21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

Fixed point theorems for noncommutative functions

Gulnara Abduvalieva[1] and Dmitry S. Kaliuzhnhyi-Verbovetskyi[2]

I. INTRODUCTION

The theory of noncommutative (nc) functions has its origin in the papers of Joseph L. Taylor [1], [2]. It was further developed by D.-V. Voiculescu [3], [4] in his fundamental work in free probability. Based on pioneering ideas of J. L. Taylor, the second author and Victor Vinnikov [5] developed the nc difference-differential calculus and used it for studying various questions of nc analysis, in particular, extending the classical (commutative) theory of analytic functions to a nc setting. A special case of nc rational functions, which is important for applications in optimization and control, where matrices are natural variables and the problems are dimension-independent (see [6], [7], [8] for a detailed discussion), can be studied independently — see [9], [10]. The theory of nc rational functions is motivated by and useful in nc semialgebraic geometry — see, e.g., [11], [12], [13], [14]. We also mention the works [15], [16], [17], [18], [19], [20], [21], [22] on various aspects of nc function theory. We establish a certain type of fixed point theorems as a useful tool in nc (free) analysis.

We begin with some basic definitions from [5]. Let $\mathcal{R}$ be a commutative unital ring. For a module $\mathcal{M}$ over $\mathcal{R}$, we define the *nc space over* $\mathcal{M}$,

$$\mathcal{M}_{\rm nc} := \coprod_{n=1}^{\infty} \mathcal{M}^{n\times n}. \tag{1}$$

A subset $\Omega \subseteq \mathcal{M}_{\rm nc}$ is called a *nc set* if it is closed under direct sums; that is, denoting $\Omega_n = \Omega \cap \mathcal{M}^{n\times n}$, we have

$$X \in \Omega_n, Y \in \Omega_m \Longrightarrow X \oplus Y := \begin{bmatrix} X & \mathbf{O}_{n\times m} \\ \mathbf{O}_{m\times n} & Y \end{bmatrix} \in \Omega_{n+m}, \tag{2}$$

where $\mathbf{O}_{p\times q}$ is the $p \times q$ matrix whose all entries are 0.

Notice that matrices over $\mathcal{R}$ act from the right and from the left on matrices over $\mathcal{M}$ by the standard rules of matrix multiplication: if $T \in \mathcal{R}^{r\times p}$ and $S \in \mathcal{R}^{p\times s}$, then for $X \in \mathcal{M}^{p\times p}$ we have

$$TX \in \mathcal{M}^{r\times p}, \quad XS \in \mathcal{M}^{p\times s}.$$

[1] Gulnara Abduvalieva is with the Department of Mathematics, Drexel University, 3141 Chestnut St., Philadelphia, PA 19104, USA, gabduv@yahoo.com; she is partially supported by the NSF grant DMS-0901628

[2] Dmitry S. Kaliuzhnhyi-Verbovetskyi is with the Department of Mathematics, Drexel University, 3141 Chestnut St., Philadelphia, PA 19104, USA, dmitryk@math.drexel.edu; he is partially supported by the USA–Israel BSF grant 2010432 and by the NSF grant DMS-0901628

In the special case where $\mathcal{M} = \mathcal{R}^d$, we identify matrices over $\mathcal{M}$ with d-tuples of matrices over $\mathcal{R}$:

$$(\mathcal{R}^d)^{p\times q} \cong (\mathcal{R}^{p\times q})^d.$$

Under this identification, for d-tuples $X = (X_1, \dots, X_d) \in (\mathcal{R}^{n\times n})^d$ and $Y = (Y_1, \dots, Y_d) \in (\mathcal{R}^{m\times m})^d$ their direct sum has the form

$$X \oplus Y = \left(\begin{bmatrix} X_1 & \mathbf{O}_{n\times m} \\ \mathbf{O}_{m\times n} & Y_1 \end{bmatrix}, \dots, \begin{bmatrix} X_d & \mathbf{O}_{n\times m} \\ \mathbf{O}_{m\times n} & Y_d \end{bmatrix} \right) \in (\mathcal{R}^{(n+m)\times(n+m)})^d;$$

and for a d-tuple $X = (X_1, \dots, X_d) \in (\mathcal{R}^{p\times p})^d$ and matrices $T \in \mathcal{R}^{r\times p}$, $S \in \mathcal{R}^{p\times s}$,

$$TY = (TY_1, \dots, TY_d) \in (\mathcal{R}^{r\times p})^d,$$
$$XS = (X_1 S, \dots, X_d S) \in (\mathcal{R}^{p\times s})^d;$$

that is, T and S act on d-tuples of matrices componentwise.

Let $\mathcal{M}$ and $\mathcal{N}$ be modules over $\mathcal{R}$, and let $\Omega \subseteq \mathcal{M}_{\rm nc}$ be a nc set. A mapping

$$f : \Omega \to \mathcal{N}_{\rm nc}$$

with the property that $f(\Omega_n) \subseteq \mathcal{N}^{n\times n}$, $n = 1, 2, \dots$, is called a *nc function* if f satisfies the following two conditions:

f *respects direct sums:*

$$f(X \oplus Y) = f(X) \oplus f(Y), \quad X, Y \in \Omega;$$

f *respects similarities:*

$$\text{if } X \in \Omega_n \text{ and } S \in \mathcal{R}^{n\times n} \text{ is invertible}$$
$$\text{with } SXS^{-1} \in \Omega_n, \text{ then } f(SXS^{-1}) = Sf(X)S^{-1},$$

or, equivalently, satisfies the single condition:

f *respects intertwinings* :

$$\text{if } X \in \Omega_n, Y \in \Omega_m, \text{ and } T \in \mathcal{R}^{n\times m}$$
$$\text{are such that } XT = TY, \text{ then } f(X)T = Tf(Y). \tag{3}$$

Condition (3) was used by J. L. Taylor in the case where $\mathcal{M} = \mathbb{C}^d$, together with an additional assumption of analyticity of $f(X)$ as a function of matrix entries $(X_i)_{j,k}$, $i = 1, \dots, d;\ j, k = 1, \dots, n$, for every $n \in \mathbb{N}$ — see [2].

Notice a certain discrepancy in our terminology (inherited from [5]): a set is nc if it respects direct sums, while a function is nc if it respects both direct sums and similarities. Nevertheless, we prefer to keep it this way, setting the minimal assumptions under which the theory of nc functions starts revealing its phenomena.

II. THE RESULTS

The results below are from our recent paper [23].

A part of our main results is valid under much more general assumptions on the sets of matrices and the corresponding functions. If $\mathcal{S}$ is a set, then we will write $\mathcal{S}^{n\times n}$ for a set of $n \times n$ matrices with entries from $\mathcal{S}$, where we consider matrices just as arrays, i.e., not assuming any algebraic structure on them. We will also extend the notation of (1) to an arbitrary set $\mathcal{S}$ in the place of $\mathcal{M}$.

Fix some element $O \in \mathcal{S}$. We define direct sums of matrices from $\mathcal{S}_{\rm nc}$ in the same way as in (2) except that $\mathbf{O}_{p\times q}$ is understood now as the $p\times q$ matrix whose all entries are equal to O.

Theorem 1: Let $\mathcal{S}$ be a set, $O \in \mathcal{S}$, and let $\Omega \subseteq \mathcal{S}_{\rm nc}$ respect direct sums of matrices. Define $\operatorname{supp}\Omega := \{n \in \mathbb{N} : \Omega_n \neq \emptyset\}$. Let $f : \Omega \to \Omega$ satisfy

- $f(\Omega_n) \subseteq \Omega_n$, $n \in \operatorname{supp}\Omega$;
- f respects direct sums: $f(X \oplus Y) = f(X) \oplus f(Y), \quad X, Y \in \Omega$;
- For every $n \in \operatorname{supp}\Omega$ the mapping $f|_{\Omega_n}$ has a unique fixed point, X_{*n}.

Let $d = \gcd\{n : n \in \operatorname{supp}\Omega\}$. Then

1. There exists $X_* \in \mathcal{S}^{d\times d}$ such that
$$X_{*n} = \bigoplus_{\alpha=1}^{n/d} X_*, \quad n \in \operatorname{supp}\Omega.$$
2. If, moreover, $\mathcal{S} = \mathcal{M}$ is a module over $\mathcal{R}$, $O = 0 \in \mathcal{M}$, and f is a nc function, then there exists a nc set $\tilde{\Omega} \supseteq \Omega$ with $\operatorname{supp}\tilde{\Omega} = \mathbb{N}d$ and a nc function $\tilde{f} : \tilde{\Omega} \to \tilde{\Omega}$ such that
   - $\tilde{f}|_\Omega = f$;
   - For every $n \in \mathbb{N}d$ the mapping $\tilde{f}|_{\tilde{\Omega}_n}$ has a unique fixed point
$$X_{*n} = \bigoplus_{\alpha=1}^{n/d} X_*.$$

Theorem 1 establishes the following general principle. Given a mapping on matrices which respects matrix sizes and direct sums of matrices, and given that this mapping has a unique fixed point in each matrix dimension, all these fixed points arise as multiple copies of a single matrix X_*; moreover, if the mapping is a nc function, then it can be extended to a nc function whose domain includes the matrix X_*. This principle can be used further to generalize *any fixed point theorem where the fixed point is unique* to the noncommutative setting; in Theorem 2, we will present a nc version of the Banach Fixed Point Theorem.

We will need the following definition of a complex (real) operator space (see [24], [25]) which gives rise to a natural topology on a nc space. Let $\mathbb{F}$ be a field, $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$. A vector space $\mathcal{V}$ over $\mathbb{F}$ is called an *operator space* if a sequence of Banach space norms $\|\cdot\|_n$ on $\mathcal{V}^{n\times n}$, $n = 1, 2, \dots$ is defined so that the following two conditions hold:

- For every $n, m \in \mathbb{N}$, $X \in \mathcal{V}^{n\times n}$ and $Y \in \mathcal{V}^{m\times m}$,
$$\|X \oplus Y\|_{n+m} = \max\{\|X\|_n, \|Y\|_m\};$$
- For every $n \in \mathbb{N}$, $X \in \mathcal{V}^{n\times n}$ and $S, T \in \mathbb{F}^{n\times n}$,
$$\|SXT\|_n \le \|S\|\,\|X\|_n\|T\|,$$
where $\|\cdot\|$ denotes the $(2, 2)$ operator norm on $\mathbb{F}^{n\times n}$.

Theorem 2: Let $\mathcal{S}$ be a set, $O \in \mathcal{S}$, and let $\Omega \subseteq \mathcal{S}_{\rm nc}$ respect direct sums of matrices. Suppose that Ω_n is a complete metric space with respect to a metric ρ_n for every $n \in \operatorname{supp}\Omega$. Let $f\colon \Omega \to \Omega$ satisfy

- $f(\Omega_n) \subseteq \Omega_n$, $n \in \operatorname{supp}\Omega$;
- f respects direct sums: $f(X \oplus Y) = f(X) \oplus f(Y), \quad X, Y \in \Omega$;
- For every $n \in \operatorname{supp}\Omega$ there exists $c_n\colon 0 \le c_n < 1$ so that
$$\rho_n(f(X), f(Y)) \le c_n\rho_n(X, Y), \quad X, Y \in \Omega_n.$$

Let $d = \gcd\{n : n \in \operatorname{supp}\Omega\}$. Then:

1. There exists $X_* \in \mathcal{S}^{d\times d}$ such that for every $n \in \operatorname{supp}\Omega$ the mapping $f|_{\Omega_n}$ has a unique fixed point $X_{*n} = \bigoplus_{\alpha=1}^{n/d} X_*$.
2. Suppose additionally that $\mathbb{F}$ is a field, $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$, $\mathcal{S} = \mathcal{V}$ is an operator space over $\mathbb{F}$, so that for every $n \in \operatorname{supp}\Omega$ one has $\rho_n(X, Y) = \|X - Y\|_n$, $X, Y \in \Omega_n$, $O = 0 \in \mathcal{V}$, and that f is a nc function. Then the conclusions of Theorem 1, part 2, hold; moreover $\tilde{\Omega}$ and $\tilde{f}$ can be chosen such that for every $n \in \mathbb{N}d$:
   - $\tilde{\Omega}_n$ is a complete metric space with respect to the metric
$$\tilde{\rho_n}(X, Y) = \|X - Y\|_n, \quad X, Y \in \tilde{\Omega}_n$$
(which extends the metric ρ_n for $n \in \operatorname{supp}\Omega$);
   - There exists $\tilde{c}_n\colon 0 \le \tilde{c}_n < 1$ (obviously, $\tilde{c}_n \ge c_n$ for $n \in \operatorname{supp}\Omega$) such that
$$\|\tilde{f}(X) - \tilde{f}(X_{*n})\|_n \le \tilde{c}_n\|X - X_{*n}\|_n, \quad X \in \tilde{\Omega}_n;$$
   - For an arbitrary $X^0 \in \tilde{\Omega}_n$, define $X^{j+1} = f(X^j)$, $j = 0, 1, \dots$. Then $X_{*n} = \lim_{j\to\infty} X^j$. Moreover,
$$\|X^j - X_{*n}\|_n \le (\tilde{c}_n)^j\|X^0 - X_{*n}\|_n, \quad j = 1, \dots.$$

Remark 3: We do not know whether one can find $\tilde{\Omega}$ and $\tilde{f}$ such that for every $n \in \mathbb{N}d$ there exists $c^\circ_n\colon 0 \le c^\circ_n < 1$ satisfying
$$\|\tilde{f}(X) - \tilde{f}(Y)\|_n \le c^\circ_n\|X - Y\|_n, \quad X, Y \in \tilde{\Omega},$$
and leave this as an open question.

Remark 4: We now give a useful interpretation of part 2 of Theorem 2. First of all, in a general setting of part 1 of Theorem 2, we define a relation $\sim$ on a $\Omega \subseteq \mathcal{S}_{\rm nc}$ as follows: $X \sim Y$ if both X and Y are direct sums of several copies of the same matrix over $\mathcal{S}$. Observe that $\sim$ is an equivalence relation on Ω, and define the quotient set $\widehat{\Omega} := \Omega/{\sim}$. We will call the equivalence class $\widehat{X} \in \widehat{\Omega}$ of $X \in \Omega$ a *noncommutative singleton*. Since $f\colon \Omega \to \Omega$ preserves the equivalence $\sim$, it gives rise to the quotient mapping $\widehat{f}\colon \widehat{\Omega} \to \widehat{\Omega}$. The conclusion of part 1 of Theorem 2 means that the fixed points X_{*n} are equivalent, and therefore the mapping

$\widehat{f}$ has a unique fixed point. However, the assumptions of part 1 are too general to have a good interpretation in terms of the mapping $\widehat{f}$. If one strengthen them as in part 2, then one defines a function $\rho\colon \Omega \times \Omega \to \mathbb{R}_+$ as follows. For any $n, m \in \operatorname{supp}\Omega$, $X \in \Omega_n$, $Y \in \Omega_m$, set

$$\rho(X,Y) = \left\| \bigoplus_{\alpha=1}^{m} X - \bigoplus_{\beta=1}^{n} Y \right\|_{nm}.$$

Since $\mathcal{S} = \mathcal{V}$ is an operator space, ρ extends ρ_n for every $n \in \operatorname{supp}\Omega$. Clearly, ρ is a pseudometric on Ω, and $\rho(X,Y) = 0$ if and only if $X \sim Y$. Observe that $\widehat{\Omega}$ is a metric space with respect to the quotient metric $\widehat{\rho}$ defined by $\widehat{\rho}(\widehat{X}, \widehat{Y}) := \rho(X,Y)$. Moreover, for a nc set $\widetilde{\Omega} \supseteq \Omega$ there is a natural embedding $\widehat{\Omega} \hookrightarrow \widehat{\Omega^\sim}$, and $\widehat{f}$ admits an extension $\widehat{f^\sim}$ to $\widehat{\Omega^\sim}$ which also has a unique fixed point, the equivalence class of a matrix $X_* \in \mathcal{V}^{d\times d}$, such that $X_{*n} = \bigoplus_{\alpha=1}^{n/d} X_*$ for every $n \in \operatorname{supp}\Omega$.

Even though part 2 of Theorem 2 can be interpreted as a unique fixed point theorem in a metric space, it does not follow directly from the classical Banach Fixed Point Theorem for two reasons. First, the possibility that the smallest possible Lipschitz constant for $\widehat{f}$, that is $\sup_{n\in\operatorname{supp}\Omega} c_n$, is equal to 1, is not ruled out. Second, the metric space $\widehat{\Omega}$ may be incomplete, as the following example shows.

Example 5: Let $\Omega = \mathbb{C}_{\rm nc}$, with the $(2,2)$ operator norm topology on $\mathbb{C}^{n\times n}$, $n = 1, \ldots$. Define recursively the following sequence in Ω:

$$X^0 := 1 \in \Omega_1, \quad X^{j+1} := \begin{bmatrix} X^j & \frac{1}{2^j} I_{2^j} \\ \frac{1}{2^j} I_{2^j} & X^j \end{bmatrix} \in \Omega_{2^{j+1}}.$$

This is a Cauchy sequence in pseudometric ρ. Indeed,

$$\rho(X^{j+k}, X^j) \le \|X^{j+k} - X^{j+k-1}\|_{2^{j+k}} + \cdots + \|X^{j+1} - X^j\|_{2^{j+1}} = \frac{1}{2^{j+k-1}} + \cdots + \frac{1}{2^j} \to 0$$

as $j, k \to \infty$. Correspondingly, the sequence $\{\widehat{X^j}\}_{j=1,\ldots} \in \widehat{\Omega}$ is Cauchy in metric $\widehat{\rho}$ because

$$\widehat{\rho}(\widehat{X^{j+k}}, \widehat{X^j}) = \rho(X^{j+k}, X^j) \to 0$$

as $j, k \to \infty$. Since matrices X^j have infinitely increasing number of entries in the first row, and the sequence X^j_{1k} is eventually constant for every fixed k, and the entries X^j_{1k} continuously depend on matrices X^j, the sequence of representatives of the class $\widehat{X} := \lim_{j\to\infty} \widehat{X^j}$ (provided the limit does exist) would have infinitely increasing number of entries in the first row. The latter is impossible for elements of $\widehat{\Omega}$, thus the sequence $\{\widehat{X^j}\}_{j=1,\ldots}$ does not converge, and the metric space $\widehat{\Omega}$ is incomplete.

Our proof of Theorem 1 is based on the following lemma (whose name is explained in Remark 4).

*Lemma 6 (**Noncommutative singleton lemma**):* Let $\mathcal{S}$ be a set, $O \in \mathcal{S}$. Let $\Omega \subseteq \mathcal{S}_{\rm nc}$ respect direct sums of matrices and be of the form $\Omega = \{X_n\}_{n\in\operatorname{supp}\Omega}$ with $X_n \in \Omega_n$. Then there exists $X \in \mathcal{S}^{d\times d}$, with $d = \gcd\{n\colon n \in \operatorname{supp}\Omega\}$, such that

$$X_n = \bigoplus_{\alpha=1}^{n/d} X, \quad n \in \operatorname{supp}\Omega.$$

We also apply Lemma 6 to obtain a nc version of another important principle of analysis.

*Theorem 7 (**The principle of nested nc sets**):* Let $\mathcal{S}$ be a set, $O \in \mathcal{S}$, and let $\Omega \subseteq \mathcal{S}_{\rm nc}$ respect direct sums of matrices. Suppose that Ω_n is a complete metric space with respect to a metric ρ_n for every $n \in \operatorname{supp}\Omega$. Given a sequence of sets

$$\Omega^j = \coprod_{n\in\operatorname{supp}\Omega} \Omega^j_n \subseteq \Omega, \quad j = 1, \ldots$$

such that

- Ω^j respects direct sum of matrices, for every $j = 1, \ldots$;
- Ω^j_n is a non-empty closed subset of Ω_n, for every $n \in \operatorname{supp}\Omega$, $j = 1, \ldots$;
- $\Omega_1 \supseteq \Omega_2 \supseteq \cdots$;
- $\operatorname{diam}\Omega^j_n := \sup_{X,Y\in\Omega^j_n} \rho_n(X,Y) \to 0$ as $j \to \infty$, for every $n \in \operatorname{supp}\Omega$,

there exists a unique $X_* \in \mathcal{S}^{d\times d}$, with $d = \gcd\{n\colon n \in \operatorname{supp}\Omega\}$, such that

$$\bigcap_{j=1}^{\infty} \Omega^j = \left\{ \bigoplus_{\alpha=1}^{n/d} X_* \right\}_{n\in\operatorname{supp}\Omega}.$$

Remark 8: Similarly to Remark 4, we can interpret the conclusion of Theorem 7 in terms of quotient sets of nc sets. Namely, in the assumptions of Theorem 7, the sequence of nested quotient sets $\widehat{\Omega^j}$ has a nonempty intersection consisting of a single point. Again, the assumptions on the metrics ρ_n are too general to have a good interpretation in terms of quotient sets. If we assume, as in part 2 of Theorem 2, that $\mathcal{S} = \mathcal{V}$ is an operator space and the metrics ρ_n are norm-induced, then we can define the corresponding pseudometric ρ on Ω which extends every ρ_n and the metric $\widehat{\rho}$ on $\widehat{\Omega}$ as in Remark 4. Clearly, the sequence $\operatorname{diam}\widehat{\Omega^j} := \sup_{\widehat{X},\widehat{Y}\in\widehat{\Omega^j}} \widehat{\rho}(\widehat{X}, \widehat{Y})$ is non-increasing. However, it may happen that it does not converge to 0. Also, the sets $\widehat{\Omega^j}$ are not necessarily complete metric spaces. So, both the assumptions of the classical principle of nested closed sets may fail in our case, as confirmed by the following example.

Example 9: Let $\Omega = \mathbb{C}_{\rm nc}$, with the $(2,2)$ operator norm topology on $\mathbb{C}^{n\times n}$, $n = 1, \ldots$. Let

$$\Omega^j_n := \left\{ X \in \mathbb{C}^{n\times n}\colon 0 \le \|X\|_n \le \frac{n}{n+j} \right\}, \quad n, j = 1, \ldots.$$

It is easy to see that the sets Ω^j_n satisfy the conditions of Theorem 7 and, for a fixed n, $0_{n\times n}$ is the unique common point of the sets Ω^j_n. Define

$$X^j_n := \frac{n}{n+j} I_n \in \Omega^j_n, \quad Y^j_n := -X^j_n \in \Omega^j_n \quad n, j = 1, \ldots.$$

We have $\|X^j_n - Y^j_n\|_n \to 2$ as $n \to \infty$, hence $\operatorname{diam}\widehat{\Omega^j} = 2$ for every j, and $\lim_{j\to\infty} \operatorname{diam}\widehat{\Omega^j} \neq 0$. The sequence $\{X^j_n\}_{n=1,\ldots}$ is Cauchy in pseudometric ρ for every fixed j, and so is the sequence $\{\widehat{X_n^j}\}_{n=1,\dots}$ in metric $\widehat{\rho}$. However, the latter has no limit in $\widehat{\Omega^j}$, since such a limit would be a class whose representative are of norm 1, which is impossible.

We now present an application of Theorem 2 to initial value problems for ODEs in nc spaces. The following theorem is a nc counterpart of (a version of) the classical existence and uniqueness theorem for solutions of ODEs.

Theorem 10: Let $\mathcal{I}$ be an interval in $\mathbb{R}$ and let t_0 be a point in the interior of $\mathcal{I}$. Let $\mathcal{V}$ be a (real or complex) operator space and let $\Xi \subseteq \mathcal{V}_{\rm nc}$ be a nc set, with Ξ_n a closed subspace of the Banach space $\mathcal{V}^{n\times n}$ for every $n \in \operatorname{supp}\Xi$. Let $X_{0n} \in \Xi_n$, $n \in \operatorname{supp}\Xi$, and let $\{X_{0n}\}_{n\in\operatorname{supp}\Xi}$ be a nc set. Suppose that $g\colon \mathcal{I}\times\Xi \to \Xi$ satisfies the conditions:

- $g(t,\cdot)$ maps Ξ_n to itself, $n \in \operatorname{supp}\Xi$, and respects direct sums of matrices, for every $t \in \mathcal{I}$;
- $g_n := g|_{\mathcal{I}\times\Xi_n}$ is continuous for every $n \in \operatorname{supp}\Xi$;
- There is a constant $C > 0$ such that
$$\|g(t,X) - g(t,Y)\|_n \le C\|X-Y\|_n$$
for every $t \in \mathcal{I}$, $n \in \operatorname{supp}\Xi$, and $X, Y \in \Xi_n$.

Then

1. There is a matrix $X_0 \in \mathcal{V}^{d\times d}$, with $d = \gcd\{n\colon n \in \operatorname{supp}\Xi\}$, such that
$$X_{0n} = \bigoplus_{\alpha=1}^{n/d} X_0, \quad n \in \operatorname{supp}\Xi.$$
2. There is a continuously differentiable function $X_*\colon \mathcal{I} \to \mathcal{V}^{d\times d}$ such that, for every $n \in \operatorname{supp}\Xi$,
$$X_{*n} = \bigoplus_{\alpha=1}^{n/d} X_*\colon \mathcal{I} \to \Xi_n$$
is a unique solution of the initial value problem for the first-order ODE
$$\dot{X} = g_n(t,X), \quad X(t_0) = X_{0n}.$$
3. Suppose that, in addition, $g(t,\cdot)$ respects similarities of matrices, thus is a nc function, for every $t \in \mathcal{I}$. Then for every $t \in \mathcal{I}$ there exist a nc set $\widetilde{\Xi}(t) \supseteq \Xi$ with $\operatorname{supp}\widetilde{\Xi}(t) = \mathbb{N}d$, so that for every $n \in \mathbb{N}d$ one has a fiber bundle Ψ_n with the total space
$$\widetilde{\Xi}_n = \coprod_{t\in\mathcal{I}} \widetilde{\Xi}(t)_n \subseteq \mathcal{I}\times\mathcal{V}^{n\times n},$$
the base space $\mathcal{I}$ and the projection $\pi_n\colon \widetilde{\Xi}_n \to \mathcal{I}$ defined by $\pi_n\colon \widetilde{X}(t) \mapsto t$; a map $\widetilde{g}$ of the set $\coprod_{t\in\mathcal{I}}\widetilde{\Xi}(t)$ to itself such that $\widetilde{g}_n := \widetilde{g}|_{\widetilde{\Xi}_n}$ is a continuous bundle endomorphism for every $n = \mathbb{N}d$, that extends the function g (where we identify all copies of Ξ_n in $\widetilde{\Xi}(t)_n$, $t \in \mathcal{I}$); and, for every $n \in \mathbb{N}d$, a unique continuously differentiable cross-section of the fiber bundle Ψ_n,
$$X_{*n} = \bigoplus_{\alpha=1}^{n/d} X_*\colon \mathcal{I} \to \widetilde{\Xi}_n,$$
which is a solution of the initial value problem for the first-order ODE
$$\dot{X} = \widetilde{g}_n(t,X), \quad X(t_0) = \bigoplus_{\alpha=1}^{n/d} X_0.$$

REFERENCES

[1] J. L. Taylor, A general framework for a multi-operator functional calculus. *Advances in Math.* 9 (1972), 183–25.
[2] J. L. Taylor, Functions of several noncommuting variables. *Bull. Amer. Math. Soc.,* 79 (1973), 1–34.
[3] D. Voiculescu, Free analysis questions. I. Duality transform for the coalgebra of $\partial_{X:B}$, *Int. Math. Res. Not.* 16 (2004), 793–822.
[4] D.-V. Voiculescu, Free analysis questions II: the Grassmannian completion and the series expansions at the origin. *J. Reine Angew. Math.* 645 (2010), 155–236.
[5] D. S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, Foundations of noncommutative function theory. arXiv:1212.6345.
[6] J. W. Helton, Manipulating matrix inequalities automatically. Mathematical systems theory in biology, communications, computation, and finance (Notre Dame, IN, 2002), 237–256, IMA Vol. Math. Appl., 134, Springer, New York, 2003.
[7] J. W. Helton, S. A. McCullough, and V. Vinnikov, Noncommutative convexity arises from linear matrix inequalities. *J. Funct. Anal.* 240 (2006), no. 1, 105–191.
[8] J. W. Helton, S. A. McCullough, M. Putinar, and V. Vinnikov, Convex matrix inequalities versus linear matrix inequalities. *IEEE Trans. Aut. Control* 54 (2009), 952–964.
[9] D. S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, Singularities of rational functions and minimal factorizations: the noncommutative and the commutative setting. *Linear Algebra Appl.* 430 (2009), no. 4, 869–889.
[10] D. S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, Noncommutative rational functions, their difference-differential calculus and realizations. *Multidimens. Syst. Signal Process.* 23 (2012), no. 1–2, 49–77.
[11] J. W. Helton, "Positive" noncommutative polynomials are sums of squares. *Ann. of Math.* (2) 156 (2002), no. 2, 675–694.
[12] J. W. Helton and S. A. McCullough, A Positivstellensatz for noncommutative polynomials. *Trans. Amer. Math. Soc.* 356 (2004), no. 9, 3721–3737 (electronic).
[13] J. W. Helton and M. Putinar, Positive polynomials in scalar and matrix variables, the spectral theorem, and optimization. Operator theory, structured matrices, and dilations, 229–306, Theta Ser. Adv. Math., 7, Theta, Bucharest, 2007.
[14] J. W. Helton, I. Klep, and S. A. McCullough, The convex Positivestellensatz in a free algebra, *Adv. Math.* 231 (2012), 516–534.
[15] J. W. Helton, I. Klep, and S. A. McCullough, Proper analytic free maps. *J. Funct. Anal.* 260 (2011), no. 5, 1476–1490.
[16] J. W. Helton, I. Klep, and S. A. McCullough, Analytic mappings between noncommutative pencil balls. *J. Math. Anal. Appl.* 376 (2011), no. 2, 407–428.
[17] G. Popescu, Free holomorphic functions on the unit ball of $B(\mathcal{H})^n$. *J. Funct. Anal.* 241 (2006), no. 1, 268–333.
[18] G. Popescu, Free holomorphic functions on the unit ball of $B(\mathcal{H})^n$. II. *J. Funct. Anal.* 258 (2010), no. 5, 1513–1578.
[19] G. Popescu, Operator theory on noncommutative domains. *Mem. Amer. Math. Soc.* 205 (2010), no. 964, vi+124 pp.
[20] P. Muhly and B. Solel, Progress in noncommutative function theory. *Science China Mathematics* 54 (2011), no. 11, 2275–2294.
[21] J. Agler and J.E. McCarthy, Global holomorphic functions in several non-commuting variables. arXiv:1305.1636.
[22] J. Agler and N.J. Young, Symmetric functions of two noncommuting variables. arXiv:1307.1588.
[23] G. Abduvalieva and D.S. Kaliuzhnyi-Verbovetskyi, Fixed point theorems for noncommutative functions. J. Math. Anal. Appl., 401 (2013), no. 1, 436-446.
[24] E. G. Effros and Zh.-J. Ruan, Operator spaces. London Mathematical Society Monographs. New Series, 23. The Clarendon Press, Oxford University Press, New York, 2000. xvi+363 pp.
[25] Zh. J. Ruan, On real operator spaces. International Workshop on Operator Algebra and Operator Theory (Linfen, 2001). *Acta Math. Sin. (Engl. Ser.)* 19 (2003), no. 3, 485–496.