

Minimal realizations of a special class of 2D codes

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Abstract—In this paper we consider a special class of 2D convolutional codes (composition codes) whose encoders $G(d_1, d_2)$ can be decomposed as the product of two 1D encoders, i.e., $G(d_1, d_2) = G_2(d_2)G_1(d_1)$. We prove that if $G_2(d_2)$ is a systematic encoder, then the composition code $\text{Im } G(d_1, d_2)$ has a minimal 2D state-space realization by means of a separable Roesser model that can be obtained from minimal state space realizations of the 1D codes $\text{Im } G_1(d_1)$ and $\text{Im } G_2(d_2)$.

I. INTRODUCTION

The problem of obtaining minimal state-space realizations for convolutional codes is a question of crucial importance not only due to implementation issues, but also because such realizations allow to construct codes with suitable properties. This issue has been solved in [4] for the one-dimensional (1D) case using the connection between coding and the behavioral approach, developed by J. C. Willems [11] for the analysis of dynamical systems. The purpose of this paper is to analyze the realization problem for two-dimensional (2D) convolutional codes, starting from their encoders.

Similarly to what happens in the 1D case this is a hard problem since there are many different encoders for the same code. Therefore it is not enough to obtain a minimal realization for an encoder, but it is also necessary to guarantee that such realization is a minimal realization of the code, i.e., it has the lowest dimension among all the minimal realizations of all the encoders for the same code. Encoders whose minimal realizations are also minimal realizations for the corresponding code are called *minimal encoders*.

A characterization of minimal 1D encoders has been given in [4]. Concerning the 2D case, a characterization of minimal 2D polynomial encoders can be found in [7] for 2D convolutional codes of rate $\frac{1}{n}$. However, generalizing the results presented in [7] for 2D convolutional codes of rate $\frac{k}{n}$, with $k > 1$, appears to be a very difficult problem. Therefore here we take another approach and restrict ourselves to a particular class of 2D convolutional codes.

Concretely, in this study we consider a particular class of 2D polynomial encoders that we call *composition encoders*; these encoders are obtained through the composition of two 1D encoders, each one in one direction/indefinite. We prove that, under certain conditions, composition encoders are minimal. Moreover, for the encoders that satisfy these minimality conditions, minimal 2D state-space realizations are obtained, which are minimal realizations of the corresponding 2D convolutional code.

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This paper is organized as follows: in the next section we present the notions of 2D convolutional codes and their encoders. Minimal realizations of an encoder/code are discussed for both 1D and 2D cases. In section III, the particular class of 2D composition encoders to be considered is presented together with the corresponding codes. In section IV sufficient conditions for the minimality of a 2D convolutional encoder are introduced and minimal realizations of composition codes are obtained. Section V contains the concluding remarks.

II. PRELIMINARIES

A. 2D convolutional codes and their encoders

In this paper we consider 2D convolutional codes constituted by sequences indexed by \mathbb{Z}^2 and taking values in \mathbb{F}^n , where \mathbb{F} is a field. Such sequences $\{\mathbf{w}(i, j)\}_{(i, j) \in \mathbb{Z}^2}$ can be represented by bilateral formal power series

$$\hat{\mathbf{w}}(d_1, d_2) = \sum_{(i, j) \in \mathbb{Z}^2} \mathbf{w}(i, j) d_1^i d_2^j.$$

For $n \in \mathbb{N}$, the set of bilateral formal power series over \mathbb{F}^n is denoted by \mathcal{F}_{2D}^n . This set is a module over the ring $\mathbb{F}[d_1, d_2]$ of 2D polynomials over \mathbb{F} . The set of matrices of size $n \times k$ with elements in $\mathbb{F}[d_1, d_2]$ will be denoted by $\mathbb{F}^{n \times k}[d_1, d_2]$.

Given a subset \mathcal{C} of sequences indexed by \mathbb{Z}^2 , taking values in \mathbb{F}^n , we denote by $\hat{\mathcal{C}}$ the subset of \mathcal{F}_{2D}^n defined by $\hat{\mathcal{C}} = \{\hat{\mathbf{w}} : \mathbf{w} \in \mathcal{C}\}$.

Definition 1: A 2D convolutional code \mathcal{C} is a subset of sequences indexed by \mathbb{Z}^2 such that $\hat{\mathcal{C}}$ is a submodule of \mathcal{F}_{2D}^n which coincides with the image of \mathcal{F}_{2D}^k (for some $k \in \mathbb{N}$) by a polynomial operator $G(d_1, d_2)$, i.e.,

$$\begin{aligned} \hat{\mathcal{C}} &= \text{Im } G(d_1, d_2) \\ &= \{\hat{\mathbf{w}}(d_1, d_2) = G(d_1, d_2)\hat{\mathbf{u}}(d_1, d_2), \hat{\mathbf{u}}(d_1, d_2) \in \mathcal{F}_{2D}^k\}. \end{aligned}$$

With some abuse of language we also write $\mathcal{C} = \text{Im } G(d_1, d_2)$.

It follows, as a consequence of [Theorem 2.2, [6]], that a 2D convolutional code can always be given as the image of a full column rank polynomial operator $G(d_1, d_2) \in \mathbb{F}^{n \times k}[d_1, d_2]$. Such polynomial operator/matrix is called an *encoder* of \mathcal{C} .

Note that this definition of code differs from the definition in [10], where only finite support codewords are considered. Moreover our definition of encoder is slightly different from the one in [3] where non full column rank 2D polynomial matrices are allowed as encoders. However, our definition is motivated by the fact that only full column rank encoders are

relevant for the purpose of obtaining minimal realizations of a code.

Two encoders, $G_1(d_1, d_2) \in \mathbb{F}^{n \times k}[d_1, d_2]$ and $G_2(d_1, d_2) \in \mathbb{F}^{n \times k}[d_1, d_2]$ are said to be *equivalent* if they generate the same code \mathcal{C} . Similar to what is proved in [4] for the 1D case, it can be shown that if $G_1(d_1, d_2)$ and $G_2(d_1, d_2)$ are equivalent encoders, there exist two square non-singular matrices over $\mathbb{F}[d_1, d_2]$, $P_1(d_1, d_2)$ and $P_2(d_1, d_2)$, such that

$$G_1 P_1 = G_2 P_2.$$

This implies that

$$G_1 = G_2 U_2 \quad \text{and} \quad G_2 = G_1 U_1,$$

with $U_2 = P_2 P_1^{-1}$ and $U_1 = P_1 P_2^{-1}$, i.e., the convolutional encoders are unique up to the post-multiplication by a square nonsingular 2D rational matrix.

If $G_1(d_1, d_2)$ is right factor prime¹ and $G_2(d_1, d_2)$ is equivalent to $G_1(d_1, d_2)$ then

$$G_2 = G_1 P,$$

for some square 2D polynomial matrix $P(d_1, d_2)$. In case $G_1(d_1, d_2)$ and $G_2(d_1, d_2)$ are both right factor prime then

$$G_2 = G_1 U,$$

for some 2D unimodular polynomial matrix $U(d_1, d_2) \in \mathbb{F}^{k \times k}[d_1, d_2]$. In this paper also 1D encoders are considered. These are defined in a similar way as the 2D encoders, but only in one indeterminate d (instead of d_1 and d_2).

B. Realization Problem

As is well-known, there exist several types of 2D state-space models [1], [2], [9]. In our study we shall consider the *separable Roesser model*. This model has the following form

$$\begin{cases} \sigma_1 x_1 = A_{11} x_1 + B_1 u \\ \sigma_2 x_2 = A_{21} x_1 + A_{22} x_2 + B_2 u \\ w = C_1 x_1 + C_2 x_2 + D u \end{cases}, \quad (1)$$

where $A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2$ and D are matrices over \mathbb{F} , $\sigma_1 x_1(i, j) = x_1(i+1, j)$ and $\sigma_2 x_2(i, j) = x_2(i, j+1)$, for all $(i, j) \in \mathbb{Z}^2$, u is the input-variable, w is the output-variable and $x = (x_1, x_2)$ is the state variable where x_1 and x_2 are the horizontal and the vertical state-variable, respectively. It is denoted by $\Sigma^{2D}(A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D)$.

¹A polynomial matrix $G(d_1, d_2) \in \mathbb{F}^{n \times k}[d_1, d_2]$ is right factor prime if for every factorization $G(d_1, d_2) = \bar{G}(d_1, d_2)T(d_1, d_2)$, with $\bar{G}(d_1, d_2) \in \mathbb{F}^{n \times k}[d_1, d_2]$ and $T(d_1, d_2) \in \mathbb{F}^{k \times k}[d_1, d_2]$, $T(d_1, d_2)$ is unimodular, i.e., is invertible in $\mathbb{F}^{k \times k}[d_1, d_2]$.

1) Encoder and code realization:

Definition 2: $\Sigma^{2D}(A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D)$ is said to be a realization of an encoder $G(d_1, d_2) \in \mathbb{F}^{n \times k}[d_1, d_2]$ if ²

$$G(d_1, d_2) = \bar{C} \bar{A}(d_1, d_2)^{-1} \bar{B}(d_1, d_2) + D,$$

where

$$\bar{C} = [C_1 \quad C_2], \quad \bar{A}(d_1, d_2) = \begin{bmatrix} I - A_{11}d_1 & 0 \\ -A_{21}d_2 & I - A_{22}d_2 \end{bmatrix}$$

and

$$\bar{B}(d_1, d_2) = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} d_1 + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} d_2.$$

This is equivalent to saying that

$$\begin{aligned} \mathcal{B}_{(u,w)} &:= \{(u, w) : \hat{w}(d_1, d_2) = G(d_1, d_2)\hat{u}(d_1, d_2)\} \\ &= \{(u, w) : \exists x = (x_1, x_2) \text{ s.t. } (u, x, w) \text{ satisfies (1)}\}; \end{aligned}$$

this fact will be here expressed by the equality $\Sigma^{2D}(A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D) = \Sigma^{2D}(G)$. Moreover, $\Sigma^{2D}(G)$ is said to be a minimal realization of $G(d_1, d_2)$ if the size of $x = (x_1, x_2)$ is minimal among all the realizations of $G(d_1, d_2)$.

Definition 3: $\Sigma^{2D}(A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D)$ is said to be a realization of the 2D convolutional code \mathcal{C} if

$$\begin{aligned} \mathcal{B}_w &:= \{w : \mathbb{Z}^2 \rightarrow \mathbb{F}^n \mid \exists x_1, x_2, u \text{ s.t. } (u, x_1, x_2, w) \text{ satisfies (1)}\} \\ &= \mathcal{C}, \end{aligned}$$

which will be denoted by $\Sigma^{2D}(A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D) = \Sigma^{2D}(\mathcal{C})$. Moreover, $\Sigma^{2D}(\mathcal{C})$ is said to be a minimal realization of the code \mathcal{C} if the size of (x_1, x_2, u) is minimal among all the realizations of \mathcal{C} .

Note that when realizing an encoder the focus is set on an input/output relation (translated by the input/output behavior $\mathcal{B}_{(u,w)}$ or, equivalently, by the input/output operator $G(d_1, d_2)$). This gives rise to an *input/state/output* (i/s/o) model. On the other hand, when realizing a code one is only interested in the system output behavior \mathcal{B}_w . This gives rise to a type of realization that has been widely considered within Willems's behavioral approach [11], and is known as *state/driving-variable* (s/dv) realization.

It is worth mentioning that in the realization of an output behavior there is some freedom in the choice of the input-variables as long as the set of output-trajectories remains the same. As shall be seen this can be exploited in order to reduce the dimension of the obtained state-space realizations. In this way code realizations can have lower dimension than encoder realizations (for which the freedom to change the inputs does not exist).

The minimal encoders are the ones for which a minimal realization is also minimal when regarded as a code realization.

In order to study the question of minimality of the class of models (1) we first recall some results established for 1D systems.

²Note that $\widehat{\sigma_i w} = d_i^{-1} \hat{w}$.

2) *Minimality of 1D realizations:* In the sequel the 1D state-space model

$$\begin{cases} \sigma x = Ax + Bu \\ w = Cx + Du \end{cases}, \quad (2)$$

where A, B, C and D are matrices over \mathbb{F} , $\sigma x(t) = x(t+1)$, for all $t \in \mathbb{Z}$, u is the input-variable, w is the output-variable and x is the state-variable, will be denoted by $\Sigma^{1D}(A, B, C, D)$.

Definition 4: $\Sigma^{1D}(A, B, C, D)$ is said to be a realization of the 1D encoder $G(d) \in \mathbb{F}^{n \times k}[d]$ if

$$G(d) = C(I - Ad)^{-1}Bd + D,$$

which is equivalent to say that

$$\begin{aligned} \mathcal{B}_{(u,w)} &:= \{(u, w) : \hat{w}(d) = G(d)\hat{u}(d)\} \\ &= \{(u, w) : \exists x \text{ s.t. } (u, x, w) \text{ satisfies (2)}\}. \end{aligned}$$

This fact is expressed by the equality $\Sigma^{1D}(A, B, C, D) = \Sigma^{1D}(G)$. Moreover, $\Sigma^{1D}(G)$ is said to be a minimal realization of $G(d)$ if the size of x is minimal among all the realizations of $G(d)$.

It follows from the previous definition that $\Sigma^{1D}(A, B, C, D)$ is a minimal realization of the encoder $G(d)$ if and only if it is a minimal i/s/o realization of the transfer function $G(d)$. As is well known, minimal realizations of transfer functions are characterized by being simultaneously controllable and observable [5].

Definition 5: $\Sigma^{1D}(A, B, C, D)$ is said to be a realization of the 1D convolutional code \mathcal{C} if

$$\mathcal{B}_w := \{w : \mathbb{Z} \rightarrow \mathbb{F}^n \mid \exists x, u \text{ s.t. } (u, x, w) \text{ satisfies (2)}\} = \mathcal{C}.$$

This is denoted by $\Sigma^{1D}(A, B, C, D) = \Sigma^{1D}(\mathcal{C})$. Moreover, $\Sigma^{1D}(\mathcal{C})$ is said to be a minimal realization of the code \mathcal{C} if the size of (x, u) is minimal among all the realizations of \mathcal{C} .

A complete characterization of minimality for 1D convolutional codes is given by [Theorem 4.2, [11]], reproduced below using the terminology of codes.

Theorem 1: [Theorem 4.2, [11]] A realization $\Sigma^{1D}(A, B, C, D) = \Sigma^{1D}(\mathcal{C})$ of a code \mathcal{C} is minimal if and only if the following conditions are satisfied.

- (i) $\begin{bmatrix} B^T & D^T \end{bmatrix}^T$ has full column rank.
- (ii) $\begin{bmatrix} A & B \end{bmatrix}$ has full row rank.
- (iii) $\ker D \subseteq \ker B$ (i.e., there exists a matrix L such that $B = LD$).
- (iv) Let $L \in \mathbb{F}^{m \times n}$ be as in (iii), and let $\Lambda \in \mathbb{F}^{(n-k) \times n}$ be a minimal left-annihilator (*mla*)³ of D . Then the pair $(A - LC, \Lambda C)$ is observable.

Note that (i) and (iii) are equivalent to (i') - D has full column rank - and (iii).

Example 1: Consider the following 1D polynomial encoder of a code \mathcal{C}

³ Λ is a *mla* of D if $\Lambda D = 0$ and for all Λ^* such that $\Lambda^* D = 0$ there exists $\tilde{\Lambda}$ satisfying $\Lambda^* = \tilde{\Lambda} \Lambda$

$$G(d) = \begin{bmatrix} 1 + d - d^3 & -1 + d^3 \\ d + d^2 - d^3 & -1 - d^2 + d^3 \\ d + d^2 & -1 - d - d^2 \end{bmatrix}$$

Clearly

$$\Sigma^{1D} \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 0 & -1 \end{bmatrix} \right)$$

is a realization of $G(d)$ which is controllable and observable and therefore is minimal.

However $\Sigma^{1D}(A, B, C, D)$ is not a minimal realization of \mathcal{C} as not all the conditions of Theorem 1 are satisfied. It easy to see that condition (ii) is fulfilled for

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Considering $\Delta = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$ a minimal left annihilator of D , we have that

$$A - LC = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Lambda C = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix},$$

are such that the pair $(A - LC, \Delta C)$ is not observable.

In some particular cases, a minimal realization of an encoder $G(d)$ is also a minimal realization of the correspondent convolutional code \mathcal{C} . As mentioned before, these encoders are called *minimal*. For 1D convolutional codes, minimal encoders are completely characterized [4]. In particular, the right-prime⁴ and column reduced⁵ encoders (called *canonical encoders*) are minimal.

3) *Minimality of 2D realizations:* Returning to the 2D case, note that every 2D polynomial encoder $G(d_1, d_2) \in \mathbb{F}^{n \times k}[d_1, d_2]$ can be factorized as follows

$$G(d_1, d_2) = G_2(d_2)G_1(d_1), \quad (3)$$

where $G_2(d_2) \in \mathbb{F}^{n \times p}[d_2]$ and $G_1(d_1) \in \mathbb{F}^{p \times k}[d_1]$, for a suitable value of $p \in \mathbb{N}$.

Indeed, writing

$$\begin{aligned} G(d_1, d_2) &= G_2^{\ell_1}(d_2)d_1^{\ell_1} + \dots + G_2^1(d_2)d_1 + G_2^0(d_2) \\ &= \tilde{G}_2(d_2)D_1(d_1), \end{aligned}$$

where ℓ_1 is the highest exponent of d_1 appearing in $G(d_1, d_2)$, $\tilde{G}_2(d_2) = \begin{bmatrix} G_2^{\ell_1}(d_2) & \dots & G_2^0(d_2) \end{bmatrix}$ and

$$D_1(d_1) = \begin{bmatrix} I_k d_1^{\ell_1} \\ \vdots \\ I_k \end{bmatrix}, \quad \text{and decomposing}$$

⁴A polynomial matrix $G(d) \in \mathbb{F}^{n \times k}[d]$ is right-prime if for every factorization $G(d) = \tilde{G}(d)U(d)$, with $\tilde{G}(d) \in \mathbb{F}^{n \times k}[d]$ and $U(d) \in \mathbb{F}^{k \times k}[d]$, $U(d)$ is unimodular, i.e., it is invertible in $\mathbb{F}^{k \times k}[d]$.

⁵A polynomial matrix $G(d) \in \mathbb{F}^{n \times k}[d]$ is column reduced if the maximum degree of its full size minors is the sum of the column degrees of $G(d)$.

$$\begin{aligned}\tilde{G}_2(d_2) &= \tilde{G}_2^{\ell_2} d_2^{\ell_2} + \dots + \tilde{G}_2^1 d_2 + \tilde{G}_2^0 \\ &= D_2(d_2)N,\end{aligned}$$

where ℓ_2 is the highest exponent of d_2 appearing in

$$G(d_1, d_2), D_2(d_2) = [I_n d_2^{\ell_2} \quad \dots \quad I_n] \text{ and } N = \begin{bmatrix} \tilde{G}_2^{\ell_2} \\ \vdots \\ \tilde{G}_2^0 \end{bmatrix},$$

yields:

$$G(d_1, d_2) = D_2(d_2)ND_1(d_1).$$

Now any factorization $N = N_2N_1$ gives rise to decomposition

$$G(d_1, d_2) = G_2(d_2)G_1(d_1)$$

of the form (3) with $G_2(d_2) = D_2(d_2)N_2$ and $G_1(d_1) = N_1D_1(d_1)$.

Furthermore, as it is shown in [8], the encoder $G(d_1, d_2)$ can be realized by means of a separable model taking advantage of the factorization above.

However, contrary to what happens in the 1D case, it seems hard to obtain necessary and sufficient conditions for the minimality of realizations of a 2D convolutional code. In [8], sufficient conditions were established that guarantee the minimality of 2D realizations of a code. These sufficient conditions are given in the following result.

Theorem 2: [8] Let \mathcal{C} be a 2D convolutional code, and let $\Sigma^{2D}(A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D) = \Sigma^{2D}(\mathcal{C})$ be a realization of \mathcal{C} . Suppose that $\Sigma^{1D}\left(A_{11}, B_1, \begin{bmatrix} A_{21} \\ C_1 \end{bmatrix}, \begin{bmatrix} B_2 \\ D \end{bmatrix}\right)$ and $\Sigma^{1D}(A_{22}, [A_{21} \ B_2], C_2, [C_1 \ D])$ satisfy the conditions of Theorem 1, i.e., they are both minimal realizations of the corresponding output behaviors. Then $\Sigma^{2D}(\mathcal{C})$ is a minimal realization of \mathcal{C} .

III. COMPOSITION ENCODERS AND COMPOSITION CODES

In this section we consider a particular class of 2D convolutional codes generated by 2D polynomial encoders that are obtained from the composition of two 1D polynomial encoders. Such encoders/codes will be called *composition encoders/codes*. The formal definition of composition encoders is as follows.

Definition 6: An encoder $G(d_1, d_2) \in \mathbb{F}^{n \times k}[d_1, d_2]$ such that

$$G(d_1, d_2) = G_2(d_2)G_1(d_1),$$

where $G_1(d_1) \in \mathbb{F}^{p \times k}[d_1]$ and $G_2(d_2) \in \mathbb{F}^{n \times p}[d_2]$ are 1D encoders, is said to be a composition encoder.

Note that the requirement that $G_i(d_i)$, for $i = 1, 2$, is a 1D encoder is equivalent to the condition that $G_i(d_i)$ is a full column rank matrix. Moreover this requirement clearly implies that $G_2(d_2)G_1(d_1)$ has full column rank, hence the composition G_2G_1 of two 1D encoders is indeed a 2D encoder.

The 2D composition code \mathcal{C} associated with $G = G_2G_1$ is given as

$$\begin{aligned}\mathcal{C} &= \text{Im } G(d_1, d_2) = G_2(d_2)(\text{Im } (G_1(d_1))) \\ &= \{\hat{\mathbf{w}}(d_1, d_2) \in \mathcal{F}_{2D}^n : \exists \hat{\mathbf{z}}(d_1, d_2) \in \text{Im } (G_1(d_1)) \\ &\quad \text{such that } \hat{\mathbf{w}}(d_1, d_2) = G_2(d_2)\hat{\mathbf{z}}(d_1, d_2)\}.\end{aligned}$$

Next we restrict our study to 2D composition encoders that admit a special structure, namely, in which $G(d_1, d_2) = G_2(d_2)G_1(d_1)$, where $G_2(d_2)$ is a systematic encoder.

Definition 7: $G(d) \in \mathbb{F}^{n \times k}[d]$ is a systematic encoder if

$$G(d) = T \begin{bmatrix} \bar{G}(d) \\ I_k \end{bmatrix}, \quad (4)$$

where $T \in \mathbb{F}^{n \times n}$ is an invertible constant matrix and $\bar{G}(d) \in \mathbb{F}^{(n-k) \times k}[d]$.

Note that this definition is slightly different from the usual one (see for instance [4]) as T is any invertible matrix rather than a permutation matrix.

Systematic encoders are right-prime, but not necessarily column reduced, and hence they are not necessarily canonical. However as stated in the following proposition they are minimal encoders.

Proposition 1: Let $G(d) \in \mathbb{F}^{n \times k}[d]$ be a polynomial encoder. If $G(d)$ is systematic then every minimal realization of $G(d)$ is a minimal realization of $\mathcal{C} = \text{Im } G(d)$.

Example 2: Consider the polynomial encoder given by

$$G(d) = \begin{bmatrix} d & 1 & d & 0 \\ 0 & d^2 & 0 & d^2 \\ d+1 & 0 & d+1 & 0 \\ 0 & d^2+1 & 0 & d^2+1 \\ 1 & 1 & 0 & 0 \\ d & d^2 & d & d^2 \end{bmatrix}.$$

$G(d)$ is a systematic encoder since

$$G(d) = T \begin{bmatrix} \bar{G}(d) \\ I_4 \end{bmatrix},$$

$$\text{with } T = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ invertible and}$$

$$\bar{G}(d) = \begin{bmatrix} d & 0 & d & 0 \\ 0 & d^2 & 0 & d^2 \end{bmatrix}.$$

Since

$$\Sigma^{1D} \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

is a 1D minimal realization of $G(d)$, it is a minimal realization of the corresponding code as well. This can also be confirmed by checking the conditions of Theorem 1.

IV. MINIMAL REALIZATIONS OF COMPOSITION CODES

Let \mathcal{C} be a composition code generated by a composition encoder $G(d_1, d_2) \in \mathbb{F}^{n \times k}[d_1, d_2]$ such that

$$G(d_1, d_2) = G_2(d_2)G_1(d_1), \quad (5)$$

where $G_2(d_2) \in \mathbb{F}^{n \times p}[d_2]$, for some $p \in \mathbb{N}$, is a systematic encoder, and $G_1(d_1) \in \mathbb{F}^{p \times k}[d_1]$ is a minimal encoder. Note that the minimality assumption on $G_1(d_1)$ is not restrictive, as $G_1(d_1)$ can be taken to be right-prime and post-multiplying $G(d_1, d_2)$ by a suitable unimodular matrix $U(d_1)$ allows putting $G_1(d_1)$ in the column reduced form, without changing the corresponding code. Let $\Sigma^{1D}(A_{11}, B_1, \bar{C}_1, \bar{D}_1)$ and $\Sigma^{1D}(A_{22}, \bar{B}_2, C_2, \bar{D}_2)$ be minimal realizations of $G_1(d_1)$ and $G_2(d_2)$, respectively. Observe that, since $G_1(d_1)$ is a minimal encoder $\Sigma^{1D}(A_{11}, B_1, \bar{C}_1, \bar{D}_1)$ is a minimal realization of the 1D code $\mathcal{C}_1 = \text{Im } G_1(d_1)$. Moreover, by Proposition 1, because $G_2(d_2)$ is systematic, $\Sigma^{1D}(A_{22}, \bar{B}_2, C_2, \bar{D}_2)$ is a minimal realization of the 1D convolutional code $\mathcal{C}_2 = \text{Im } G_2(d_2)$.

Connecting in series $\Sigma^{1D}(A_{11}, B_1, \bar{C}_1, \bar{D}_1)$ and $\Sigma^{1D}(A_{22}, \bar{B}_2, C_2, \bar{D}_2)$ yields the following 2D realization of $G(d_1, d_2)$:

$$\begin{cases} \sigma_1 x_1 = A_{11}x_1 + B_1 u \\ \sigma_2 x_2 = A_{21}x_1 + A_{22}x_2 + B_2 u \\ w = C_1 x_1 + C_2 x_2 + Du \end{cases}, \quad (6)$$

where $A_{21} = \bar{B}_2 \bar{C}_1$, $B_2 = \bar{B}_2 \bar{D}_1$, $C_1 = \bar{D}_2 \bar{C}_1$ and $D = \bar{D}_2 \bar{D}_1$.

As we shall see, under the technical condition that $\begin{bmatrix} \bar{C}_1 & \bar{D}_1 \end{bmatrix}$ is invertible, the minimality of $\Sigma^{1D}(A_{11}, B_1, \bar{C}_1, \bar{D}_1)$ and $\Sigma^{1D}(A_{22}, \bar{B}_2, C_2, \bar{D}_2)$ as code realizations implies that $\Sigma^{1D}(A_{11}, B_1, E, F)$ and $\Sigma^{1D}(A_{22}, J, C_2, H)$, with

$$E = \begin{bmatrix} A_{21} \\ C_1 \end{bmatrix} = \begin{bmatrix} \bar{B}_2 \\ \bar{D}_2 \end{bmatrix} \bar{C}_1, \quad F = \begin{bmatrix} B_2 \\ D \end{bmatrix} = \begin{bmatrix} \bar{B}_2 \\ \bar{D}_2 \end{bmatrix} \bar{D}_1$$

and

$$\begin{aligned} J &= [A_{21} \quad B_2] = \bar{B}_2 [\bar{C}_1 \quad \bar{D}_1] \\ H &= [C_1 \quad D] = \bar{D}_2 [\bar{C}_1 \quad \bar{D}_1] \end{aligned}$$

are minimal code realizations that satisfy the conditions for minimality of Theorem 1. By Theorem 2, this in turn allows to conclude that the realization $\Sigma^{2D}(A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D)$ given by (6) is a minimal realization of the composition code \mathcal{C} , as stated in the following result.

Theorem 3: Let $G(d_1, d_2) \in \mathbb{F}^{n \times k}[d_1, d_2]$ be a composition encoder such that

$$G(d_1, d_2) = G_2(d_2)G_1(d_1),$$

where $G_2(d_2) \in \mathbb{F}^{n \times p}[d_2]$ is systematic and $G_1(d_1) \in \mathbb{F}^{p \times k}[d_1]$, for some $p \in \mathbb{N}$, is a minimal 1D encoder. Moreover, let $\Sigma^{1D}(A_{11}, B_1, \bar{C}_1, \bar{D}_1)$ and $\Sigma^{1D}(A_{22}, \bar{B}_2, C_2, \bar{D}_2)$

be two 1D minimal realization of $G_2(d_2)$ and $G_1(d_1)$, respectively, and assume that $\begin{bmatrix} \bar{C}_1 & \bar{D}_1 \end{bmatrix}$ is square and invertible. Then $\Sigma^{2D}(A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D)$, where $A_{21} = \bar{B}_2 \bar{C}_1$, $B_2 = \bar{B}_2 \bar{D}_1$, $C_1 = \bar{D}_2 \bar{C}_1$ and $D = \bar{D}_2 \bar{D}_1$ is a minimal realization of \mathcal{C} .

Proof: Let $\Sigma^{1D}(A_{11}, B_1, \bar{C}_1, \bar{D}_1)$ and $\Sigma^{1D}(A_{22}, \bar{B}_2, C_2, \bar{D}_2)$ be both 1D minimal realizations of $\text{Im } G_1(d_1)$ and $\text{Im } G_2(d_2)$, respectively. By Theorem 1 (and the remark thereafter) this means that:

Condition 1: \bar{D}_1 and \bar{D}_2 have full column rank.

Condition 2: (A_{11}, B_1) and (A_{22}, \bar{B}_2) are both controllable pairs.

Condition 3: $\text{Ker } \bar{D}_1 \subseteq \text{Ker } B_1$ and $\text{Ker } \bar{D}_2 \subseteq \text{Ker } \bar{B}_2$ (i.e., there exist matrices L_1 and L_2 such that $B_1 = L_1 \bar{D}_1$ and $\bar{B}_2 = L_2 \bar{D}_2$).

Condition 4: Let L_1 and L_2 be defined as in Condition 3, and let Λ_1 and Λ_2 be minimal left-annihilators (*m.l.a.*) of \bar{D}_1 and \bar{D}_2 , respectively. Then the pairs $(A_{11} - L_1 \bar{C}_1, \Lambda_1 \bar{C}_1)$ and $(A_{22} - L_2 C_2, \Lambda_2 C_2)$ are both observable.

Firstly we show that the conditions of Theorem 1 for the minimality of $\Sigma^{1D}(A_{11}, B_1, E, F)$ as a code realization are satisfied. For this purpose we prove that:

(i) F has full column rank

Since Condition 1 and Condition 3 hold,

$$F = \begin{bmatrix} \bar{B}_2 \\ \bar{D}_2 \end{bmatrix} \bar{D}_1 = \begin{bmatrix} L_2 \\ I \end{bmatrix} \bar{D}_2 \bar{D}_1$$

has full column rank as its factors \bar{D}_1 , \bar{D}_2 and $\begin{bmatrix} L_2 \\ I \end{bmatrix}$ have full column rank.

(ii) (A_{11}, B_1) is controllable

This condition trivially holds due to Condition 2, i.e., (A_{11}, B_1) is a controllable pair.

(iii) There exists a matrix \bar{L}_1 such that $B_1 = \bar{L}_1 F$

Taking into account that

$$F = \begin{bmatrix} B_2 \\ D \end{bmatrix}, \quad D = \bar{D}_2 \bar{D}_1 \quad \text{and} \quad B_2 = \bar{B}_2 \bar{D}_1, \quad (7)$$

the claim to be shown is equivalent to the existence of a matrix \bar{L}_1 such that

$$B_1 = \bar{L}_1 \begin{bmatrix} \bar{B}_2 \bar{D}_1 \\ \bar{D}_2 \bar{D}_1 \end{bmatrix} = \bar{L}_1 \begin{bmatrix} \bar{B}_2 \\ \bar{D}_2 \end{bmatrix} \bar{D}_1. \quad (8)$$

Since $\bar{B}_2 = L_2 \bar{D}_2$ and \bar{D}_2 has full column rank, $\begin{bmatrix} L_2 \\ I \end{bmatrix} \bar{D}_2$ has full column rank, then there exists a left inverse, U , such that

$$U \begin{bmatrix} L_2 \\ I \end{bmatrix} \bar{D}_2 = I. \quad (9)$$

On the other hand, there exists L_1 such that $B_1 = L_1 \bar{D}_1$. Therefore, from (7), (8) and (9) we obtain that

$$B_1 = \bar{L}_1 F, \quad (10)$$

where $\bar{L}_1 = L_1U$.

- (iv) $(A_{11} - \bar{L}_1E, \bar{\Lambda}_1E)$ is observable,
with \bar{L}_1 s.t. $B_1 = \bar{L}_1F$ and $\bar{\Lambda}_1$ is a *mla* of F

To prove this, consider $\bar{L}_1 = L_1U$, as defined above.
Moreover note that

$$\begin{aligned} \Lambda_1UF &= \Lambda_1U \begin{bmatrix} \bar{B}_2 \\ \bar{D}_2 \end{bmatrix} \bar{D}_1 \\ &= \Lambda_1U \begin{bmatrix} L_2 \\ I \end{bmatrix} \bar{D}_2\bar{D}_1 \\ &= \Lambda_1\bar{D}_1 \\ &= 0 \end{aligned}$$

due to (9) and to the fact that Λ_1 is, by definition, a *mla* of \bar{D}_1 .

This implies that a *mla* of F can be obtained by (if necessary) adding extra rows to Λ_1U .

Let then $\bar{\Lambda}_1 = \begin{bmatrix} \Lambda_1U \\ T \end{bmatrix}$, for a suitable matrix T , be a *mla* of F . Now, the pair $(A_{11} - \bar{L}_1E, \bar{\Lambda}_1E)$ is given by

$$\left(A_{11} - \bar{L}_1 \begin{bmatrix} L_2 \\ I \end{bmatrix} \bar{D}_2\bar{C}_1, \bar{\Lambda}_1 \begin{bmatrix} L_2 \\ I \end{bmatrix} \bar{D}_2\bar{C}_1 \right),$$

which is equal to

$$\left(A_{11} - L_1U \begin{bmatrix} L_2 \\ I \end{bmatrix} \bar{D}_2\bar{C}_1, \begin{bmatrix} \Lambda_1U \\ T \end{bmatrix} \begin{bmatrix} L_2 \\ I \end{bmatrix} \bar{D}_2\bar{C}_1 \right),$$

or equivalently,

$$\left(A_{11} - L_1\bar{C}_1, \begin{bmatrix} \Lambda_1\bar{C}_1 \\ M \end{bmatrix} \right),$$

where $M = T \begin{bmatrix} L_2 \\ I \end{bmatrix} \bar{D}_2\bar{C}_1$.

Since, by Condition 4, the pair $(A_{11} - L_1\bar{C}_1, \Lambda_1\bar{C}_1)$ is observable, then the pair

$$\left(A_{11} - L_1\bar{C}_1, \begin{bmatrix} \Lambda_1\bar{C}_1 \\ M \end{bmatrix} \right)$$

is also observable. In this way we conclude that $(A_{11} - \bar{L}_1E, \bar{\Lambda}_1E)$ is observable, as desired.

Therefore all the conditions of Theorem 1 are satisfied and $\Sigma^{1D}(A_{11}, B_1, E, F)$ is minimal as a code realization.

Finally, note that $\Sigma^{1D}(A_{22}, J, C_2, H)$ is given by

$$\Sigma^{1D}(A_{22}, \bar{B}_2 [\bar{C}_1 \ \bar{D}_1], C_2, \bar{D}_2 [\bar{C}_1 \ \bar{D}_1])$$

which corresponds to making an invertible input transformation, associated to $[\bar{C}_1 \ \bar{D}_1]$, in

$\Sigma^{1D}(A_{22}, \bar{B}_2, C_2, \bar{D}_2)$. Thus it is clear that the former model realizes the same code as the latter, with the same dimension.

So $\Sigma^{1D}(A_{22}, J, C_2, H)$ is a minimal code realization. ■

Example 3: Consider the following composition encoder

$$G(d_1, d_2) = \begin{bmatrix} d_2 + d_1d_2 & 1 \\ 0 & d_2^2 + d_1d_2^2 \\ d_2 + d_1d_2 + d_1 + 1 & 0 \\ 0 & d_2^2 + d_1d_2^2 + d_1 + 1 \\ 1 & 1 \\ d_2 + d_1d_2 & d_2^2 + d_1d_2^2 \end{bmatrix}.$$

It is easy to factorize $G(d_1, d_2)$ as in (5) where

$$G_2(d_2) = \begin{bmatrix} d_2 & 1 & d_2 & 0 \\ 0 & d_2^2 & 0 & d_2^2 \\ d_2 + 1 & 0 & d_2 + 1 & 0 \\ 0 & d_2^2 + 1 & 0 & d_2^2 + 1 \\ 1 & 1 & 0 & 0 \\ d_2 & d_2^2 & d_2 & d_2^2 \end{bmatrix}$$

and

$$G_1(d_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ d_1 & 0 \\ 0 & d_1 \end{bmatrix},$$

which is canonical and therefore minimal. $G_2(d_2)$ is a systematic encoder since

$$G_2(d_2) = T \begin{bmatrix} \bar{G}_2(d_2) \\ I_4 \end{bmatrix},$$

with

$$T = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

invertible and

$$\bar{G}_2(d_2) = \begin{bmatrix} d_2 & 0 & d_2 & 0 \\ 0 & d_2^2 & 0 & d_2^2 \end{bmatrix}.$$

Moreover $\Sigma^{1D} = (A_{11}, B_1, \bar{C}_1, \bar{D}_1)$, where

$$A_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \bar{C}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \bar{D}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and $\Sigma^{1D} = (A_{22}, \bar{B}_2, C_2, \bar{D}_2)$, where

$$A_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \bar{B}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \bar{D}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

are both 1D minimal realizations of $G_1(d_1)$ and $G_2(d_2)$, respectively, and $[\bar{C}_1 \ \bar{D}_1] = I_4$ is invertible. Thus, by Theorem 3,

$$\Sigma_{2D} = (A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D),$$

where

$$A_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

is a minimal realization of the 2D convolutional code generated by $G(d_1, d_2)$.

V. CONCLUSION

In this paper we have analyzed the minimality of realizations for a special class of 2D composition codes, namely for codes that admit encoders which can be factorized as the product of a systematic 1D encoder and a minimal 1D encoder. The series connection of minimal realizations of those 1D encoders yields a minimal realization of the 2D convolutional code.

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