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Inverse Nonlinear Fourier Transforms Via Interpolation: The Ablowitz-Ladik Case

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***Abstract*—Nonlinear Fourier transforms arise when nonlinear evolution equations are solved with the inverse scattering method. In this paper, the inverse nonlinear Fourier transform that arises during the solution of the nonlinear Schrödinger equation on the real line is investigated. The inverse problem consists in the synthesis of a potential for the Zakharov-Shabat differential operator such that the spectrum of the operator fulfills certain prescribed properties. Instead of discretizing an exact solution for the continuous case, a numerical discretization of the forward transform is inverted exactly in this paper. The focus is on the case of finitely many samples, and the problems at hand take the form of interpolation problems. A connection between analytic interpolation with degree constraint and fiber Braggs grating design is established, leading to a new synthesis method. A new method to compute reflectionless potentials is presented as well.**

***Index Terms*—Nonlinear Schrödinger Equation, Nonlinear Fourier Transform, Inverse Scattering, Interpolation**

I. Introduction

The inverse scattering method for solving nonlinear evolution equations goes back to Gardner et al. [1]. It can be thought of as a generalization of Fourier's method for solving the heat equation [2]. The idea is that the time-evolution of a function governed by some evolution equation might be trivial in the (nonlinear) Fourier domain. To solve the Cauchy problem, a suitable nonlinear Fourier transform of the initial condition is taken and the time-evolution is performed in the nonlinear Fourier domain. Eventually, an inverse nonlinear Fourier transform is used to recover the function at the desired time. The nonlinear Fourier transforms arising in this method are indeed true generalizations of the common Fourier transform. They enable the analysis and synthesis of functions in terms of non-sinusoidal wave forms [2].

In this paper, having recent applications in mind, we derive an especially simple approach to the inverse nonlinear Fourier transform for the nonlinear Schrödinger equation. The inverse transform will be not computed by discretizing an exact solution to the continuous inverse problem, but instead by inverting a numerical forward transform that maps samples of the continuous function to an approximation of the nonlinear Fourier spectrum. This idea is not new [3], [4], but with the discretization process in mind and a special interest in the practically relevant case of finitely many samples new interpolation problems arise. The results derived in this paper turn out to have some interesting applications in optical filter design and optical communications.

In the following, the continuous inverse scattering method is introduced in Section II. The Ablowitz-Ladik discretization is reviewed afterwards in Section III. Then, in Section IV, new interpolation formulations of the first step of the inverse nonlinear Fourier transform are presented. Section V briefly discusses the second step of the inverse transform. Section VI gives some pointers on computational aspects, before finally an outlook on future research is given in Section VII.

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II. The Continuous Nonlinear Schrödinger Equation and the Inverse Scattering Method

The continuous *nonlinear Schrödinger equation*

$$\mathrm{i}\,\partial_t q + \partial_{xx} q + 2\kappa |q|^2 q = 0, \quad \kappa \in \{\pm 1\}, \tag{1}$$

where $q : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{C}$, $(x,t) \mapsto q(x,t)$, is a sufficiently differentiable function, is a well-known nonlinear evolution equation. If $\kappa = -1$, the equation is known as the *focusing* nonlinear Schrödinger equation, while it is called *defocusing* if $\kappa = +1$. The nonlinear Schrödinger equation has several interesting applications. For example, it describes the evolution of light in optical fiber with perfect loss compensation [5, Ch. 6.1]. It also serves as a model for certain water waves [6], [7]. The nonlinear Schrödinger equation can be solved in closed-form with the inverse scattering method [8]. In the following, the three steps of the inverse scattering method will be outlined: the nonlinear Fourier transform, the time evolution of the nonlinear Fourier spectrum, and the inverse transform. Eventually, the physical interpretation of the nonlinear Fourier spectrum will be discussed. For details, the reader is referred to [2] and [8]. Other useful references include [9], [10], [11] and [12].

A. The Nonlinear Fourier Transform

The nonlinear Fourier transform is derived from the *scattering operator*

$$\mathbf{L} := \mathrm{i} \begin{bmatrix} \frac{d}{dx} & q(\cdot, t_0) \\ \kappa \bar{q}(\cdot, t_0) & -\frac{d}{dx} \end{bmatrix}.$$

The term scattering operator refers to the fact that $\mathbf{L}$ describes how a wave evolves in a one-dimensional medium that causes reflections [13, p. 319]. The function $q(\cdot, t_0)$ is called the *potential* of the operator. The nonlinear Fourier transform takes $q(\cdot, t_0)$ and maps it to a special representation of the operator's spectrum, the *nonlinear Fourier spectrum*. (Often, the terms *scattering transform* and *scattering data* are used instead.)

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Let us start with the discrete spectrum, which consists of the $z \in \mathbb{C}$ for which the eigenrelation $\mathbf{L}\mathbf{v} = z\mathbf{v}$ admits a square-integrable solution $\mathbf{v} \neq \mathbf{0}$. A rearrangement of $\mathbf{L}\mathbf{v} = z\mathbf{v}$ leads to

$$\frac{d}{dx}\mathbf{v} = \begin{bmatrix} -\mathrm{i}\, z & -q(\cdot, t_0) \\ \kappa\bar{q}(\cdot, t_0) & \mathrm{i}\, z \end{bmatrix} \mathbf{v}. \tag{2}$$

This is an ordinary differential equation which can be solved for any complex z if q is sufficiently often differentiable. The question is for which z square-integrable solutions can be found. Consider the canonical solutions $\boldsymbol{\phi}, \boldsymbol{\phi}^\star, \boldsymbol{\psi}, \boldsymbol{\psi}^\star$ of (2) that correspond to the boundary conditions

$$\begin{aligned}
\boldsymbol{\phi}(x) &= \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\, zx} \\ 0 \end{bmatrix} + o(1), \quad x \to -\infty, \\
\boldsymbol{\phi}^\star(x) &= \begin{bmatrix} 0 \\ \mathrm{e}^{\mathrm{i}\, zx} \end{bmatrix} + o(1), \quad x \to -\infty, \\
\boldsymbol{\psi}(x) &= \begin{bmatrix} 0 \\ \mathrm{e}^{\mathrm{i}\, xz} \end{bmatrix} + o(1), \quad x \to +\infty, \\
\boldsymbol{\psi}^\star(x) &= \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\, xz} \\ 0 \end{bmatrix} + o(1), \quad x \to +\infty.
\end{aligned}$$

Any solution to (2) can be written as a linear combination of $\boldsymbol{\phi}$ and $\boldsymbol{\phi}^\star$ as well as of $\boldsymbol{\psi}$ and $\boldsymbol{\psi}^\star$. In particular, there exist unique scalars $\alpha(z), \beta(z), \gamma(z)$ and $\delta(z)$ such that

$$\boldsymbol{\phi} = \alpha(z)\boldsymbol{\psi}^\star + \beta(z)\boldsymbol{\psi}, \quad \boldsymbol{\phi}^\star = \gamma(z)\boldsymbol{\psi}^\star + \delta(z)\boldsymbol{\psi}.$$

The function $\boldsymbol{\phi}$ will be square-integrable only if it vanishes for $x \to \pm\infty$. It will vanish at $-\infty$ if and only if $\Im(z) > 0$. After evaluation of the limit at $+\infty$, one finds that

$$\lim_{x \to \pm\infty} \|\boldsymbol{\phi}(x)\| \to 0 \quad \Longleftrightarrow \quad \Im(z) > 0,\ \alpha(z) = 0. \tag{3}$$

If $q(\cdot, t_0)$ vanishes sufficiently rapidly, then $\boldsymbol{\phi}$ will do the same and (3) becomes also sufficient for $\boldsymbol{\phi}$ being square-integrable. A symmetry argument shows that $\delta(z) = \bar{\alpha}(\bar{z}^{-1})$, which implies the same conditions for $\boldsymbol{\phi}^\star$ being square-integrable. Since any solution of (2) is a linear combination of $\boldsymbol{\phi}$ and $\boldsymbol{\phi}^\star$, the discrete spectrum of $\mathbf{L}$ is given by the roots

$$\{\mu_k\} = \{\mu \in \mathbb{C} : \alpha(\mu) = 0,\ \Im(\mu) > 0\}. \tag{4}$$

Together with the *norming constants*[1]

$$\tilde{q}(\mu_k) := \frac{\beta(\mu_k)}{\alpha'(\mu_k)}, \quad \left(\alpha' = \frac{d\alpha}{dz}\right) \tag{5}$$

which encode some information about an canonical eigenfunction, they form the first part of the nonlinear Fourier spectrum. The continuous spectrum of the scattering operator can be shown to consist of the real line. The *reflection coefficient*

$$\hat{q}(z) := \frac{\beta(z)}{\alpha(z)}, \quad z \in \mathbb{R}, \tag{6}$$

which encodes information about the eigenfunctions, forms the second part of the nonlinear Fourier spectrum.

[1]The given formula for the norming constants requires that all roots are simple and that $\beta(z)$ is analytic in the upper halfplane, which is only ensured if $q(\cdot, t_0)$ vanishes sufficiently rapidly. See [2, Sec. IV.A] for sufficient conditions. In general, the formula [2, Eq. 4.20a] has to be used.

B. Time-Evolution of the Nonlinear Fourier Spectrum

The functions $\alpha(z)$ and $\beta(z)$ have been derived from the scattering operator for a fixed time t_0. Let us write $\alpha(z, t)$ and $\beta(z, t)$ to denote the $\alpha(z)$ and $\beta(z)$ that correspond to the time $t_0 = t$. Then, the time evolution of these functions is $\alpha(z, t) = \alpha(z, 0)$, $\beta(z, t) = \mathrm{e}^{-4\,\mathrm{i}\, z^2 t}\, \beta(z, 0)$ [2, Eq. 3.8].

C. Inverse Nonlinear Fourier Transform

There are two main methods to recover the potential q from the nonlinear Fourier spectrum. The first is to solve a system of integral equations, the *Marchenko equations* [2, Sec. IV.B]. The second method involves the solution of a *Riemann-Hilbert problem* [11, Sec. 2.3]. Specialized solutions for rational functions q have been given in [14], [15], [16], [17] and [18]. The discrete inverse problem will later be derived from first principles, which is why we need not discuss the continuous case here. See [19] and [20] for recent publications on the numerical continuous inverse scattering problem.

D. Nonlinear Fourier Analysis

The original interpretation of the nonlinear Fourier spectrum came from the scattering interpretation of the operator $\mathbf{L}$. However, the nonlinear Fourier spectrum can also be interpreted as a nonlinear version of the Fourier transform, leading to nonlinear Fourier analysis [2]. This is most easily understood for the reflection coefficient because it reduces to the linear Fourier transform as the amplitude of a potential becomes smaller and smaller [21, p. 257]. In the limit, they both coincide. The reflection coefficient is said to describe the *radiation* component of the potential. The discrete spectrum has no linear counterpart. It describes the *solitonic* components (traveling localized, particle-like waves) in a potential. See Figure 1 for an example. The real and the imaginary part of roots z_k describe the amplitude and the velocities of the solitons. The norming constants determine phase shifts. Asymptotically, for $t \to \infty$, the radiation components of the potential vanish and only solitons remain. Solitons can arise only for the focusing nonlinear Schrödinger equation.

Example 1 (Soliton [11], p. 252)**.** Consider the potential

$$q(x, t) = -\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}(2\xi x - 4(\xi^2 - \eta^2)t + \psi)}\, 2\eta\, \mathrm{sech}(2\eta x - 8\xi\eta t - \delta).$$

Figure 1 depicts the case $\xi = \eta = 1$, $\delta = \ln\frac{1}{2}$, $\psi = 0$. The discrete spectrum of this potential consists of exactly one pair $(z_0, \tilde{q}(z_0))$, which is related to the potential by

$$z_0 = \xi + \mathrm{i}\,\eta, \quad \mathrm{e}^{\delta} = \frac{|\tilde{q}(z_0)|}{2\eta}, \quad \mathrm{e}^{\mathrm{i}\,\psi} = \frac{\tilde{q}(z_0)}{|\tilde{q}(z_0)|}.$$

The amplitude of $q(x, t)$ is determined by the imaginary part η of the eigenvalue z_0, while the velocity is determined through the real part ξ. The norming constant $\tilde{q}(z_0)$ causes a constant offset in the phase of the overall potential and offsets the argument in the sech term. The reflection coefficient is trivial in this example, $\hat{q}(z) = 0$ for all $z \in \mathbb{R}$.

For a more in-depth discussion of the physical interpretation of the spectra, see [2, Sec. V].

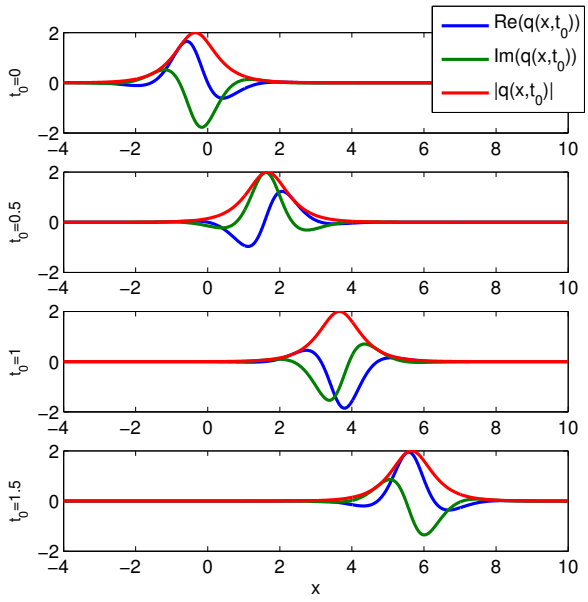

Figure 1. A single soliton.

III. The Ablowitz-Ladik Discretization and the Discrete Nonlinear Fourier Transform

Often, only samples of the potential q are known and the nonlinear Schrödinger equation has to be discretized. In this section, the Ablowitz-Ladik discretization [4] will be introduced and its relation to the continuous case will be discussed. For details, we refer to [3], [4], [10] and [11].

A. The Ablowitz-Ladik Discretization

Let $\varepsilon > 0$ denote a step size and assume that the samples

$$Q[n](\tau) := \varepsilon q(n\varepsilon, \tau\varepsilon^2), \quad n \in \mathbb{Z},\ \tau \in \mathbb{R}_{\geq 0},$$

are known. Then, the Ablowitz-Ladik discretization of the continuous nonlinear Schrödinger equation (1) is [4, Eq. 1.7]

$$\begin{aligned} 0 = \mathrm{i}\,\partial_\tau Q[n] + Q[n+1] - 2Q[n] + Q[n-1] \\ + \kappa |Q[n]|^2 (Q[n+1] + Q[n-1]). \end{aligned} \tag{7}$$

For $\varepsilon \to 0$ and $n\varepsilon \to x$, the solution of (7) converges towards the solution of the continuous nonlinear Schrödinger equation [11, Eq. 3.3]. There are of course other ways to discretize the nonlinear Schrödinger equation [22], but the Ablowitz-Ladik discretization has the advantage that it is amenable to a discrete version of the inverse scattering method [3], [4]. In particular, there is a discrete nonlinear Fourier transform that approximates the continuous case for $\varepsilon \to 0$.

B. The Discrete Nonlinear Fourier Transform

There are different ways to formulate the nonlinear Fourier transform. Here, we will use the definitions from Tao and Thiele [23] and Tsai [24] for the defocusing and the focusing case, respectively. Suppose that $Q[\cdot](\tau_0)$ is known at a fixed time $\tau = \tau_0$. In order to keep the notation simple, we will use the shorthand $Q[n]$ for $Q[n](\tau_0)$. We shall simplify the problem by making the often reasonable assumption

$$\forall n < 0: \quad Q[n] = 0. \tag{8}$$

Furthermore, in the defocusing case $\kappa = +1$, we have to assume that $|Q[n]| < 1$ for all n. This means no loss of generality since any solution to (7) has to satisfy this condition. The condition ensures the well-definedness of the normalization factor

$$\theta[n] := \sqrt{1 - \kappa |Q[n]|^2}.$$

Starting with $a_{-1}(w) := 1$ and $b_{-1}(w) := 0$, $w \in \mathbb{C}$, define the following two sequences of Laurent polynomials:

$$\begin{bmatrix} a_n(w) \\ b_n(w) \end{bmatrix} := \frac{1}{\theta[n]} \begin{bmatrix} 1 & \kappa \bar{Q}[n] w^{-n} \\ Q[n] w^n & 1 \end{bmatrix} \begin{bmatrix} a_{n-1}(w) \\ b_{n-1}(w) \end{bmatrix}, \tag{9}$$

where $n \in \mathbb{N}$. Their point-wise limits

$$a_\infty(w) := \lim_{n\to\infty} a_n(w), \quad b_\infty(w) := \lim_{n\to\infty} b_n(w),$$

are well-defined whenever $\sum_{n=0}^{\infty} |Q[n]| < \infty$ [23, Ch. 1.3], [24, Ch. 2]. Note that a_∞ and b_∞ will be Laurent polynomials for any $Q[\cdot]$ with finite support. It can be shown that a_∞ and b_∞ are approximations of the functions α and β arising in the continuous-time case subject to the coordinate transform

$$w = \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\varepsilon z} \tag{10}$$

because (9) implements a first-order discretization of a differential equation that describes the evolution of [22, p. 8]

$$\begin{bmatrix} \mathrm{e}^{\mathrm{i}xz} & \\ & \mathrm{e}^{-\mathrm{i}xz} \end{bmatrix} \boldsymbol{\phi}(x) = \begin{bmatrix} \alpha(z) \\ \beta(z) \end{bmatrix} + o(1), \quad x \to \infty,$$

where $\boldsymbol{\phi}$ was defined in Sec. II-A.[2] Thus,

$$\lim_{\varepsilon\to 0} a_\infty(w) = \alpha(z), \quad \lim_{\varepsilon\to 0} b_\infty(w) = \beta(z) \tag{11}$$

for well-behaved potentials q. In [23] and [24], the nonlinear Fourier transform of $Q[\cdot]$ has been defined as the tuple (a, b), but having the continuous case in mind we shall prefer to define the discrete-time versions of the reflection coefficient,

$$\hat{Q}(w) := \frac{b_\infty(w)}{a_\infty(w)}, \quad |w| = 1, \tag{12}$$

as well as of the discrete spectrum

$$\{\lambda_k\}_{k=0}^{K} := \{\lambda \in \mathbb{C} : a_\infty(\lambda) = 0,\ |\lambda| > 1\}, \tag{13}$$

and the norming constants

$$\tilde{Q}(\lambda_k) := \frac{b_\infty(\lambda_k)}{a'_\infty(\lambda_k)}, \tag{14}$$

where $a'_\infty = da_\infty/dw$. In light of (11), these quantities are related to their continuous-time counterparts by

$$\lim_{\varepsilon\to 0} \hat{Q}(w) = \hat{q}(z), \quad \lim_{\varepsilon\to 0} \frac{\tilde{Q}(w)}{-\mathrm{i}\varepsilon w} = \tilde{q}(z).$$

[2] This differential equation is a normalized version of (2). In [3] and [4], a direct discretization of (2) is used [22, Sec. III.E]. Details follow in Remark 2.

The roots of $a_\infty(w)$ will include approximations of the roots of $\alpha(z)$, but numerical experiments have shown that similar discretizations contain spurious roots $a(w) = 0$ that have no correspondance in the continuous spectrum [22, p. 9], [25, Sec. VII]. However, these are usually well-separated from the meaningful roots and can thus be easily removed.

Remark 2 (Related Discretizations). The original Ablowitz-Ladik discretization [3, Eq. 2.1], which is a direct discretization of (2), results if the transfer matrix in (9) is replaced with

$$\begin{bmatrix} w & -\bar{Q}[n] \\ \kappa\bar{Q}[n] & w^{-1} \end{bmatrix}.$$

In some applications, also the closely related transfer matrix

$$\begin{aligned} &\begin{bmatrix} w^{-\frac{1}{2}} & 0 \\ 0 & w^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 1 & -\bar{Q}[n] \\ \kappa Q[n] & 1 \end{bmatrix} \\ &= \begin{bmatrix} w^{-\frac{1}{2}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w^{\frac{1}{2}} & -\bar{Q}[n] \\ \kappa\bar{Q}[n] & w^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} w^{\frac{1}{2}} & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{15}$$

has been used [26, Eqs. 28+29], [27, Eqs. 33+34].

C. The Range of the Discrete Nonlinear Fourier Transform

Since the focus of this paper will be on the inverse nonlinear Fourier transform, the range of the discrete nonlinear Fourier transform is of interest. The following two theorems summarize some relevant results from the literature.

Theorem 3 (Defocusing Case)**.** *Let $\kappa = +1$. Then, there exist scalar coefficients $a_{n,j}$ and $b_{n,j}$ such that*

$$a_n(w) = \sum_{j=-n}^{0} a_{n,j} w^j, \quad b_n(w) = \sum_{j=0}^{n} b_{n,j} w^j. \tag{16}$$

For any function $w \mapsto f(w)$, denote the para-conjugate $w \mapsto \bar{f}(\bar{w}^{-1})$ by $f^\sim$. With this notation, the coefficients $a_{n,j}$ are completely determined by the $b_{n,j}$ through the conditions

$$\forall w \in \mathbb{C} \text{ s.t. } |w| > 1 : \; a_n(w) \neq 0, \quad a_n a_n^\sim = 1 + b_n b_n^\sim, \tag{17}$$

and $\mathbb{R} \ni a_{n,0} > 0$. It is $a_{n,0} = \prod_{j=0}^{n} \theta[n]^{-1}$, and the map

$$\begin{bmatrix} Q[0] & \dots & Q[n] \end{bmatrix} \mapsto \begin{bmatrix} b_{n,0} & \dots & b_{n,n} \end{bmatrix}$$

is a bijection between $\{z \in \mathbb{C} : |z| < 1\}^{n+1}$ and $\mathbb{C}^{n+1}$.

Proof: See [23, Ch. 1.2], or [26] for results w.r.t. (15). ∎

In order to understand the condition (17) better, let us recall a few well-known facts about spectral factorization.

Remark 4 (Spectral Factorization [28]). Let $\Psi(w) = \sum_{j=-n}^{n} \Psi_j w^j$ denote a non-zero Laurent polynomial such that $\mathbb{R} \ni \Psi(w) \geq 0$ whenever $|w| = 1$. Then, there exists a polynomial $S(w) = \sum_{j=0}^{n} S_j w^j$ such that $\Psi = SS^\sim$ and S has no roots in the unit disc $|w| < 1$. This factorization is known as *spectral factorization*, and any such S is a *spectral factor* of Ψ. Spectral factors are unique up to a scalar factor with absolute value one. The *canonical spectral factor* is the one with $\mathbb{R} \ni S(0) > 0$. The condition on the roots of S is equivalent to $S^\sim(w) = \bar{S}(\bar{w}^{-1}) = \sum_{j=-n}^{0} \bar{S}_{-j} w^j$ having no roots strictly outside the unit circle and $S^\sim(\infty) = \bar{S}_0 \neq 0$.

Hence, Theorem 3 states that for any polynomial b_n there is a unique a_n such that the pair (a_n, b_n) lies in the range of the nonlinear Fourier transform. This a_n is the para-conjugate of the canonical spectral factor of $1 + b_n b_n^\sim$.

In the focusing case, the situation is more complicated.

Theorem 5 (Focusing Case)**.** *Let $\kappa = -1$. Then, there exist scalar coefficients $a_{n,j}$ and $b_{n,j}$ such that (16) is satisfied. The two Laurent polynomials in (16) satisfy*

$$a_n a_n^\sim + b_n b_n^\sim = 1, \quad \mathbb{R} \ni a_{n,0} > 0. \tag{18}$$

It is $a_{n,0} = \prod_{j=0}^{n} \theta[n]^{-1}$, and the map

$$\begin{bmatrix} Q[0] & \dots & Q[n] \end{bmatrix} \mapsto (a_n, b_n)$$

is a bijection between $\mathbb{C}^{n+1}$ and the set of all Laurent polynomials of the form (16) that satisfy (18).

Proof: See [24, Ch. 2], or [27] for results w.r.t. (15). ∎

Note that $a_n^\sim$ does not have to be a spectral factor of $1 - b_n b_n^\sim$ in this case. Nevertheless, for any polynomial b_n the corresponding canonical spectral factor S of $1 - b_n b_n^\sim$ is such that the pair $(S^\sim, b_n)$ lies in the range of the nonlinear Fourier transform. Since this solution will have no roots outside the unit circle, it corresponds to a potential without solitonic components. By reflecting the roots of $S^\sim$ with respect to the unit circle (i.e., $w \mapsto \bar{w}^{-1}$), other Laurent polynomials a such that (a, b_n) lies in the range of the nonlinear Fourier transform can be obtained [27, p. 258]. If a is not constrained to Laurent polynomials, Blaschke products may be used to create an a with arbitrary roots outside the unit circle [24, Th. 4.1].

IV. First Step of the Inverse Transform: Finding $a(w)$ and $b(w)$ From Spectral Data

The process of constructing a potential $Q[\cdot]$ with prescribed nonlinear Fourier spectrum (12) and/or (13)–(14) will be carried out in two steps. First, suitable polynomials a and b that lie in the range of the discrete nonlinear Fourier transform (9) will be constructed. Several variants of this first step will be discussed in the following. The second step, the reconstruction of $Q[\cdot]$ from a and b, will be discussed in the next section.

A. Defocusing Case: Prescribed Reflection Coefficients

The inverse nonlinear Fourier problem for the defocusing nonlinear Schrödinger equation arises in the design of a certain type of optical filters, *fiber Bragg gratings* [26], [29].

Problem 6. Let $w_k, \hat{Q}(w_k) \in \mathbb{C}$, $k \in \{1, \dots, K\}$, denote given problem data such that $w_k \neq w_l$ whenever $k \neq l$, $|w_k| = 1$, and $|\hat{Q}(w_k)| < 1$ for all k. Then, find Laurent polynomials

$$a(w) = \sum_{j=-n}^{0} a_j w^j, \quad b(w) = \sum_{j=0}^{n} b_j w^j \tag{19}$$

such that $b(w_k) = a(w_k)\hat{Q}(w_k)$ for all k, $a(\lambda) = 0$ only if $|\lambda| \leq 1$, and $aa^\sim = 1 + bb^\sim$.

In [26, Sec. III], an algorithm that tries to match a and b in the case where the problem is overdetermined (i.e., $K > n$) has been given. Naturally, since a and b are constrained to be Laurent polynomials, b/a will not match the interpolation conditions exactly, in general. Another issue is that the obtained Laurent polynomials in [26, Sec. III] may not exactly satisfy the condition $aa^\sim = 1 + bb^\sim$. Hence, they do not have to lie in the range of the nonlinear Fourier transform. In [29], first a solution to the interpolation problem is found without any consideration of the range of the nonlinear Fourier transform. Then, in a second step, the result of the interpolation has to be forced into the range of the nonlinear Fourier transform, which introduces new interpolation errors. In the following, a new method for the approximate solution of Problem 6 will be proposed. Solutions found by the new method are guaranteed to lie in in the range of the nonlinear Fourier transform

Consider the following family of *Pick matrices*:

$$\mathbf{P}_\rho := \left[\frac{1-\bar{\hat{Q}}_k\hat{Q}_l}{1-\rho^2 w_k\bar{w}_l}\right]_{k,l=1}^{K}, \quad 0 \le \rho < 1.$$

As was noted in [30], the diagonal entries of this matrix are real and diverge towards $+\infty$ as $\rho \to 1$, while the absolute values of the remaining entries are upper bounded independently of the parameter ρ. Hence, $\mathbf{P}_\rho$ will be positive definite for $\rho < 1$ large enough. In that case, [31, Thm. 1] ensures the existence of a unique pair of polynomials,

$$r_\rho(w) = \sum_{j=0}^{K} r_{\rho,j} w^j, \quad s_\rho(w) = \sum_{j=0}^{K} s_{\rho,j} w^j,$$

with all roots on or outside the complex unit circle such that

$$\frac{s_\rho(\rho w_k)}{r_\rho(\rho w_k)} = w_k^K \bar{\hat{Q}}_k \tag{20}$$

for all k and $r_\rho r_\rho^\sim = 1 + s_\rho s_\rho^\sim$. The choice

$$a_\rho(w) := r_\rho^\sim(w), \quad b_\rho(w) := w^K s_\rho^\sim(w),$$

satisfies all conditions in Problem 6 except for the interpolation conditions. Regarding the interpolation error, note that

$$\frac{b_\rho(w_k\rho^{-1})}{a_\rho(w_k\rho^{-1})} = w_k^K \frac{s_\rho^\sim(w_k\rho^{-1})}{r_\rho^\sim(w_k\rho^{-1})} = w_k^K \frac{\bar{s}_\rho(\rho w_k)}{\bar{r}_\rho(\rho w_k)} = \hat{Q}_k$$

for all k. Since a_ρ and b_ρ are continuous, it seems to be common practice to assume that

$$\frac{b_\rho(w_k)}{a_\rho(w_k)} \approx \hat{Q}_k$$

whenever $\rho \approx 1$ [32, Sec. III]. However, the authors are currently not aware of a formal proof that the interpolation error $\max_k |b_\rho(w_k)/a_\rho(w_k) - \hat{Q}(w_k)|$ vanishes as $\rho \to 1$. Also, the behavior of the polynomials a_ρ and b_ρ themselves in the limiting case $\rho \to 1$ is currently unclear. The discussions in [33, Ex. 6.1], [34, p. 861] and [35] however indicate that this process may be well-conditioned at least in certain cases.

Remark 7. Finding r_ρ and s_ρ can be formulated as a convex optimization problem [36]. The relation between the problem formulation in [36] and the special case considered in [31] can be found, e.g, in [37, Sec. 5.2].

B. Defocusing Case: Simplified Problem

Consider the following slight modification of Problem 6.

Problem 8. Let $w_k, \hat{Q}(w_k) \in \mathbb{C}$, $k \in \{1, \dots, K\}$, denote given problem data such that $w_k \neq w_l$ whenever $k \neq l$, $|w_k| = 1$, and $|\hat{Q}(w_k)| < 1$ for all k. Then, find Laurent polynomials as in (19) such that $b(w_k) = |a(w_k)|\hat{Q}(w_k)$ for all k, $a(\lambda) = 0$ only if $|\lambda| \le 1$, and $aa^\sim = 1 + bb^\sim$.

Surprisingly, it is straightforward to solve this problem exactly for the fixed degree $n = K - 1$. Proceed as follows. First, use polynomial interpolation to find the unique polynomial b of degree $n = K-1$ that satisfies the conditions

$$b(w_k) = \frac{\hat{Q}(w_k)}{\sqrt{1 - |\hat{Q}(w_k)|^2}} \tag{21}$$

for all k, and solve the spectral factorization problem $1 + bb^\sim = SS^\sim$ for the canonical spectral factor S. Using that

$$1 + |b(w)|^2 = S(w)S^\sim(w) = |S(w)|^2, \quad |w| = 1,$$

one finds that $a := S^\sim$ satisfies the interpolation conditions:

$$\begin{aligned}\frac{b(w_k)}{|a(w_k)|} &= b(w_k)\frac{1}{\sqrt{1+|b(w_k)|^2}} \\ &= \frac{\hat{Q}(w_k)}{\sqrt{1-|\hat{Q}(w_k)|^2}} \frac{1}{\sqrt{1+\frac{|\hat{Q}(w_k)|^2}{1-|\hat{Q}(w_k)|^2}}} = \hat{Q}(w_k) \quad \forall k.\end{aligned}$$

The function a satisfies the other constraints in Problem 8 as well. Hence, a and b form a solution to Problem 8.

The advantage of the formulation in Problem 8 is that the major physical properties of the potential (the magnitude of the reflection coeffficient) can be controlled while at the same time all degrees of freedom in the potential can be modulated because the relationship between the interpolation data and the coefficients of b is one-to-one (cf. Theorem 3).

C. Focusing Case: Non-Solitonic Potentials

In the focusing case, it is known that a continuous-time potential with $\int_{-\infty}^{\infty} |q(x,t)|dx \le \pi/2$ cannot contain solitons [38]. In such cases, α in (4) will have no roots with $\Im(z) > 0$ and only the reflection coefficient (5) remains to be assigned. The discretized version of the inverse problem where the values of the polynomial b in (12) are prescribed on the unit circle has been discussed in the context of optical filter design, for so-called *grating-assisted codirectional couplers* [27], [39]. Theorem 3 shows that for an arbitrary b, the unique admissable a_0 (up to a phase factor) that has no roots outside the unit circle can be found via spectral factorization.

The more involved problem to prescribe reflection coefficients has recently been considered in the context of optical information transmission [40], where an iterative method was designed to generate a potential whose reflection coefficient matches a prescribed linear Fourier spectrum. The same could be achieved by solving the following problem.

Problem 9. Let $w_k, \hat{Q}(w_k) \in \mathbb{C}$, $k \in \{1, \dots, K\}$, denote given problem data such that the w_k are mutually different and $|w_k| = 1$ for all k. Then, find Laurent polynomials as in (19) such that $b(w_k) = a(w_k)\hat{Q}(w_k)$ for all k, $a(\lambda) = 0$ only if $|\lambda| \geq 1$, and $aa^\sim + bb^\sim = 1$.

We have not yet investigated whether or not the method from Section IV-A carries over to Problem 9. However, the method in Section IV-B can easily be modified to solve the simplified variant of Problem 9, where the interpolation condition is $b(w_k) = |a(w_k)|\hat{Q}(w_k)$. This approach seems especially interesting for optical information transmission [40] because it is simple, but at the same time all degrees of freedom can be used for the modulation of information. Furthermore, in contrast to the direct modulation of information in the coefficients of the polynomial b, it allows us to restrict a potential to certain bands within the nonlinear spectrum. This is a very valuable property in multiuser scenarios [41, Sec. II.C].

D. Focusing Case: Reflectionless Potentials

Whenever a solution to the focusing nonlinear Schrödinger equation contains solitonic components, they manifest themselves in the discrete spectrum (4) through the roots of the polynomial a. If, in addition, the continuous spectrum is trivial (i.e., the reflection coefficient (6) is identically zero), the potential is called *reflectionless* or, if there are only finitely many solitons, an *N-soliton solution*. The nonlinear Fourier transform for reflectionless potentials has well-known explicit solutions [8], [15], [18, Sec. III]. An interesting application of reflectionless potentials lies in the area of optical communications, where solitons have been considered as carriers of information [5], [42], [43]. In particular, the use of reflectionless potentials for information transmission has been investigated in [41] and [44]. Another application is nuclear magnetic resonance spectroscopy [15].

We consider the following discretized version of the reflectionless soliton assignment problem.

Problem 10. Let $\lambda_k \in \mathbb{C}$, $k \in \{1, \dots, K\}$, denote given problem data such that the λ_k are mutually distinct and $|w_k| > 1$ for all k. Then, find a Laurent series

$$a(w) = \sum_{j=-\infty}^{0} a_j w^j, \quad |w| \geq 1,$$

such that $a(\lambda) = 0$ and $|\lambda| > 1$ if and only if $\lambda \in \{\lambda_k\}$, all λ_k are simple roots, and $aa^\sim = 1$.

The proof of Theorem 4.1 in [24] shows that Problem 10 has a straight-forward solution: the finite Blaschke product

$$a(w) := \mathrm{e}^{\mathrm{i}\,\varphi} \prod_{k=1}^{K} \frac{w - \lambda_k}{1 - w\bar{\lambda}_k}, \quad w \in \mathbb{C}, \tag{22}$$

where $\varphi \in [-\pi, \pi)$ is such that $\mathbb{R} \ni a(\infty) \geq 0$. It clearly has the right roots and satisfies $aa^\sim = 1$. By applying the geometric series to a partial fraction expansion

$$a(w) = \frac{c_1}{1 - w^{-1}\bar{\lambda}_1^{-1}} + \dots + \frac{c_K}{1 - w^{-1}\bar{\lambda}_K^{-1}}, \quad c_k \in \mathbb{C},$$

one also finds that a has a series expansion

$$a(w) = \sum_{j=-\infty}^{0} a_j w^j, \quad |w| \geq 1, \tag{23}$$

such that $|a_j| \in O(\varrho^j)$ for $\varrho := \max_k |\lambda_k^{-1}|$.

Remark 11. The solution to Problem 10 presented here is extremely simple. Together with the layer peeling method in the next section and the next remark, a simple and efficient method to synthesize arbitrary almost reflectionless potentials is obtained. As such, it is an appealing alternative to the more complicated methods discussed in [41, Sec. III.B], where the soliton assignment problem for reflectionless potentials is approached from the continuous-time viewpoint. It is also simpler than the approaches presented in [15, Sec. III] and [18].

Remark 12. Let a solve Problem 10. Then, $a_\epsilon(w) := \sqrt{1-\epsilon}a(w)$ and $b_\epsilon(w) := \sqrt{\epsilon}w^n$, $0 < \epsilon < 1$, lie in the range of the nonlinear Fourier transform for any n and ϵ. Furthermore, a_ϵ has exactly the same roots as a and the associated potential will be almost reflectionless if ϵ is small. The impact of ϵ on the energy of the associated potential can be assessed with [24, (3.1)].

Remark 13. The only solutions $a(w) = \sum_{j=-n}^{0} a_j w^j$ to $aa^\sim = 1$ with finite n are the monomials $a(w) = w^{-n}$. (Use that $a^\sim$ is an entire function with constant modulus on the unit circle and [45, Ex. 6.12].) Thus, in contrast to the Problems 6 and 8, Problem 10 can only be solved with an infinite series.

Remark 14. Consider any sequence $\eta_j \in \mathbb{C}$ with $|\eta_j| < 1$, $j \in \{0, 1, 2, \dots\}$, that satifies the *Blaschke condition* $\sum_{n=1}^{\infty}(1 - |\eta_j|) < \infty$. Then, $a(w) := \prod_{k=1}^{K} \frac{w-\lambda_k}{1-w\bar{\lambda}_k} \prod_{j=1}^{\infty} \frac{w-\eta_j}{1-w\bar{\eta}_j}$ solves Problem 10 as well. Hence, the solution (22) is not unique.

Remark 15. Problem 10 does not specify norming constants because they will always be zero in light of (14). At this point, note that the potential in Example 1 is reflectionless but has non-zero reflection coefficients. This happens because (14) in not valid in Example 1 for the reasons explained in Footnote 1. The series expansion (23) however always decays sufficiently rapidly for (14) to be valid.

V. Second Step of the Inverse Transform: Layer Peeling Applied to $a(w)$ and $b(w)$

The potential $Q[0], \dots, Q[n]$ in (9) is easily recovered from the functions $a = a_n$ and $b = b_n$ by performing *layer peeling*. Layer peeling procedures for the discretization (9) have been described in [23] and [24] for the defocusing and the focusing case, respectively. Procedures for the discretization (15) have been given in [27], [29] and [46]. More related algorithms for similar settings can be found in [13], [47] and the references therein. Here, we follow [26, p. 1240].

Let a_n and b_n be known for finite n. Multiply (9) by

$$\frac{1}{\theta[n]} \begin{bmatrix} 1 & -\kappa\bar{Q}[n]w^{-n} \\ -Q[n]w^n & 1 \end{bmatrix}$$

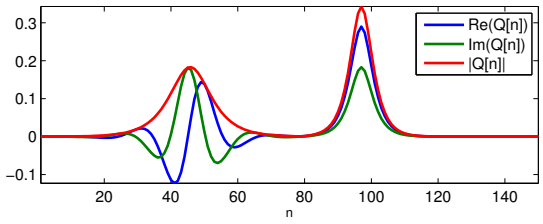

Figure 2. Almost reflectionless potential with two solitons.

from the left to find that

$$\begin{bmatrix} a_{n-1}(w) \\ b_{n-1}(w) \end{bmatrix} = \frac{-1}{\theta[n]} \begin{bmatrix} -1 & \kappa\bar{Q}[n]w^{-n} \\ Q[n]w^n & -1 \end{bmatrix} \begin{bmatrix} a_n(w) \\ b_n(w) \end{bmatrix}. \tag{24}$$

The second line in this equation can be written as

$$-\theta[n] \sum_{j=0}^{n-1} b_{n-1,j} w^j = \sum_{j=0}^{n} (Q[n] a_{n,j-n} - b_{n,j}) w^j.$$

By comparing the coefficients of w^n, one finds $0 = Q[n]a_{n,0} - b_{n,n}$. The Theorems 3 and 5 ensure $a_{n,0} \neq 0$, so that

$$Q[n] = \frac{b_{n,n}}{a_{n,0}}.$$

With $Q[n]$ known, a_{n-1} and b_{n-1} can be found via (24). By repeating this process $n-1$ times, the complete potential $Q[n], Q[n-1], \ldots, Q[0]$ can eventually be reconstructed.

Remark 16. The method discussed here does not apply to the reflectionless potentials in Section IV-D. Those are always of infinite length and have $b = 0$. A truncation of a and b to finite length a_n and b_n results in $b_n = 0$, and the iteration generates $Q[n] = \cdots = Q[0] = 0$. Nevertheless, layer peeling may be applied to the finite approximations of a_ϵ and b_ϵ given in Remark 12.

Example 17. Figure 2 depicts an almost reflectionless potential, which has been computed by applying layer peeling to the finite approximations of a_ϵ and b_ϵ given in Remark 12. The parameters used during the computation were $\lambda_1 = 1.2\,\mathrm{e}^{0.3\,\mathrm{i}}$, $\lambda_2 = 1.4$, $n = 149$, and $\epsilon = 10^{-16}$.

VI. Computational Aspects And Fast Transforms

Most of the methods derived in this paper have straightforward numerical implementations, which we would like to recall in this section for the readers convenience.

A. The Defocusing Case (Problem 6)

The approximate solution for Problem 6 is exceptional because its numerical implementation is neither straightforward nor fast. Algorithms that solve (20) subject to the conditions in Section IV-A have been discussed in [34], [36] and [48].

B. The Simplified Defocusing and the Non-solitonic Focusing Case (Problems 8 and 9)

The *barycentric Lagrange method*, which is a numerically reliable method that requires $O(K^2)$ *floating point operations (flops)*, may be used to solve the interpolation problem (21) [49]. The *fast inverse non-equidistant Fourier transform* [50], [51] poses a fast alternative with $O(K \operatorname{polylog} K)$ flops (because $|w_k| \equiv 1$). Other $O(K \operatorname{polylog} K)$ algorithms exploit the structure of Vandemonde matrices [52], [53].

Regarding the consecutive spectral factorization problem $1 + bb^\sim = SS^\sim$, we refer to the surveys [28] and [54]. Some classical algorithms like the *Cepstral method* [54, Sec. 2.5] can be implemented with the fast Fourier transform, but this approach is not always reliable. For recent works on fast spectral factorization algorithms, see [55], [56] and [57].

C. The Reflectionless Focusing Case (Problem 10)

The goal is to find the coefficients $a_0, \ldots, a_n$ in (23) for n large. The coefficients of the numerator as well as of the denominator in (22) can be found with $O(K \log^2 K)$ flops using the approach described in [25, Alg. 1] and [58, Sec. V]. Then, the series expansion of the rational function can be approximated with the help of the fast Fourier transform.

VII. Outlook

The layer peeling method is currently the computational bottleneck with a complexity of $O(n^2)$ flops. For future research, it would be interesting to see if and how the *doubling technique* mentioned in [47, p. 18], [59, p. 376] can be used to break that bottleneck. The discussion in Section VI suggests that this might enable fast inverse nonlinear Fourier transforms in $O(n \operatorname{polylog} n)$ flops, similar to the recently obtained fast nonlinear Fourier transforms in [25] and [58].

The behavior of the interpolation error in Section IV-A on interpolation of the reflection coefficient in the defocusing case also deserves further study. Although the impact of a single new root that is added to the problem close to the unit circle has been investigated in [60], this does not help in the current case where all roots approach the unit circle simultaneously. The question of if and how the approach carries over to the focusing case is of interest as well.

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