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Controlled invariance for behavioral systems

Ricardo Pereira[1] and Paula Rocha[2]

***Abstract*— In this work we introduce and characterize the notion of controlled invariance in the framework of the behavioral appproach. This concept is based on the definition of behavioral invariance given in [3].**

***Index Terms*— Behaviors, partial interconnections, controlled invariance, (AMS subject classification: 93C05).**

I. BACKGROUND: CONTROL BY PARTIAL INTERCONNECTION

Similar to what happens for state space systems, controlled invariance means “invariance after control”. In the behavioral approach, control is nothing but interconnecting (intersecting) a given behavior with a suitable controller behavior in order to obtain a desired controlled behavior. This can be done essentially in two ways, namely by full interconection (where all the system variables are available for control) or by partial interconnection (where the variables are divided into *to-be-controlled variables* and *control variables*, [4], [1]). Here we consider the case of control by partial interconnection.

In the sequel we denote the to-be-controlled variables by w and the control variables by c. In the 1D case, we assume that the joint behavior of these variables, i.e., the (w, c)-behavior, is given as:

$$\mathcal{B}_{(w,c)} := \{(w, c) \in \mathcal{U}^{\mathtt{w}} \times \mathcal{U}^{\mathtt{c}} \mid R(\sigma)w = M(\sigma)c\}, \qquad (1)$$

where, for $\mathtt{q} \in \mathbb{N}$, $\mathcal{U}^{\mathtt{q}} := \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$, σ denotes the differential operator $\frac{d}{dt}$, and $R(s) \in \mathbb{R}^{g\times\mathtt{w}}[s], M(s) \in \mathbb{R}^{g\times\mathtt{c}}[s]$ are polynomial matrices.

The w-behavior induced by $\mathcal{B}_{(w,c)}$ - i.e, $\mathcal{B}_w = \pi_w\left(B_{(w,c)}\right)$, where π_w denotes the projection into $\mathcal{U}^{\mathtt{w}}$ - is obtained by eliminating c from the equation $R(\sigma)w = M(\sigma)c$, which is achieved by premultiplying both sides of the equation by a minimal left annihilator $L(\sigma)$ of $M(\sigma)$. This yields $\mathcal{B}_w = \ker(LR)$.

The control action then consists in restricting the behavior of the control variables c in order to obtain a desired effect on w, this is, given a behavior to be controlled $\mathcal{B}_{(w,c)} \subset \mathcal{U}^{\mathtt{w}} \times \mathcal{U}^{\mathtt{c}}$ and a desired behavior $\mathcal{K}_w \subset \mathcal{U}^{\mathtt{w}}$, a controller behavior $\mathcal{C}_c \subset \mathcal{U}^{\mathtt{c}}$ (given by $\mathcal{C}_c = \{c \in \mathcal{U}^{\mathtt{c}} : K(\sigma)c = 0\} = \ker K$, for some adequate polynomial matrix $K(s)$) has to be determined such that

$$\mathcal{K}_w = \pi_w\left(\mathcal{B}_{(w,c)} \cap \mathcal{C}^*_{(w,c)}\right), \qquad (2)$$

where $C^*_{(w,c)}$ stands for the lifted behavior

$$\mathcal{C}^*_{(w,c)} := \{(w, c) \in \mathcal{U}^{\mathtt{w}} \times \mathcal{U}^{\mathtt{c}} \mid w \text{ is free and } c \in \mathcal{C}_c\}.$$

If (2) holds, we say that $\mathcal{K}_w$ is *achievable* by partial interconnection from $\mathcal{B}_{(w,c)}$.

Regular controllers play an important role in this context. They are characterized by imposing restrictions on the control variables that do not overlap with the ones already implied by the laws of the original behavior $\mathcal{B}_{(w,c)}$.

Partial interconnection with a regular controller is called *regular partial interconnection*. In terms of the polynomial matrices $R(s), M(s)$ and $K(s)$ that describe the to-be-controlled behavior $\mathcal{B}_{(w,c)}$ and the controller $\mathcal{C}_c$, the regularity of the corresponding partial interconnection is equivalent to the following condition:

$$\text{rank}\begin{bmatrix} R(s) & M(s) \\ 0 & K(s) \end{bmatrix} = \text{rank}\begin{bmatrix} R(s) & M(s) \end{bmatrix} + \text{rank}\begin{bmatrix} 0 & K(s) \end{bmatrix}.$$

Thus, every controller $\mathcal{C}_c = \ker K$ is regular if the polynomial matrix $R(s)$ has full row rank. In turn, this condition means that all the control variables are free in the to-be-controlled behavior $\mathcal{B}_{(w,c)}$. The case where $R(s)$ is not full row rank can also be treated, but leads to more cumbersome computations. Therefore, for simplicity, in the sequel we shall assume that $R(s)$ always has full row rank, and, hence, interconnection is to be understood as regular interconnection.

It is not difficult to see that only sub-behaviors $\mathcal{K}_w$ of $\mathcal{B}_w$ are achievable from $\mathcal{B}_{(w,c)}$ by partial interconnection. Moreover, the smallest sub-behavior of $\mathcal{B}_w$ achievable by partial interconnection is clearly obtained by setting all the control variables to be zero. This gives rise to the behavior

$$\mathcal{N}_w := \left\{w \in \mathcal{U}^{\mathtt{w}} \mid (w, 0) \in \mathcal{B}_{(w,c)}\right\},$$

whose kernel representation is $\mathcal{N}_w = \ker R$, known as *hidden behavior*, [4].

II. BEHAVIORAL INVARIANCE

As in [3], we adopt the following definition for behavioral invariance.

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[1] CIDMA, Department of Mathematics, University of Aveiro, Portugal `ricardopereira@ua.pt`

[2] CIDMA and Faculty of Engineering, University of Porto, Portugal `mprocha@fe.up.pt`

Definition 1: Given a behavior $\mathcal{B}_w$, a sub-behavior $\mathcal{V}_w$ of $\mathcal{B}_w$ is said to be $\mathcal{B}_w$*-invariant* if, for every $t_0 \in \mathbb{R}$, the following condition holds:

$$[w \in \mathcal{B}_w,\ w_{|(-\infty,t_0]} \in \mathcal{V}_{w|(-\infty,t_0]}] \Rightarrow [w \in \mathcal{V}_w],$$

where, as usual, $w_{|(-\infty,t_0]}$ denotes the restriction of w to the interval $(-\infty, t_0]$ and $\mathcal{V}_{w|(-\infty,t_0]}$ is the set of restrictions of the trajectories $v \in \mathcal{V}_w$ to the that same interval.

This means that the trajectories of $\mathcal{B}_w$ whose past is in $\mathcal{V}_w$ remain forever in $\mathcal{V}_w$, i.e., $\mathcal{B}_w$ is "autonomous up to $\mathcal{V}_w$", as stated in the next proposition.

Proposition 1: A sub-behavior $\mathcal{V}_w$ of $\mathcal{B}_w$ is $\mathcal{B}_w$-invariant if and only if the quotient behavior $\mathcal{B}_w/\mathcal{V}_w$ is autonomous.

We refer the reader to [2], [5], [6] for the definition of the quotient of two behaviors and for a proof of the fact that this quotient has the structure of a behavior.

III. BEHAVIORAL CONTROLLED-INVARIANCE

In the previous setting, controlled-invariance is defined as follows.

Definition 2: Let $\mathcal{B}_{(w,c)} \subset \mathcal{U}^{\mathtt{w}} \times \mathcal{U}^{\mathtt{c}}$. A sub-behavior $\mathcal{V}_w$ of the induced w-behavior $\mathcal{B}_w \subset \mathcal{U}^{\mathtt{w}}$ is said to be *controlled-invariant* if $\mathcal{V}_w$ is $\mathcal{K}_w$-invariant, for some behavior $\mathcal{K}_w$ achievable by partial interconnection from $\mathcal{B}_{(w,c)}$.

As expected, not every sub-behavior of $\mathcal{B}_w$ is controlled-invariant. Indeed, if $\mathcal{K}_w$ is achievable by partial interconnection from $\mathcal{B}_{(w,c)}$, then, as mentioned before, $\mathcal{K}_w$ must contain $\mathcal{N}_w$.

Now, if in particular $\mathcal{V}_w \subset \mathcal{N}_w \subset \mathcal{B}_w$ is a sub-behavior which is not $\mathcal{N}_w$-invariant, then $\mathcal{N}_w/\mathcal{V}_w$ is not autonomous, and hence neither is $\mathcal{K}_w/\mathcal{V}_w$, as it contains $\mathcal{N}_w/\mathcal{V}_w$. Therefore, by Proposition 1, the following result holds.

Proposition 2: Let $\mathcal{B}_{(w,c)} \subset \mathcal{U}^{\mathtt{w}} \times \mathcal{U}^{\mathtt{c}}$. Then, if $\mathcal{V}_w \subset \mathcal{N}_w$:

$$\mathcal{V}_w \text{ is controlled-invariant } \Leftrightarrow \mathcal{V}_w \text{ is } \mathcal{N}_w - invariant.$$

In the general case, i.e., when $\mathcal{V}_w$ is not necessarily a subset of $\mathcal{N}_w$, the following result holds.

Proposition 3: Let $\mathcal{B}_{(w,c)} \subset \mathcal{U}^{\mathtt{w}} \times \mathcal{U}^{\mathtt{c}}$, and $\mathcal{V}_w \subset \mathcal{B}_w$. Then:

$$\mathcal{V}_w \text{ is controlled-invariant } \Leftrightarrow \bar{\mathcal{B}}_w/\mathcal{V}_w \text{ is autonomous,}$$

where $\bar{\mathcal{B}}_w$ is the smallest achievable behavior by partial interconnection from $\mathcal{B}_{(w,c)}$ containing $\mathcal{V}_w$, i.e.,

$$\bar{\mathcal{B}}_w = \mathcal{N}_w + \mathcal{V}_w.$$

In our talk we characterize the controllers that achieve the invariance of a given sub-behavior $\mathcal{V}_w$. In order to make contact with the classical approach, the obtained results are interpreted in terms of state space systems.

The overall problem of behavioral control invariance will also be considered for nD systems, for which the situation is considerably more involved, as the regularity of the partial interconnection cannot in general be assumed without loss of generality.

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