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A Design Methodology for Robust Distributed Controllers for Networked Systems

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Abstract— **This paper presents a novel algorithm for the synthesis of robust distributed controllers for interconnected linear discrete time systems. For a network of interconnected and uncertain systems, the distributed controller achieves robust stability and a guaranteed level of robust performance in a well defined $\mathcal{H}_\infty$ sense. The theory is developed for networks of linear discrete time systems where model uncertainty is described in the setting of linear fractional representations. A computationally tractable synthesis algorithm is proposed that employs the use of linear matrix inequalities in an iterative D-K type of algorithm. Convergence properties of the algorithm are inferred.**

I. INTRODUCTION

The need for novel controller synthesis algorithms for networked systems is well motivated by the consideration that in many applications a centralized communication with network components is not desirable or practically infeasible. Indeed, the logistic configuration of a network may prevent from a centralized communication between one controller and all subsystems in the network. In addition, a model based synthesis of a centralized controller may become computationally infeasible due to the complexity or the geographical distribution of the network. Decentralized control designs may remedy this problem. However, decentralized control architectures typically ignore the dynamical interactions among subsystems through communication channels in the network and therefore fall short in guaranteeing stability and robustness of the network.

A distributed controller consists of multiple controller modules, each of which influences the behavior of individual network components while, in addition, a well defined information and communication protocol among the modules allows to accomplish a global property or task of the network. This paper addresses the problem of distributed controller synthesis for a network of interconnected linear discrete time systems with uncertainty. We propose a computationally tractable algorithm for the synthesis of distributed controllers for such a network that achieves both robust stability and robust performance in a well defined $\mathcal{H}_\infty$ sense. The main results involve solving Linear Matrix Inequalities (LMI's) to yield explicit distributed controllers. As a specific motivation of this work, we emphasize that networks of interconnected discrete time systems naturally allow to analyze and incorporate delays or lags in communication channels. It is well known that communication delays may have a substantial impact on network stability. We will illustrate this usage in the example section.

This work continuous a line of research that was initiated in [1], [2] and [3]. The theory is largely based on a generalized notion of dissipation that was initiated by [4] and further employed for the synthesis of continuous time distributed controllers in [5] and for robust distributed control in [6]. In [7] robustness against communication delays was studied, albeit with reasonably small communication delays. Other studies on delays in interconnected systems can be found in [8]–[10]. Specific synthesis results on discrete time systems have been reported in [11] and with important generalizations to random communication packet losses in [12]. Recent work on gain scheduling for distributed control has been reported in [13].

This paper is organized as follows. In Section II, the properties and structure of a distributed system and controller are explained in detail. Problem formulations are presented in Section III. Section IV presents analysis results in the form of sufficient conditions for robust stability and performance of a distributed system. Section V presents numerical procedures and algorithms to synthesize robust distributed controllers. Simulations are presented and discussed in Section VI. Conclusions are deferred to Section VII.

A. Notation

We use largely standard notation. $\mathbb{R}$ and $\mathbb{N}$ denote the sets of reals and positive integers, respectively. The set of non-negative integers is denoted by $\mathbb{N}_0$. The set of $n \times n$ real symmetric matrices is denoted by $\mathbb{R}^n_S$. The notation $\mathbb{N}_{\geq c_1}$ and $\mathbb{N}_{(c_1,c_2]}$ is used to denote the sets $\{k \in \mathbb{N}_0 \mid k \geq c_1\}$ and $\{k \in \mathbb{N}_0 \mid c_1 < k \leq c_2\}$, respectively. The cardinality of a finite set $\mathcal{V}$ is denoted by $card(\mathcal{V})$. The notation $\mathbf{diag}_{i\in\mathbb{N}_{[k,\ell]}} A_i$ is used to denote the block diagonal matrix that has matrices $A_k, \ldots, A_\ell$ on its block diagonal. The Hölder p-norm of a vector x is denoted by $||x||_p$ for $p \in \mathbb{N}_{[1,\infty]}$. The inertia of a matrix $M \in \mathbb{R}^n_S$ is denoted by $in(M) : \mathbb{R}^n_S \to \mathbb{N}^3_0$ and is defined as the triplet $in(M) := (a_-, a_0, a_+)$ of negative, zero and positive eigenvalues of M, respectively. Finally, the space of n-dimensional vector-valued and square summable sequences ℓ^n_2 consist of all sequences $x : \mathbb{N}_0 \to \mathbb{R}^n$ such that $\|x\|^2_2 := \sum_{i=0}^{\infty} ||x(i)||^2_2 < \infty$.

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II. DISCRETE TIME DISTRIBUTED SYSTEMS

Following the approach in [2], [5], [6] we will view a distributed system as a graph in which L arbitrary dynamical systems are interconnected. Here, we focus on interconnections of discrete time, linear time-invariant and uncertain dynamical systems. In this section, the structure and properties of such a distributed system are discussed.

A. System representation

The structure of an arbitrary distributed system G is represented by a graph $\mathcal{G}_G := (\mathcal{V}_G, \mathcal{E}_G)$, where $\mathcal{V}_G$ is the set of vertices, and $\mathcal{E}_G \subseteq \mathcal{V}_G \times \mathcal{V}_G$ is the set of edges between the vertices [14]. For a distributed system that consists of L subsystems, the vertices $\mathcal{V}_G = \{G^1, \ldots, G^L\}$ consist of L transfer functions and the non-oriented edges $\mathcal{E}_G$ consist of pairs (G^i, G^j) where $(G^i, G^j) \in \mathcal{E}_G$ if and only if the systems G^i and G^j are connected. The set of indices of neighbors of the ith subsystem is $\mathcal{N}^i := \{j \in \mathbb{N} \mid (G^i, G^j) \in \mathcal{E}_G\}$. Note that a subsystem can be self-connected.

For *uncertain* distributed systems G_Δ, the graph $\mathcal{G}_{G_\Delta} = (\mathcal{V}_{G_\Delta}, \mathcal{E}_{G_\Delta})$ has vertices $\{G^1_\Delta, \ldots, G^L_\Delta\}$ that represent uncertain dynamical systems (referred to as *subsystems*) where the uncertainty admits a well defined linear-fractional representation, as defined in the next subsection.

The ith subsystem in G_Δ consists of an interconnection of a nominal multi-channel LTI system $G^i_0 : \mathbb{R}^{n^i_G + n^i_\Delta + n^i_d + n^i_u} \rightarrow \mathbb{R}^{n^i_G + n^i_\Delta + n^i_z + n^i_y}$ and an operator $\Delta^i : \mathbb{R}^{n^i_\Delta} \rightarrow \mathbb{R}^{n^i_\Delta}$, that is assumed to belong to a class $\boldsymbol{\Delta}^i$ of LTI causal operators with bounded gain, representing the uncertainty. Specifically, for all $i = 1, \ldots, L$, the dynamics of the uncertain discrete time LTI subsystem G^i_Δ is represented by

$$p^i(k) = \Delta^i(q^i(k)) \text{ for some } \Delta^i \in \boldsymbol{\Delta}^i \tag{1}$$

$$\begin{bmatrix} x^i(k+1) \\ w^i(k) \\ q^i(k) \\ z^i(k) \\ y^i(k) \end{bmatrix} = \begin{bmatrix} A^i_{xx} & A^i_{xv} & B^i_{xp} & B^i_{xd} & B^i_{xu} \\ A^i_{wx} & A^i_{wv} & B^i_{wp} & B^i_{wd} & B^i_{wu} \\ C^i_{qx} & C^i_{qv} & D^i_{qp} & D^i_{qd} & D^i_{qu} \\ C^i_{zx} & C^i_{zv} & D^i_{zp} & D^i_{zd} & D^i_{zu} \\ C^i_{yx} & C^i_{yv} & D^i_{yp} & D^i_{yd} & D^i_{yu} \end{bmatrix} \begin{bmatrix} x^i(k) \\ v^i(k) \\ p^i(k) \\ d^i(k) \\ u^i(k) \end{bmatrix} \tag{2}$$

in which we distinguish a

$$\begin{array}{llll} \textit{performance channel} & (d^i, z^i) & \in & \mathbb{R}^{n^i_d + n^i_z}, \\ \textit{interconnection channel} & (v^i, w^i) & \in & \mathbb{R}^{2n^i_G}, \\ \textit{uncertainty channel} & (p^i, q^i) & \in & \mathbb{R}^{2n^i_\Delta}, \\ \textit{control channel} & (u^i, y^i) & \in & \mathbb{R}^{n^i_u + n^i_y}. \end{array}$$

Here, $x^i(k) \in \mathbb{R}^{m^i_G}$ with $m^i_G = m^i \in \mathbb{N}$ is the state variable. The nominal state space representation (2) is denoted by G^i_0 and we remark that G^i_Δ is a linear fractional representation of the nominal system G^i_0 with Δ^i. The sets $\boldsymbol{\Delta}^i$ are assumed to be known.

If G^i_Δ is interconnected with G^j_Δ, the interconnection channel (v^i, w^i) of subsystem i is further partitioned such that $(v^{ij}, w^{ij}) \in \mathbb{R}^{2n^{ij}_G}$ denotes the interconnection channel between subsystem i and j. To simplify the analysis, the matrix A^i_{wv} is assumed to be square by constraining v^{ij} and w^{ij} to share the same dimension, say n^{ij}_G. This can always be achieved without loss of generality. When the control channels (u^i, y^i), $i = 1, \ldots, L$, are not taken into account, we refer to G_Δ as the *open loop system*.

B. Dissipation

For the analysis of stability and performance of an uncertain distributed system, we make use of concepts from the theory of dissipative systems initiated by [4], [15].

Definition 1 (Discrete time dissipativity): A nominal discrete time subsystem G^i_0 as in (2) is dissipative with respect to a supply function $\Phi^i(\cdot)$, depending on all input and output signals of the system, if there exists a storage function $V^i : \mathbb{R}^{m^i_G} \rightarrow \mathbb{R}$, depending on the state of the system, such that, for all integers $M \geq 0$,

$$V^i(x^i(M)) - V^i(x^i(0)) \leq \sum_{k=0}^{M-1} \Phi^i(k) \tag{3}$$

holds for all possible system trajectories generated by (1)-(2). Here, $\Phi^i(k)$ represents the supply delivered to the system at time k, viewed as function of the signal variables at time instance k. The system is said to be strictly dissipative if the inequality in (3) is strict.

For the nominal system G^i_0 we consider its aggregated supply function $\Phi^i(k)$ to be defined as

$$\Phi^i(k) := S^i(k) + P^i(k) + U^i(k) \tag{4}$$

where

$$S^i(d^i, z^i) := \gamma||d^i||^2 - \frac{1}{\gamma}||z^i||^2, \tag{5}$$

$$U^i(q^i, p^i) := -\begin{bmatrix} q^i \\ p^i \end{bmatrix}^T D^i_\Delta \begin{bmatrix} q^i \\ p^i \end{bmatrix}, \tag{6}$$

$$P^i(v^i, w^i) := \sum_{j \in \mathcal{N}^i} P^{ij}(v^{ij}, w^{ij}), \tag{7}$$

$$\text{with } P^{ij}(v^{ij}, w^{ij}) := \begin{bmatrix} w^{ij} \\ v^{ij} \end{bmatrix}^T X^{ij} \begin{bmatrix} w^{ij} \\ v^{ij} \end{bmatrix},$$

with $\gamma > 0$, $X^{ij} \in \mathbb{R}^{n^{ij}_G}_S$ and $D^i_\Delta \in \mathbb{R}^{n^i_\Delta}_S$. These define quadratic supply functions on the performance, interconnection and uncertainty channels, respectively. The supply functions represent the intuitive idea on how the nominal system exchanges power with its environment through its different channels. Here, for the time being, the control channel is disregarded. In (4) all variables have dimensions that match the dimensions of the time-varying signals with the same name in (2).

Throughout, we suppose that the matrix $D^i_\Delta \in \mathbb{R}^{n^i_\Delta}_S$, used in (6), is partitioned accordingly with the uncertainty channel as

$$D^i_\Delta = \begin{bmatrix} D^i_{11} & D^i_{12} \\ (D^i_{12})^T & D^i_{22} \end{bmatrix}. \tag{8}$$

Generally, we will assume well-posedness of the networked system. A sufficient condition for well-posedness of (1)-(2)

is that

$$C^i_{yv} = 0, \quad D^i_{yp} = 0, \quad D^i_{yu} = 0, \quad B^i_{wp} = 0. \tag{9}$$

These conditions are rather mild and can be enforced, for example, by placing low-pass filters on the appropriate channels [2], [6].

We consider quadratic storage functions of the form

$$V^i(x^i) := (x^i)^T X^i_T x^i \tag{10}$$

with $X^i_T \in \mathbb{R}^{m^i}_S$. The interconnection channels (v, w) satisfy

$$\begin{bmatrix} w^{ij}(k) \\ v^{ij}(k) \end{bmatrix} = \begin{bmatrix} v^{ji}(k) \\ w^{ji}(k) \end{bmatrix} \quad \forall\, i \geq j,\ \forall k \geq 0. \tag{11}$$

If G^i_Δ is dissipative with respect to supply function P^{ij} and its neighboring subsystem G^j_Δ is dissipative with respect to supply function P^{ji}, then their interconnection satisfies the algebraic constraint (11) together with the neutrality condition

$$P^{ij}(v^{ij}, w^{ij}) + P^{ji}(v^{ji}, w^{ji}) = 0. \tag{12}$$

The latter reflects that no power is lost nor injected in the interconnection channel [15]. Using (7) and (11) it is easily shown that the neutrality condition (12) is equivalent to

$$X^{ij} = -\begin{bmatrix} 0 & I_{n^{ij}_G} \\ I_{n^{ij}_G} & 0 \end{bmatrix} X^{ji} \begin{bmatrix} 0 & I_{n^{ij}_G} \\ I_{n^{ij}_G} & 0 \end{bmatrix} \tag{13}$$

which we require for all i and j. If X^{ij} is partitioned accordingly with the signals (v^{ij}, w^{ij}) into four $n^{ij}_G \times n^{ij}_G$ blocks as in

$$X^{ij} := \begin{bmatrix} X^{ij}_{11} & X^{ij}_{12} \\ \left(X^{ij}_{12}\right)^T & X^{ij}_{22} \end{bmatrix}, \quad \text{for all } i, j = 1, \ldots, L \tag{14}$$

then (13) implies that

$$X^{ij}_{11} = \left(X^{ij}_{11}\right)^T = -X^{ji}_{22}, \quad \left(X^{ij}_{12}\right)^T = -X^{ji}_{12}. \tag{15}$$

Hence, the set $\{X^{ij} \in \mathbb{R}^{2n^{ij}_G}_S \mid (13) \text{ holds for all } i, j = 1, \ldots, L\}$ is parametrized by the two sets

$$\begin{aligned} &\{X^{ij}_{11} \in \mathbb{R}^{n^{ij}_G}_S \mid i, j = 1, \ldots, L\}, \\ &\{X^{ij}_{12} \in \mathbb{R}^{n^{ij}_G \times n^{ij}_G} \mid 1 \leq j \leq i \leq L\}. \end{aligned} \tag{16}$$

C. Controller representation

A distributed controller K for a distributed system G is represented by a graph $\mathcal{G}_K = (\mathcal{V}_K, \mathcal{E}_K)$, where we identify the vertices $\mathcal{V}_K$ with a set of local controllers $\{K^1, \ldots, K^L\}$ and the edges $\mathcal{E}_K$ with the interconnections between the controllers. More specifically, we assume $card(\mathcal{V}_K) = card(\mathcal{V}_G)$ and only allow connections between controllers if their respective plants are also connected. This is formalized by requiring that edges $(K^i, K^j) \in \mathcal{E}_K$ if and only if edges $(G^i_\Delta, G^j_\Delta) \in \mathcal{E}_{G_\Delta}$. For $i = 1, \ldots, L$, a local controller $K^i : \mathbb{R}^{n^i_K + n^i_y} \to \mathbb{R}^{n^i_K + n^i_u}$ is an LTI system that is represented in state space form by

$$\begin{bmatrix} x^i_K(k+1) \\ w^i_K(k) \\ u^i(k) \end{bmatrix} = \begin{bmatrix} \left(A^i_{xx}\right)_K & \left(A^i_{xv}\right)_K & \left(B^i_{xy}\right)_K \\ \left(A^i_{wx}\right)_K & \left(A^i_{wv}\right)_K & \left(B^i_{wy}\right)_K \\ \left(C^i_{ux}\right)_K & \left(C^i_{uv}\right)_K & \left(D^i_{uy}\right)_K \end{bmatrix} \begin{bmatrix} x^i_K(k) \\ v^i_K(k) \\ y^i(k) \end{bmatrix} \tag{17}$$

Here, $x^i_K \in \mathbb{R}^{m^i_K}$ with $m^i_K = m^i_G = m^i$ denotes the state variable, $(v^i_K, w^i_K) \in \mathbb{R}^{2n^i_K}$ is the *controller interconnection channel*, and $(u^i, y^i) \in \mathbb{R}^{n^i_u + n^i_y}$ is the *control channel* of the ith controller. The controller interconnection channel is further partitioned into $(v^{ij}_K, w^{ij}_K) \in n^{ij}_K$ for any pair (i, j) for which $(K^i, K^j) \in \mathcal{E}_K$. This is equivalent to the partitioned interconnection channels in the subsystems of G_Δ.

The interconnection of the uncertain distributed system G_Δ and the distributed controller K defines the *controlled system* or *closed loop system* $(G_\Delta)_C : d := col(d^1, \ldots, d^L) \to z := col(z^1, \ldots, z^L)$ with graph $\mathcal{G}_{(G_\Delta)_C} = (\mathcal{V}_{(G_\Delta)_C}, \mathcal{E}_{(G_\Delta)_C})$. The vertices $\mathcal{V}_{(G_\Delta)_C}$ correspond to the set $\{\left(G^i_\Delta\right)_C := S(G^i_\Delta, K^i), i = 1, \ldots, L\}$, where $S(\cdot, \cdot)$ denotes the Redheffer star product [16]. It follows that edges $((G^i_\Delta)_C, (G^j_\Delta)_C) \in \mathcal{E}_{(G_\Delta)_C}$ if and only if $(G^i_\Delta, G^j_\Delta) \in \mathcal{E}_{G_\Delta}$ if and only if $(K^i, K^j) \in \mathcal{E}_K$.

Given a distributed controller, a local interconnection of a subsystem and its corresponding controller $\left(G^i_\Delta\right)_C$ at node i, admits the representation

$$p^i(k) = \Delta^i(q^i(k)) \text{ for some } \Delta^i \in \boldsymbol{\Delta}^i \tag{18}$$

$$\begin{bmatrix} x^i_C(k+1) \\ w^i_C(k) \\ q^i(k) \\ z^i(k) \end{bmatrix} = \left(G^i_0\right)_C \begin{bmatrix} x^i_C(k) \\ v^i_C(k) \\ p^i(k) \\ d^i(k) \end{bmatrix} \tag{19}$$

with

$$\left(G^i_0\right)_C = \begin{bmatrix} (A^i_{xx})_C & (A^i_{xv})_C & (B^i_{xp})_C & (B^i_{xd})_C \\ (A^i_{wx})_C & (A^i_{wv})_C & (B^i_{wp})_C & (B^i_{wd})_C \\ (C^i_{qx})_C & (C^i_{qv})_C & (D^i_{qp})_C & (D^i_{qd})_C \\ (C^i_{zx})_C & (C^i_{zv})_C & (D^i_{zp})_C & (D^i_{zd})_C \end{bmatrix}$$

where the closed loop states and interconnection signals are now the stacked states and interconnection signals of the ith subsystem and ith controller. Hence, $x^i_C = \text{col}(x^i, x^i_K) \in \mathbb{R}^{2m^i}$, $v^i_C = \text{col}(v^i, v^i_K) \in \mathbb{R}^{n^i_C}$ and $w^i_C = \text{col}(w^i, w^i_K) \in \mathbb{R}^{n^i_C}$, $n^i_C = n^i_G + n^i_K$.

Throughout, we will assume that the interconnections are well-posed. We will exploit the property that the nominal controlled closed loop system $\left(G^i_0\right)_C$ is an affine function of the controller parameters Θ^i in the sense that

$$\left(G^i_0\right)_C = R^i + \left(V^i\right)^T \Theta^i U^i \tag{20}$$

where

$$R^i := \begin{bmatrix} A^i_{xx} & 0 & A^i_{xv} & 0 & B^i_{xp} & B^i_{xd} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ A^i_{wx} & 0 & A^i_{wv} & 0 & B^i_{wp} & B^i_{wd} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ C^i_{qx} & 0 & C^i_{qv} & 0 & D^i_{qp} & D^i_{qd} \\ C^i_{zx} & 0 & C^i_{zv} & 0 & D^i_{zp} & D^i_{zd} \end{bmatrix}, \tag{21}$$

$$U^i := \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ C^i_{yx} & 0 & C^i_{yv} & 0 & D^i_{yp} & D^i_{yd} \end{bmatrix}, \tag{22}$$

$$\Theta^i := \begin{bmatrix} \left(A^i_{xx}\right)_K & \left(A^i_{xv}\right)_K & \left(B^i_{xy}\right)_K \\ \left(A^i_{wx}\right)_K & \left(A^i_{wv}\right)_K & \left(B^i_{wy}\right)_K \\ \left(C^i_{ux}\right)_K & \left(C^i_{uv}\right)_K & \left(D^i_{uy}\right)_K \end{bmatrix}, \tag{23}$$

$$\left(V^i\right)^T := \begin{bmatrix} 0 & 0 & B^i_{xu} \\ I & 0 & 0 \\ 0 & 0 & B^i_{wu} \\ 0 & I & 0 \\ 0 & 0 & D^i_{qu} \\ 0 & 0 & D^i_{zu} \end{bmatrix}. \tag{24}$$

D. Stability and performance

Robust stability [17] is defined as follows.

Definition 2 (Robust stability): The (controlled) system with uncertainties taken from $\mathbf{\Delta}$, satisfying (1)-(2) or (18)-(19), is called *robustly stable* if, for all $\Delta \in \mathbf{\Delta}$ and for all initial conditions and for any (disturbance) input in the class ℓ_2, the state and all outputs belong to ℓ_2.

For a robustly stable system, the robust performance for the uncertain system $G_\Delta : d \in \ell_2^{n_d} \to z \in \ell_2^{n_z}$, defined by (1)-(2) or (18)-(19), is defined as follows.

Definition 3 (Robust performance): The uncertain (controlled) system G_Δ achieves *robust performance of level* γ if it is robustly stable and, with initial state set to zero, satisfies

$$||G_\Delta||_{2,2} := \sup_{0 \neq d \in \ell_2^{n_d}, \Delta \in \mathbf{\Delta}} \frac{||z||_2}{||d||_2} < \gamma.. \tag{25}$$

Robust performance therefore yields a guaranteed bound on the worst case network gain, measured in ℓ_2 sense, from (disturbance) input d to (performance) output z in view of all possible uncertainties that effect the system. Also note that for a stable LTI system G it holds that $||G||_{2,2} = \sup_\omega \sigma_{\max}(G(i\omega)) =: ||G||_\infty$ and therefore we achieve performance in $\mathcal{H}_\infty$ sense.

III. PROBLEM FORMULATIONS

With the previous definitions we are now in a position to formulate the problems that we address in this paper.

Problem 1 (Open loop analysis): Given an uncertain distributed system G_Δ defined as in Section II, find computationally verifiable conditions to test whether the open loop uncertain distributed system G_Δ is robustly stable and achieves robust performance of level $\gamma > 0$.

Problem 2 (Synthesis): Given an uncertain distributed system G_Δ defined as in Section II, find a computationally tractable synthesis procedure that generates a distributed controller K, with structure as given in Section II-C, that renders the controlled uncertain distributed system $(G_\Delta)_C$, as defined in Section II-C, well-posed, robustly stable while achieving robust performance of level γ.

The solution of Problem 1 is relevant to allow for a proper analytical verification of robustness properties of a network G_Δ that consists of uncertain dynamical systems. Problem 2 will be solved by proposing an explicit algorithm for controller synthesis.

IV. ANALYSIS

A. Open loop analysis

We first present an open loop analysis on the properties of robust stability and robust performance. For this, we consider the case where no control is applied, i.e., the input channel $u^i = 0$ and the output y^i is ignored for all subsystems.

Theorem 1: Let G_Δ be an uncertain distributed system with subsystems admitting realization (1)-(2) where $\Delta^i \in \mathbf{\Delta}^i$ are LTI causal operators in the sense that $p^i = \Delta^i q^i$. Then, the system is well-posed, stable and achieves robust performance γ if $\forall i \in \mathbb{N}_{[1,L]}$ there exist matrices $X^i_T \succ 0$, $D^i_\Delta \in \mathbb{R}_S^{2n^i_\Delta}$ and $X^{ij}_{11} \in \mathbb{R}_S^{n^{ij}_G}$ for all $i, j = 1 \dots L$, and matrices $X^{ij}_{12} \in \mathbb{R}^{n^{ij}_G \times n^{ij}_G}$ for all $i \geq j$ such that $D^i_{11} \succ 0$, $D^i_{22} \prec 0$ and $\forall \Delta^i \in \mathbf{\Delta}^i$

$$(T^i)^T M^i T^i \prec 0, \tag{26}$$

$$\begin{bmatrix} I \\ \Delta^i \end{bmatrix}^T \begin{bmatrix} D^i_{11} & D^i_{12} \\ (D^i_{12})^T & D^i_{22} \end{bmatrix} \begin{bmatrix} I \\ \Delta^i \end{bmatrix} \succeq 0, \tag{27}$$

with

$$T^i := \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ A^i_{xx} & A^i_{xv} & B^i_{xp} & B^i_{xd} \\ \hline A^i_{wx} & A^i_{wv} & B^i_{wp} & B^i_{wp} \\ 0 & I & 0 & 0 \\ \hline C^i_{qx} & C^i_{qv} & D^i_{qp} & D^i_{qd} \\ 0 & 0 & I & 0 \\ \hline C^i_{zx} & C^i_{zv} & D^i_{zp} & D^i_{zd} \\ 0 & 0 & 0 & I \end{array}\right], \tag{28}$$

$$\begin{aligned} M^i := \mathbf{diag}\Bigg(& \begin{bmatrix} -X^i_T & 0 \\ 0 & X^i_T \end{bmatrix}, \begin{bmatrix} Z^i_{11} & Z^i_{12} \\ \left(Z^i_{12}\right)^T & Z^i_{22} \end{bmatrix}, \\ & \begin{bmatrix} D^i_{11} & D^i_{12} \\ \left(D^i_{12}\right)^T & D^i_{22} \end{bmatrix}, \begin{bmatrix} \frac{1}{\gamma} I & 0 \\ 0 & -\gamma I \end{bmatrix} \Bigg), \end{aligned} \tag{29}$$

$$Z^i_{11} := -\mathbf{diag}_{j \in \mathbb{N}_{[1,L]}} X^{ij}_{11}, \quad Z^i_{22} := \mathbf{diag}_{j \in \mathbb{N}_{[1,L]}} X^{ji}_{11}, \tag{30}$$

$$Z^i_{12} := \mathbf{diag}\left(-\mathbf{diag}_{j \in \mathbb{N}_{[1,i]}} X^{ij}_{12}, \mathbf{diag}_{j \in \mathbb{N}_{(i,L]}} (X^{ji}_{12})^T \right). \tag{31}$$

Proof: (In brief). Note that (26) has a block diagonal structure where the blocks relate to the state, interconnection, uncertainty and performance channels, respectively. Let $V^i(x^i)$ be as in (10). Define $V(x) := \sum_{i=1}^L V^i(x^i)$ with $x = \text{col}(x^1, \dots, x^L)$ the column vector of stacked state variables. Post- and pre-multiply (26) with $\text{col}(x^i(k), v^i(k), p^i(k), d^i(k))$ and its transpose to get

$$V^i(x^i(k+1)) - V^i(x^i(k)) < P^i(k) + U^i(k) + S^i(k). \tag{32}$$

From this, we see that each local subsystem is strictly dissipative with respect to the aggregated supply function $\Phi^i(k)$ as defined in Section II-B. Summing over i now yields

$$\begin{aligned} V(x(k+1)) - V(x(k)) < & \sum_{i=1}^L P^i(v^i, w^i) + \sum_{i=1}^L U^i(p^i, q^i) \\ & + \sum_{i=1}^L S^i(d^i, z^i). \end{aligned} \tag{33}$$

Since $(\sum_{i=1}^{L} P^i) = 0$ because of the neutrality condition (13) and $\sum_{i=1}^{L} U^i(p^i, q^i) \leq 0$ by (6) and (27), we infer that

$$V(x(k+1)) - V(x(k)) < \sum_{i=1}^{L} S^i(d^i, z^i). \tag{34}$$

To prove stability, first set $d^i = 0$. By (5) and the assumption that $\gamma > 0$, this implies that $\sum_{i=1}^{L} S^i(0, z^i) \leq 0$. Hence,

$$V(x(k+1)) - V(x(k)) < 0. \tag{35}$$

Since $X_T^i \succ 0$ we also have that $V(x(k)) > 0$. Hence, V is a Lyapunov function for the uncertain distributed system, rendering it stable under all possible uncertainties $\Delta^i \in \boldsymbol{\Delta}^i$. By construction of $V(x)$, we have that

$$\begin{aligned}\lambda_{min}(\mathbf{diag}_{i\in\mathbb{N}_{[1,L]}} X_T^i)||x||_2^2 \leq V(x) &= x^T(\mathbf{diag}_{i\in\mathbb{N}_{[1,L]}} X_T^i)x \\ &\leq \lambda_{max}(\mathbf{diag}_{i\in\mathbb{N}_{[1,L]}} X_T^i)||x||_2^2\end{aligned}$$

and the uncertain system is therefore globally exponentially stable in Lyapunov sense.

To prove robust performance, suppose that $d \neq 0$ and set $x(0) = 0$. Taking the sum over all time instances k as defined in (3), (34) becomes

$$\lim_{k\to\infty} V(x(k)) - V(0) < \gamma||d||_2^2 - \frac{1}{\gamma}||z||_2^2. \tag{36}$$

Since $V(0) = 0$, $V(x) \geq 0$ and $x \in \ell_2$, we get

$$||z||_2^2 < \gamma^2||d||_2^2 \tag{37}$$

and it follows that $||G_\Delta||_{2,2} < \gamma$. That is, γ is a robust performance bound for the system. ■

Theorem 1 provides LMI conditions to verify the robust stability and robust performance of an uncertain distributed system. Thus, Theorem 1 yields a solution to Problem 1. we emphasize that matrices X_{11}^{ij} and X_{12}^{ij} and variable γ are shared between the subsystems. This means that the verification of robust stability and robust performance of the distributed system is not decomposable and needs to be verified in a centralized way.

B. Closed loop analysis

The following result on robust stability and performance of the closed-loop system now follows from the closed loop system as defined in Section II-C and Theorem 1.

Proposition 1: Let $(G_\Delta)_C$ be an uncertain distributed system with subsystems admitting realization (18)-(19). Then, the system is well-posed, stable and achieves robust performance γ if $\forall i \in \mathbb{N}_{[1,L]}$ there exist matrices $\left(X_T^i\right)_C \in \mathbb{R}_S^{2m^i}$, $Z_C^i \in \mathbb{R}_S^{2n_C^i}$ and $D_\Delta^i \in \mathbb{R}_S^{2n_\Delta^i}$, such that $\left(X_T^i\right)_C \succ 0$, $D_{11}^i \succ 0$, $D_{22}^i \prec 0$ and for all $\Delta^i \in \boldsymbol{\Delta}^i$

$$(T_C^i)^T M_C^i T_C^i \prec 0, \tag{38}$$

$$\begin{bmatrix} I \\ \Delta^i \end{bmatrix}^T \begin{bmatrix} D_{11}^i & D_{12}^i \\ (D_{12}^i)^T & D_{22}^i \end{bmatrix} \begin{bmatrix} I \\ \Delta^i \end{bmatrix} \succeq 0, \tag{39}$$

with

$$T_C^i := \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ \left(A_{xx}^i\right)_C & \left(A_{xv}^i\right)_C & \left(B_{xp}^i\right)_C & \left(B_{xd}^i\right)_C \\ \hline \left(A_{wx}^i\right)_C & \left(A_{wv}^i\right)_C & \left(B_{wp}^i\right)_C & \left(B_{wd}^i\right)_C \\ 0 & I & 0 & 0 \\ \hline \left(C_{qx}^i\right)_C & \left(C_{qv}^i\right)_C & \left(D_{qp}^i\right)_C & \left(D_{qd}^i\right)_C \\ 0 & 0 & I & 0 \\ \hline \left(C_{zx}^i\right)_C & \left(C_{zv}^i\right)_C & \left(D_{zp}^i\right)_C & \left(D_{zd}^i\right)_C \\ 0 & 0 & 0 & I \end{array}\right], \tag{40}$$

$$\begin{aligned} M_C^i := \mathbf{diag}\Bigg(&\begin{bmatrix} -\left(X_T^i\right)_C & 0 \\ 0 & \left(X_T^i\right)_C \end{bmatrix}, \begin{bmatrix} \left(Z_{11}^i\right)_C & \left(Z_{12}^i\right)_C \\ \left(Z_{12}^i\right)_C^T & \left(Z_{22}^i\right)_C \end{bmatrix}, \\ &\begin{bmatrix} D_{11}^i & D_{12}^i \\ \left(D_{12}^i\right)^T & D_{22}^i \end{bmatrix}, \begin{bmatrix} \frac{1}{\gamma}I & 0 \\ 0 & -\gamma I \end{bmatrix}\Bigg) \end{aligned} \tag{41}$$

and with $\left(Z_{11}^i\right)_C$ partitioned as

$$\left(Z_{11}^i\right)_C := \begin{bmatrix} \left(Z_{11}^i\right)_G & \left(Z_{11}^i\right)_{GK} \\ \left(Z_{11}^i\right)_{GK}^T & \left(Z_{11}^i\right)_K \end{bmatrix}, \tag{42}$$

and $\left(Z_{12}^i\right)_C$, $\left(Z_{22}^i\right)_C$ and $\left(X_T^i\right)_C$ defined analogously.
The above proposition is a direct extension of Theorem 1, where the Lyapunov matrices X_T^i and interconnection matrices Z^i are extended to their closed loop counterparts $\left(X_T^i\right)_C$ and $\left(Z^i\right)_C$.

Note that matrices $\left(Z^i\right)_G$, $\left(Z^i\right)_K$ and $\left(Z^i\right)_{GK}$ model both the interconnections between plants and between controllers, and their cross-terms. The construction of these matrices from the individual channel matrices $\left(X^{ij}\right)_G$ (and $\left(X^{ij}\right)_K$ and $\left(X^{ij}\right)_{GK}$, respectively) is straightforward but somewhat tedious, and the reader is referred to [2, Proposition 2] for details.

Proposition 1 gives a LMI test, for a given distributed system and a given distributed controller, of sufficient conditions for stability and robust performance.

V. SYNTHESIS

The results on controller synthesis are twofold. Firstly, we provide sufficient conditions for the existence of a distributed controller satisfying Proposition 1. Secondly, we explicitly construct the controller.

A. Controller existence

The conditions in Proposition 1 yield non-linear conditions when solving for both the unknown matrices in M^i and the unknown controller parameters Θ^i at the same time. To get LMI conditions, we eliminate the controller parameters Θ^i from the closed loop conditions using the following lemma from [18].

Lemma 1 (Elimination lemma): Let M be a symmetric matrix with $in(M) = (a_-, 0, a_+)$ and matrix $R^i \in \mathbb{R}^{a_+ \times a_-}$. Then the matrix inequality

$$\begin{bmatrix} I \\ R + V^T\Theta U \end{bmatrix}^T M \begin{bmatrix} I \\ R + V^T\Theta U \end{bmatrix} \prec 0 \tag{43}$$

in the unstructured unknown Θ has a solution if and only if

$$U_\perp^T \begin{bmatrix} I \\ R \end{bmatrix}^T M \begin{bmatrix} I \\ R \end{bmatrix} U_\perp \prec 0 \tag{44}$$

$$V_\perp^T \begin{bmatrix} -R^T \\ I \end{bmatrix}^T M^{-1} \begin{bmatrix} -R^T \\ I \end{bmatrix} V_\perp \succ 0 \tag{45}$$

with $U_\perp$ and $V_\perp$ matrices whose columns are minimal spanning sets of the nullspace of U and V, respectively. Furthermore, if a solution exists, Θ can be explicitly reconstructed from the matrices in (44) and (45).

This lemma provides us with LMI conditions for the existence of a distributed controller and the possibility to construct the controller if it exists. The LMI conditions for existence of a controller are collected in the following theorem.

Theorem 2: There exist matrices such that the conditions of Proposition 1 are satisfied with $n_K^{ij} = 3n_G^{ij}$ if and only if $\forall i \in \mathbb{N}_{[1,L]}$ there exist symmetric matrices $M^i, \tilde{M}^i \in \mathbb{S}^{2(m^i+n_G^i+n_\Delta^i)+n_d^i+n_z^i}$ such that for all $\Delta^i \in \boldsymbol{\Delta}^i$, $D_{11}^i \succ 0$, $D_{22}^i \prec 0, \left(X_T^i\right)_G, \left(Y_T^i\right)_G \succ 0, q^i \in \mathbb{R}^{n_\Delta^i}$,

$$(U_G^i)_\perp^T (T^i)^T M_G^i T^i (U_G^i)_\perp \prec 0, \tag{46}$$

$$(V_G^i)_\perp^T (T_\perp^i)^T \tilde{M}_G^i T_\perp^i (V_G^i)_\perp \succ 0, \tag{47}$$

$$\begin{bmatrix} (X_T)_G^i & I \\ I & (Y_T)_G^i \end{bmatrix} \succeq 0, \tag{48}$$

$$\begin{bmatrix} I \\ \Delta^i \end{bmatrix}^T \begin{bmatrix} D_{11}^i & D_{12}^i \\ (D_{12}^i)^T & D_{22}^i \end{bmatrix} \begin{bmatrix} I \\ \Delta^i \end{bmatrix} \succeq 0. \tag{49}$$

Here,

$$T^i := \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ A_{xx}^i & A_{xv}^i & B_{xp}^i & B_{xd}^i \\ \hline A_{wx}^i & A_{wv}^i & B_{wp}^i & B_{wd}^i \\ 0 & I & 0 & 0 \\ \hline C_{qx}^i & C_{qv}^i & D_{qp}^i & D_{qd}^i \\ 0 & 0 & I & 0 \\ \hline C_{zx}^i & C_{zv}^i & D_{zp}^i & D_{zd}^i \\ 0 & 0 & 0 & I \end{array}\right],$$

$$M_G^i := \mathbf{diag}\left(\begin{bmatrix} -(X_T)_G^i & 0 \\ 0 & (X_T)_G^i \end{bmatrix}, \begin{bmatrix} \left(Z_{11}^i\right)_G & \left(Z_{12}^i\right)_G \\ \left(Z_{12}^i\right)_G^T & \left(Z_{22}^i\right)_G \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} D_{11}^i & D_{12}^i \\ \left(D_{12}^i\right)^T & D_{22}^i \end{bmatrix}, \begin{bmatrix} \frac{1}{\gamma} I & 0 \\ 0 & -\gamma I \end{bmatrix} \right),$$

$$\tilde{M}_G^i := \mathbf{diag}\left(\begin{bmatrix} -(Y_T)_G^i & 0 \\ 0 & (Y_T)_G^i \end{bmatrix}, \begin{bmatrix} \left(\tilde{Z}_{11}^i\right)_G & \left(\tilde{Z}_{12}^i\right)_G \\ \left(\tilde{Z}_{12}^i\right)_G^T & \left(\tilde{Z}_{22}^i\right)_G \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} D_{11}^i & D_{12}^i \\ \left(D_{12}^i\right)^T & D_{22}^i \end{bmatrix}^{-1}, \begin{bmatrix} \gamma I & 0 \\ 0 & -\frac{1}{\gamma} I \end{bmatrix} \right),$$

$(U_G^i)_\perp^T$ spans the nullspace of $[C_{yx}^i \; C_{yv}^i \; D_{yp}^i \; D_{yd}^i]$,
$(V_G^i)_\perp^T$ spans the nullspace of $[(B_{xu}^i)^T \; (B_{wu}^i)^T \; (D_{qu}^i)^T \; (D_{zu}^i)^T]$,

$$\left(Z_{11}^i\right)_G := -\mathbf{diag}_{j \in \mathcal{N}^i} \left(X_{11}^{ij}\right)_G, \quad \left(\tilde{Z}_{11}^i\right)_G := -\mathbf{diag}_{j \in \mathcal{N}^i} \left(Y_{11}^{ij}\right)_G,$$

and $\left(Z_{12}^i\right)_G, \left(Z_{22}^i\right)_G, \left(\tilde{Z}_{12}^i\right)_G, \left(\tilde{Z}_{22}^i\right)_G$ defined analogously.

Proof: (Only if): We consider Proposition 1 and note that (38) is satisfied if and only if

$$T^T M T \prec 0 \quad \text{for} \quad T = \mathbf{diag}_i(T^i), \; M = \mathbf{diag}_i(M^i) \tag{50}$$

To see that Lemma 1 can be applied to (38) we thus consider the inertia of M, which is given by

$$in(M) = \sum_{i=1}^{L} in\left(\begin{bmatrix} -\left(X_T^i\right)_C & 0 \\ 0 & \left(X_T^i\right)_C \end{bmatrix} \right)$$
$$+ \sum_{i,j=1}^{L} in(-(X^{ij})_C) + \sum_{i=1}^{L} in(D_\Delta^i) + \sum_{i=1}^{L} (n_d^i, 0, n_z^i).$$

Since $\left(X_T^i\right)_C \succ 0$ and $in(D_\Delta^i) = in(D_{11}^i) + in(D_{22}^i - \left(D_{12}^i\right)^T \left(D_{11}^i\right)^{-1} D_{12}^i)$ with $D_{11}^i \succ 0, D_{22}^i \prec 0$ and due to the definition of the interconnection multipliers in (16), this is equal to

$$in(M) = (\sum_{i=1}^{L} (2m^i + n_C^i + n_\Delta^i + n_d^i),$$
$$0, \sum_{i=1}^{L} (2m^i + n_C^i + n_\Delta^i + n_z^i)).$$

Since the inertia requirement is met on M by construction, (50) can be brought into the form of Lemma 1 through matrix permutation. By eliminating the controller parameters, permuting the lines and columns back to the block-diagonal representation and selecting only the rows and columns associated with interconnection and Lyapunov matrices $(\cdot)_G$, we then arrive at (46)-(47) with M_G^i and $\tilde{M}_G^i$ truncated versions of M_C^i and $\left(M_C^i\right)^{-1}$, respectively.

If: The proof for sufficiency is exactly the same as in [6, Theorem 2]. The main line of the proof is that for $n_K^{ij} = 3n_G^{ij}$ and (48) the partial and dual results can always be used to extend the interconnection and Lyapunov matrices to full-block matrices satisfying Proposition 1. ■

Similar to the continuous time case, the result of Theorem 2 does not define a convex problem in all variables: Applying the elimination lemma on the analysis equations in Proposition 1 removed the controller parameters from the equations, thus removing the non-linearity obtained from multiplying controller parameters and multipliers. However, equations (46) and (47) give a set of LMIs in both D_Δ^i and $\left(D_\Delta^i\right)^{-1}$ which renders the combined set of equations non-convex. This is equivalent to the continuous time case. An algorithm for solving the robust distributed control problem is suggested in Section V-B.

B. Controller synthesis

By Lemma 1, a controller can be directly reconstructed from a solution to Theorem 2. The partial and dual results can always be used to extend the interconnection and Lyapunov matrices to full-block matrices satisfying Proposition 1, for details see [2], [5]. Through an injective mapping a controller can be calculated from those matrices and the system matrices, for details see [18].

Both Proposition 1 and Theorem 2 give non-convex conditions for controller synthesis. However, for known (non-singular) D_Δ^i representing the uncertainty, Theorem 2 gives convex conditions and for known controller K, Proposition 1 gives convex conditions. A D-K iterative algorithm as suggested by [6] can be used to find a solution to Problem 2. The algorithm is as follows.

Algorithm 1: Initialisation: $j = 0$, $j_{max} \in \mathbb{N}$ and $\epsilon > 0$.

1) Initial K-step: Apply Theorem 2 for the nominal system G_0 and with empty D^i_Δ for all i to find an initial $\gamma_{K,0}$ and a distributed controller K_0. Set $j = 1$.
2) Initial D-step: Apply Proposition 1 for the closed loop uncertain system G_Δ controlled by K_0 to find matrices $(D^i_\Delta)_1$, $i = 1, \dots, L$, with minimal performance bound $\gamma_{D,1}$.
3) D-K iteration:
    a) K-step: Apply Theorem 2 for the nominal system G_0 and $(D^i_\Delta)_j$ for all i from the previous D-step to find a robust distributed controller K_j obtaining performance $\gamma_{K,j}$. Set $j = j + 1$.
    b) D-step: Apply Proposition 1 for the closed loop uncertain system G_Δ controlled by K_{j-1} to find $(D^i_\Delta)_j$, $i = 1, \dots, L$, with minimal performance bound $\gamma_{D,j}$.
    c) Termination: End the iteration if the decrease in γ_K is smaller than tolerance ϵ or if $j > j_{max}$.

This algorithm has guaranteed convergence properties as is shown in the next result.

Theorem 3: Let G_Δ be an uncertain distributed system. Suppose that there exist $K_0, \gamma_{D,1}$ and $\left(D^i_\Delta\right)_1$ for all i as defined in Algorithm 1. Then the solution of Algorithm 1 has the following properties:

1) $\gamma_{K,j} \leq \gamma_{K,j-1}$ for all $1 < j \leq j_{max}$.
2) Problem 2 is solvable with robust performance bound $\gamma = \lim_{j\to\infty} \gamma_{K,j}$ and $\gamma = \gamma_{K,j_{max}}$.
3) The robust performance bound $\gamma_{K,j}$ is also guaranteed for any group of subsystems $(G_\Delta)^{sub}_C$ with subgraph $\mathcal{G}_{(G_\Delta)^{sub}_C}$ of $\mathcal{G}_{(G_\Delta)_C}$ with vertices $\mathcal{V}_{(G_\Delta)^{sub}_C} \subset \mathcal{V}_{(G_\Delta)_C}$ and associated edges $\mathcal{E}_{(G_\Delta)^{sub}_C} \subset \mathcal{E}_{(G_\Delta)_C}$.

Hence, every robust controller K_j inferred from step 3a solves Problem 2. Theorem 3 proves monotonic convergence of the performance bound γ provided the algorithm passes step 2. The algorithm does not necessarily converge to a global optimum. Also note that step 1 gives a solution to Problem 2 for the case without uncertainty, i.e., for the nominal system. Theorem 3 property 3) shows that the controllers are robust to full interconnection failure, i.e. whenever a complete edge is removed from the system.

Remark 1: The number of decision variables in the synthesis procedure can be determined to give an indication of the numerical complexity. The matrices X^i_T, $(X^i_T)_K$, $(X^i_T)_{GK}$ and $(X^i_T)_C$, corresponding to the state, contain $((m^i)^2 + m^i)/2$, $((m^i)^2 + m^i)/2$, $(m^i)^2$ and $2(m^i)^2 + m^i$ unknown variables per subsystem i, respectively. The matrices D^i_Δ, corresponding to the uncertainty, contain $2(n^i_\Delta)^2 + n^i_\Delta$ unknown variables per subsystem i. For the interconnections, we have L^2 matrices X^{ij}_{11} (for $i, j = 1, \dots, L$), each having $((n^{ij}_G)^2 + n^{ij}_G)/2$ independent variables, or $((n^{ij}_C)^2 + n^{ij}_C)/2$ for the closed loop variables $(X^{ij}_{11})_C$, and we have $(L^2 + L)/2$ matrices X^{ij}_{12} (for $1 \leq j \leq i \leq L$), each having $(n^{ij}_G)^2$ unknown variables, or $(n^{ij}_C)^2$ for the closed loop variables $(X^{ij}_{12})_C$. Recall that $n^{ij}_C = n^{ij}_G + n^{ij}_K$ with $n^{ij}_K = 3n^{ij}_G$.

VI. SIMULATION AND RESULTS

To illustrate the computational feasibility of the presented framework, consider the example of a network of $L = 3$ fully interconnected systems with communication delays where the dynamics of subsystem i is described by

$$\begin{aligned} x^i(k+1) &= p^i x^i(k) + \sum_j v^{ij}(k) + d^i(k) + u^i(k) \\ w^i(k) &= [a^{j_1 i} x^i(k) \quad a^{j_2 i} x^i(k)]^T \\ z^i(k) &= x^i(k) \\ y^i(k) &= [x^i(k) \quad v^{j_1}(k) \quad v^{j_2}(k)]^T \end{aligned}$$

Here, j_1, j_2 denote the indices of the two subsystems connected to subsystem i. The outputs w^i are stacked such that the interconnections are given by $v^{ij} = w^{ji}$. The parameters p^i represent uncertain variables $p^1 \in \mathbb{R}_{[0.5,1.5]}$ and $p^2 \in \mathbb{R}_{[1.5,3]}$ with nominal values $(p^1_{nom}, p^2_{nom}, p^3) = (1, 2, 1)$. The amplification of the outgoing variable a^{ji} is given by $a^{21} = a^{31} = a^{13} = -1$, $a^{12} = a^{32} = 3$ and $a^{23} = 1$.

We introduce delays in the interconnections as shown in Figure 1. The delays are taken as input delays for the receiving subsystem and modeled by extending the state dimension of the corresponding subsystem. In fact, delays can also be modeled via the output variables. This results in a system description that matches the framework used in this paper. For ease of exposition, we assume that the delays are equal in both directions of the interconnection lines, but this assumption is certainly not a limitation for the method.

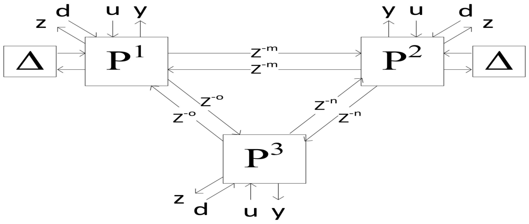

Fig. 1. Fully interconnected delay system. Delays are represented by powers of z^{-1}.

Controllers have been synthesized with delays of 5, 3 and 4 time units for the interconnections between P^1-P^2, P^1-P^3 and P^2-P^3, respectively. Again, a centralized $\mathcal{H}_\infty$ controller, a nominal distributed controller and a robust distributed controller were synthesized. A robust controller of total order $m_K = m_G = 27$ is obtained by Algorithm 1. In Table I, the resulting $\mathcal{H}_\infty$-norm of the closed loop transfer functions $d \to z$ are shown for each type of controller.

It can be seen from Table I that, for the perturbed plant, the centralized and nominal distributed controllers violate the performance bound given by the controller synthesis procedure, whereas the robust bound is not violated and even achieves a better closed loop performance. In Figure 2 the maximum singular values of the closed loop transfer functions for each controller are shown.

TABLE I
PERFORMANCE BOUNDS OF THE RESULTING CLOSED LOOP SYSTEMS FOR THE INTERCONNECTION DELAY SYSTEM.

Controller	γ_K	$\|\|(G_0)_C\|\|_\infty$	$\|\|(G_\Delta)_C\|\|_\infty$
Nominal Centralized $K_{\mathcal{H}_\infty}$	1.00e+00	1.00e+00	1.39e+00
Nominal Distributed K	1.01e+00	1.00e+00	1.39e+00
Robust Distributed K	8.66e+00	1.26e+00	1.95e+00

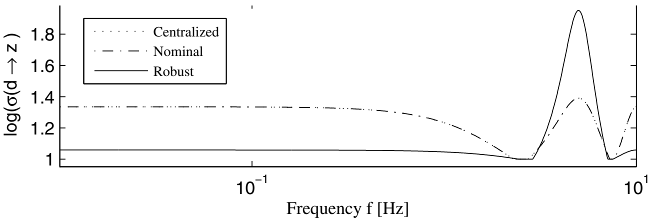

Fig. 2. Comparison of the maximum singular values for different control strategies.

The closed loop system is simulated using a stepwise unit load increase in subsystem P^2 at time $t = 10$ and a unit load decrease in P^3 at $t = 20$. Time-domain simulations can be seen in Figure 3. The plots show that, indeed, the robust controller performs better for the perturbed plant.

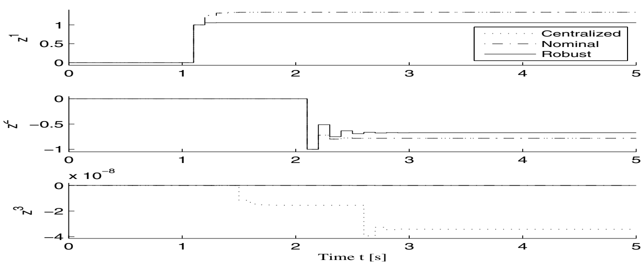

Fig. 3. Comparison of responses of different control strategies of the perturbed system.

VII. CONCLUSIONS

In this paper we proposed a novel algorithm for the synthesis of distributed robust controllers for networks of interconnected uncertain discrete time LTI systems. Analysis and synthesis results for distributed robust $\mathcal{H}_\infty$ control of these systems have been derived. The main result involves a synthesis algorithm that is based on feasibility tests involving linear matrix inequalities (LMIs). The algorithm resembles a so called D-K type of iterative procedure that is known in μ-controller synthesis. The resulting distributed controller renders an uncertain distributed network of interconnected discrete time LTI systems well-posed, robustly stable with a guaranteed robust $\mathcal{H}_\infty$ performance level γ. The synthesis of the communication channels among the controllers is part of the synthesis procedure. The algorithm is computationally tractable for moderate size networks. The discrete time nature of the problem setting allows an investigation of communication delays that has been illustrated in a simulation example. We recognize that the scope of the work is mainly limited by available computer power. More specifically, by the number of decision variables that can be handled in the interior-point optimizations that solve the linear matrix inequalities that are derived in the main results of this paper. A partial remedy to this problem lies in the possibility to reduce the problem complexity by fixing interconnection variables, fixing supply functions in the communication channels or by fixing Lyapunov functions in the stability assessments. Currently, research is being conducted to achieve these complexity reductions in a computationally tractable way [19].

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