies

$$\mathrm{Re}\,(\lambda_p(M)) < \delta_{\mathrm{ds}} < \mathrm{Re}\,(\lambda_{p+1}(M))\,. \tag{31}$$

Proof: The proof of this theorem follows the one of Theorem 4. Firstly, the matrix M being dichotomically separable, the shifted matrix $M - \delta_{\mathrm{ds}} I_{n+p}$, with δ_{ds} chosen to verify the relation (31) have exactly p stable eigenvalues and n anti-stable ones. This yields that $-(M - \delta_{\mathrm{ds}} I_{n+p})$ has exactly n stable eigenvalues and p anti-stable ones. In addition, the NARE being linear with respect to the weighting matrix M, changing the sign of M does not affect the NARE (1). The solution K is then the strongly stabilizing solution associated to the characteristic matrix $-(M - \delta_{\mathrm{ds}} I_{n+p})$. ■

The next section will emphasize the numerical efficiency and the suitability of the provided results to compute strongly stabilizing, dichotomic and reverse dichotomic solutions via the Matrix Sign Function.

V. Illustrations

Consider the academic example, with $n = 1$; $p = 3$ and where

$$M = \left[\begin{array}{c:ccc} 1 & 3 & -1 & 3 \\ \hdashline 0 & 2 & 3 & 4 \\ 3 & 1 & 3 & 5 \\ 5 & 0 & -1 & -2 \end{array}\right],$$

which eigenvalues are

$$\Lambda(M) = \left\{ \begin{array}{r} 5.9624 \\ 0.9011 + 2.5832i \\ 0.9011 - 2.5832i \\ -3.7645 \end{array} \right\}.$$

The matrix M has one stable eigenvalue and three anti-stable eigenvalues. Due to Proposition 2, it admits a strongly stabilizing solution. The Matrix Sign Function algorithm stops after 8 iterations, because $\|M_{k+1} - M_k\| \leq 10^{-16}$ at this step as shown on Figure 1.

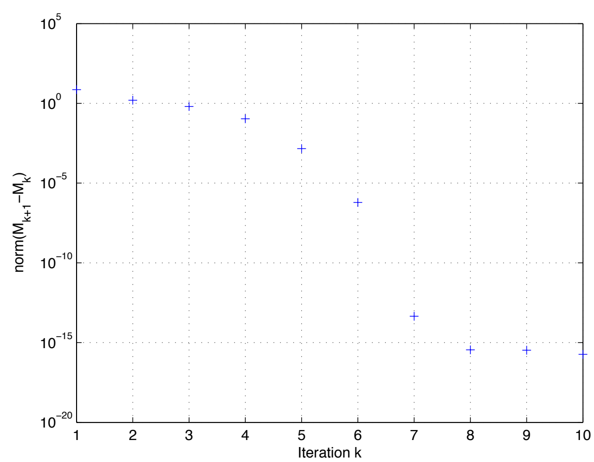

Fig. 1. Evolution of the error $\|M_{k+1} - M_k\|$ in function of the iteration k.

The Matrix Sign Function of M is obtained as

$$\mathrm{sign}\,(M) = \begin{bmatrix} 0.3614 & 0.3584 & -0.1501 & 0.6986 \\ -0.6374 & 1.3577 & -0.1498 & 0.6973 \\ -0.6767 & 0.3797 & 0.8410 & 0.7402 \\ 1.4261 & -0.8003 & 0.3351 & -0.5601 \end{bmatrix}$$

and its spectrum is $\{-1;1;1;1\}$ as predicted. The overdetermined linear system in K defined by Equation (24) is given by

$$\begin{bmatrix} W_{12} \\ W_{22}+I_p \end{bmatrix} = \begin{bmatrix} 0.3584 & -0.1501 & 0.6986 \\ 2.3577 & -0.1498 & 0.6973 \\ 0.3797 & 1.8410 & 0.7402 \\ -0.8003 & 0.3351 & 0.4399 \end{bmatrix}$$

and

$$-\begin{bmatrix} W_{11}+I_n \\ W_{21} \end{bmatrix} = \begin{bmatrix} -1.3614 \\ 0.6374 \\ 0.6767 \\ -1.4261 \end{bmatrix}.$$

The strongly stabilizing solution K is obtained by

$$K = \begin{bmatrix} 0.9981 \\ 1.0596 \\ -2.2331 \end{bmatrix}.$$

As a verification, we have

$$M_{11} + M_{12}K = -3.7645 = \lambda_1(M).$$

Let us now consider the following example, with $n = 2$, $p = 4$ and

$$M = \left[\begin{array}{cc:cccc} 1 & 3 & 1 & 0 & 1 & 4 \\ 2 & 1 & 3 & 2 & -1 & -3 \\ \hdashline 1 & 0 & -2 & 0 & 0 & 0 \\ 2 & 1 & 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 & -3 & 0 \\ 3 & 1 & 0 & 0 & 0 & -3 \end{array}\right],$$

which eigenvalues are

$$\Lambda(M) = \left\{\begin{array}{r} 4.5535 \\ 0 \\ -1.9991 + 0.2001i \\ -1.9991 - 0.2001i \\ -3.1037 \\ -5.4516 \end{array}\right\}.$$

The matrix M is singular, dichotomically separable and reverse dichotomically separable. Let us notice that M being singular, sign(M) does not exist by definition.

In order to obtain the reverse dichotomic solution, via choosing the following shift $\delta_{\text{rds}} = \frac{1}{2}(\text{Re}(\lambda_4(M) + \lambda_5(M)) = -2.5514$ associated with Theorem 4. The resulting Matrix Sign Function is obtained in a few iterations (7 iterations are only necessary to reach $\|M_{k+1} - M_k\| \leq 10^{-16}$, as depicted on Figure 2.

Solving the overdetermined linear system of equations (24) for the shift characteristic matrix $M - \delta_{\text{rds}} I_{n+p}$ leads to the reverse dichotomic solution

$$K = \begin{bmatrix} -0.2332 & 0.0974 \\ -0.8568 & -0.7678 \\ 11.7004 & 20.9855 \\ -4.5335 & -6.1135 \end{bmatrix}.$$

As a verification, we have

$$\Lambda(M_{11} + M_{12}K) = \left\{\begin{array}{c} -5.4516 \\ -3.1037 \end{array}\right\},$$

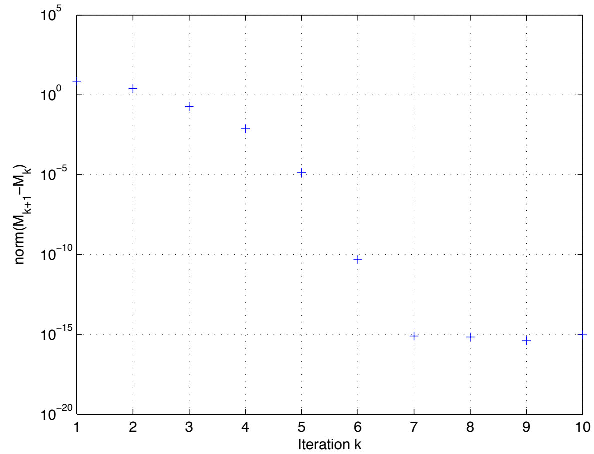

Fig. 2. Evolution of the error $\|M_{k+1} - M_k\|$ in function of the iteration k.

which are the expected values, that is the $\lambda_1(M)$ and $\lambda_2(M)$.

To compute via the Matrix Sign Function the dichotomic solution, we choose the shift $\delta_{\text{ds}} = \frac{1}{2}\text{Re}(\lambda_2(M) + \lambda_3(M))$. We compute the strongly stabilizing solution of the characteristic matrix $-(M - \delta_{\text{ds}} I_{n+p})$. The Matrix Sign Function sign$(-(M - \delta_{\text{ds}} I_{n+p}))$ is obtained after 6 iterations (see Figure 3) and leads to

$$K = \begin{bmatrix} 0.2464 & -0.1690 \\ 0.3521 & 0.0681 \\ 0.1628 & -0.5581 \\ 0.4786 & -0.0143 \end{bmatrix}.$$

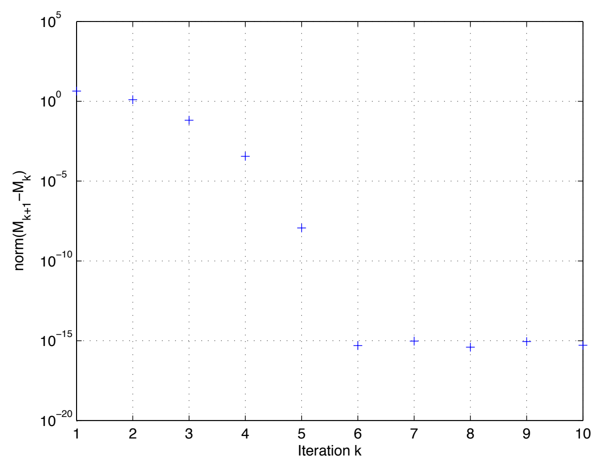

Fig. 3. Evolution of the error $\|M_{k+1} - M_k\|$ in function of the iteration k.

We verify that

$$\Lambda(M_{11} + M_{12}K) = \left\{\begin{array}{c} 4.5535 \\ 0.0000 \end{array}\right\},$$

which are $\lambda_5(M)$ and $\lambda_6(M)$.

VI. Conclusion

The numerical solving of strongly stabilizing, dichotomic and reverse dichotomic solutions of Non-symmetric Algebraic Riccati Equations (NARE) has been investigated in this paper. The idea is to use the Matrix Sign Function of a characteristic matrix because it allows firstly to preserve the invariant subspaces and secondly to linearize the NARE into a linear overdetermined system. The dichotomic and reverse dichotomic solutions are reformulated as the strongly stabilizing solution of a soundly shifted characteristic matrix. Numerical examples are given to illustrate the provided results.

References

[1] B. Anderson and J. B. Moore, *Optimal control: linear quadratic methods*. Prentice-Hall International Editions, 1989.

[2] P. Dorato, C. T. Abdallah, and V. Cerone, *Linear-Quadratic Control, An Introduction*. Ed. Prentice-Hall, 1995.

[3] A. Bunse-Gerstner, R. Byers, and V. Mehrmann, "Numerical methods for algebraic Riccati equations," in *Workshop on The Riccati Equation in Control Systems and Signals*, S. Bittanti, Ed. Como Italy: Pitagora Editrice, Bologna, June 1989, pp. 107–115.

[4] P. Lancaster and L. Rodman, *Algebraic Riccati equations*. Oxford science publications, 1995.

[5] S. Bittanti, J. A. Laub, and J.-C. Willems, *The Riccati Equation*. Springer Verlag, 1991.

[6] H. Abou-Kandil, G. Freiling, V. Ionescu, and G. Jank, *Matrix Riccati Equations in Control and Systems Theory*. Birkhäuser, 2003.

[7] T. Başar and G. J. Olsder, *Dynamic Noncooperative Game Theory*. SIAM, 1995.

[8] Y. C. Ho, "Survey paper: Differential games, dynamic optimization and generalized control theory," *Journal of Optimization Theory and Applications*, vol. 6, no. 3, pp. 179–209, 1970.

[9] J. Juang, "Existence of algebraic matrix Riccati equations arising in transport theory," *Linear Algebra and its Applications*, vol. 230, pp. 89–100, 1995.

[10] L.-Z. Lu, "Newton iterations for a non-symmetric algebraic Riccati equation," *Numerical Linear Algebra with Applications*, vol. 12, pp. 191–200, 2005.

[11] P. C.-Y. Weng, H.-Y. Fan, and E. K.-W. Chu, "Low-rank approximation to the solution of a nonsymmetric algebraic Riccati equation from transport theory," *Applied Mathematics and Computation*, vol. 219, pp. 729–740, 2012.

[12] J. C. Engwerda, *LQ Dynamic Optimization and Differential Games*. Wiley, 2005.

[13] J. Medanic, "Geometric properties and invariant manifolds of the Riccati equation," *IEEE Transactions on Automatic Control*, vol. 27, no. 3, pp. 670–677, June 1982.

[14] P. M. V. Dooren, "A generalized eigenvalue approach for solving Riccati equations," *SIAM J. Sci. Stat. Comput.*, vol. 2, pp. 121–135, 1981.

[15] M. Jungers and C. Oară, "Continuous-time non-symmetric algebraic Riccati theory: A matrix pencil approach," *European Journal of Control*, vol. 18, no. 1, pp. 74–81, 2012.

[16] M. Jungers, C. Oară, H. Abou-Kandil, and R. Ştefan, "General matrix pencil techniques for solving non–symmetric algebraic Riccati equations." *SIAM, J. Matrix Anal. & Appl.*, vol. 31, no. 3, pp. 1257–1278, 2009.

[17] N. R. Sandell, "On Newton's method for Riccati equation solution," *IEEE Transactions on Automatic Control*, vol. 19, no. 3, pp. 254–255, 1974.

[18] A. J. Laub, "A Schur method for solving the algebraic Riccati equation," *IEEE Transactions on Automatic Control*, vol. 30, pp. 97–108, 1979.

[19] J. Roberts, "Linear model reduction and solution of the algebraic Riccati equation by use of the sign function," *International Journal of Control*, vol. 32, pp. 677–687, 1980.

[20] E. Denman and A. Beavers, "The matrix sign function and computations in systems," *Applied Mathematics and Computation*, vol. 2, no. 1, pp. 63–94, 1976.

[21] J. D. Gardiner and A. J. Laub, "A generalization of the matrix-sign-function solution for algebraic Riccati equations," *International Journal of Control*, vol. 44, pp. 823–832, 1986.

[22] D. A. Bini, B. Iannazzo, and B. Meini, *Numerical Solution of Algebraic Riccati Equations*, ser. Fundamentals of Algorithms. SIAM, 2012, no. 9.

[23] J. Radon, "Über die oszillationstheoreme der kunjugierten punkte beim probleme von Lagrange. (german)," *Münchener Sitzungsberichte*, vol. 57, pp. 243–257, 1927.

[24] ——, "Zum problem von Lagrange. (german)," *Hamburger Math. Einzelschr.*, vol. 6, no. 1, pp. 273–299, 1928.

[25] J. C. Engwerda, "On the open-loop Nash equilibrium in LQ-games," *Journal of Economic Dynamics and Control*, vol. 22, pp. 729–762, 1998.

[26] D. Kremer, "Non-symmetric Riccati theory and noncooperative games," Ph.D. dissertation, RWTH Aachen, 2003.

[27] C. S. Kenney and A. J. Laub, "The matrix sign function," *IEEE Transactions on Automatic Control*, vol. 40, no. 8, pp. 1330–1348, 1995.

[28] R. Byers, "Solving the algebraic Riccati equation with the matrix sign function," *Linear Algebra and its Applications*, vol. 85, pp. 267–280, 1987.

[29] N. J. Higham, "Stable iterations for the matrix square root," *Numerical Algorithms*, vol. 15, pp. 227–242, 1997.

[30] E. S. Quintana-Orti and V. Hernandez, "Parallel Solution of Riccati Matrix Equations with the Matrix Sign Function," *Automatica*, vol. 34, no. 2, pp. 151–156, 1998.

[31] P. Pandey, C. Kenney, and A. J. Laub, "A parallel algorithm for the matrix sign function," *International Journal of High Speed Computing*, vol. 2, no. 2, pp. 181–192, 1990.

[32] A. Barraud, "Investigation autour de la fonction signe d'une matrice, application à l'équation de Riccati," *R.A.I.R.O. Automatique*, vol. 13, pp. 335–368, 1979.

[33] ——, "Produit étoile et fonction signe de matrice, application à l'équation de Riccati dans le cas discret," *R.A.I.R.O. Automatique*, vol. 14, pp. 55–85, 1980.

[34] M. Jungers, H. Abou-Kandil, E. B. Castelan, and E. R. D. Pieri, "A gametheoretic approach for non–uniform poles shifting and poles homothety," *Automatica*, vol. 49, no. 1, pp. 238–244, 2013.

[35] M. Jungers, "On Stackelberg linear quadratic games with time preference rate," *IEEE Transactions on Automatic Control*, vol. 53, no. 2, pp. 621–625, March 2008.