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Some Ergodic Control Problems for Linear Stochastic Equations in a Hilbert Space with Fractional Brownian Motions

T. E. Duncan
Department of Mathematics
University of Kansas
Lawrence, Kansas 66045, USA
duncan@math.ku.edu

B. Maslowski
Faculty of Mathematics and Physics
Charles University
Prague, Czech Republic
maslow@karlin.mff.cuni.cz

B. Pasik-Duncan
Department of Mathematics
University of Kansas
Lawrence, Kansas 66045, USA
bozenna@math.ku.edu

***Abstract*— A linear-quadratic control problem with an infinite time horizon for some infinite dimensional controlled stochastic differential equations driven by a fractional Brownian motion is formulated and solved. The feedback form of the optimal control and the optimal cost are given explicitly. The optimal control is the sum of the well known linear feedback control for the associated infinite dimensional deterministic linear-quadratic control problem and a suitable prediction of the adjoint optimal system response to the future noise. The noise covariance operator as well as the control operator in the system equation can in general be unbounded, so the results can also be applied where the noise or the control are on the boundary of the domain or at discrete points in the domain. A continuous dependence of the optimal cost on some parameters of the systems is also verified. Some examples of controlled stochastic partial differential equations are given.**

I. Problem Formulation

Consider the infinite dimensional controlled linear stochastic equation

$$dX(t) = (AX(t)+Bu(t))dt+dB_H(t) \quad \text{(I.1)}$$

$$X(0) = x \quad \text{(I.2)}$$

where $x \in V$, $X(t) \in V$, V is an infinite dimensional real separable Hilbert space with inner product $<\cdot,\cdot>$ and norm $|\cdot|$. The process $(B_H(t), t \geq 0)$ is a V-valued fractional Brownian motion with the Hurst parameter $H \in (\frac{1}{2}, 1)$ and having the incremental covariance $\widetilde{Q}$ where $\widetilde{Q}$ is trace class $(Tr(\widetilde{Q}) < \infty)$ so that

$$\mathbb{E} < B_H(t), x >< B_H(s), y > = \frac{1}{2} < \widetilde{Q}x, y > (t^{2H} + s^{2H} - |t-s|^{2H}). \quad \text{(I.3)}$$

for $x, y \in V$. The operator $A : Dom(A) \to V$ with $Dom(A) \subset V$ is a linear, densely defined operator on V which is the infinitesimal generator of a strongly continuous semigroup $(S(t), t \geq 0)$. Let $\mathscr{U} = (U, <\cdot,\cdot>_U, |\cdot|_U)$ be another Hilbert space, the state space of controls, and assume that $B \in \mathscr{L}(U, V)$. Furthermore consider the family of admissible controls, $\mathscr{U}$, defined as follows

$$\mathscr{U} = \{u : \mathbb{R}_+ \times \Omega \to U, u \text{ is progressively measurable, } \mathbb{E}\int_0^T |u(t)|_U^2 dt < \infty \text{ for all } T > 0\}$$

where the progressive measurability is understood with respect to the stochastic basis $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}(t), t \geq 0))$ where $(\mathscr{F}(t), t \geq 0)$ is the natural filtration induced by the fractional Brownian motion $(B_H(t), t \geq 0)$.

The solution of the equation (I.1) is defined as the mild solution, that is,

$$X(t) = S(t)x + \int_0^t S(t-s)Bu(t)dt + \int_0^t S(t-s)dB_H(t) \quad \text{(I.4)}$$

for $t \geq 0$ and it is known that with the above assumptions there is one and only one V- continuous solution to (I.1) (cf. [2], [3] and the references therein). Now the cost functional is defined for the control problem. Let J_T be given as follows

$$J_T(x,u) := \frac{1}{2}\int_0^T (|LX(s)|^2 + < Ru(s), u(s) >_U) ds \quad \text{(I.5)}$$

where $L \in \mathscr{L}(V)$, $R \in \mathscr{L}(U)$, R is self-adjoint and invertible. The control problem is to minimize the following ergodic cost

$$lim\ sup_{T\to\infty} \frac{1}{T}\mathbb{E}J_T(x,u). \quad \text{(I.6)}$$

The following standard conditions are imposed.

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(A1) There are $K \in \mathscr{L}(V), M_K > 0,$ and $\omega_K > 0$ such that

$$|e^{(A+KL)t}|_{\mathscr{L}(V)} \leq M_K e^{-\omega_K t} \tag{I.7}$$

for all $t > 0$ (detectability).

(A2) There are $F \in \mathscr{L}(V,U), M_F > 0,$ and $\omega_F > 0$ such that

$$|e^{(A+BF)t}|_{\mathscr{L}(V)} \leq M_F e^{-\omega_F t} \tag{I.8}$$

for all $t > 0$ (stabilizability).

II. Main Result

In this section the ergodic control problem described by ((I.1) - (I.6)) is solved. The following result on the solution of the algebraic Riccati equation for the control problem here is used ([9], Theorem 4.4 or [1], Theorem 6..2.7).

Proposition 1: If (A1)-(A2) are satisfied then there is a unique self-adjoint nonnegative operator $P \in \mathscr{L}(V)$ satisfying the following algebraic Riccati equation

$$\begin{aligned} &< Px, Ay > + < Ax, Py > + < L^* Lx, y > \\ &\quad - < R^{-1} B^* Px, B^* Py > = 0 \end{aligned} \tag{II.1}$$

for all $x, y \in Dom(A)$. Moreover the strongly continuous semigroup $(\Phi(t), t \geq 0)$ generated by $A_P = A - BR^{-1}B^*P$ is exponentially stable, that is

$$|\Phi(t)|_{\mathscr{L}(V)} \leq M_P e^{-\widetilde{\omega} t} \tag{II.2}$$

for some constants $M_P > 0$ and $\widetilde{\omega} > 0$.

Let $\Psi(t) = \Phi^*(t)$ be the adjoint semigroup of $(\Phi(t)), t \geq 0)$ that is generated by A_P^*. It is well known [3] that the stochastic integral

$$\varphi_T(t) = \int_t^T \Psi(s-t) P dB_H(s) \tag{II.3}$$

for $t \in [0,T]$ is a well defined centered Gaussian process in $L^p(\Omega \times (0,T), V)$for each $p \in [1, \infty)$. Define V_T and V as

$$\begin{aligned} V_T(t) &= \mathbb{E}[\varphi_T(t)|\mathscr{F}(t)] \\ V(t) &= \mathbb{E}[\varphi(t)|\mathscr{F}(t)] \end{aligned} \tag{II.4}$$

where

$$\varphi(t) = \int_t^\infty \Psi(s-t) P dB_H(s). \tag{II.5}$$

By (II.2) the semigroup $(\Psi(t), t \geq 0)$ is exponentially stable so it is easy to see that $(\varphi(t), t \geq 0)$ and $(V(t), t \geq 0)$ are well-defined processes with values in $L^p(\Omega \times (0,T); V)$ for each $T > 0$ and $1 \leq p < \infty$. The following theorem is the main result of this section.

Theorem 1: Let (A1)-(A2) be satisfied and let $u \in \mathscr{U}$ be a control satisfying

$$lim_{T\to\infty} \frac{1}{T} \mathbb{E} < PX^u(T), X^u(T) >= 0 \tag{II.6}$$

where $(X^u(T), T \in [0, \infty))$ is the solution to (I.1) with the control $u \in \mathscr{U}$. Then

$$lim\ sup_{T\to\infty} \frac{1}{T} \mathbb{E} J_T(x,u) \geq J_\infty \tag{II.7}$$

where

$$\begin{aligned} J_\infty := lim\ sup_{T\to\infty} \frac{1}{2T} \mathbb{E} \int_0^T |R^{\frac{1}{2}} B^* V(s)|_U^2 ds \qquad &\text{(II.8)} \\ + \int_0^\infty Tr(\widetilde{Q} P \Phi(t)) \phi_H(r) dr \qquad &\text{(II.9)} \end{aligned}$$

for each $x \in V$ where $\phi(r) = H(2H-1)|r|^{2H-2}, r \in \mathbb{R}$. Moreover, the feedback control $\hat{u}(t) = -R^{-1}B^*(PX^{\hat{u}}(s) + V(s))$ is admissible, satisfies the condition (II.6) and

$$lim\ sup_{T\to\infty} \frac{1}{T} \mathbb{E} J_T(x, \hat{u}) = J_\infty \tag{II.10}$$

for each $x \in V$. Thus $\hat{u}$ is an optimal ergodic control and J_∞ is the optimal cost for the ergodic control problem ((I.1)-(I.6)).

An unknown parameter can be introduced with A in the infinitesimal generator A in the system equation (I.1) and continuity properties of the optimal cost can be established with respect to this parameter and an adaptive control problem can be solved.

III. Examples

Example 1 (stochastic parabolic equation): Consider the $2m$-th order controlled stochastic parabolic equation

$$\frac{\partial y}{\partial t}(t,\xi) = (L_{2m}y)(t,\xi) + (Bu_t)(\xi) + \eta^H(t,\xi) \qquad \text{(III.1)}$$

for $(t,\xi) \in \mathbb{R}_+ \times D$ with the initial condition

$$y(0,\xi) = x(\xi) \qquad \text{(III.2)}$$

for $\xi \in D$ and the Dirichlet boundary conditions

$$\frac{\partial^k}{\partial \nu^k} y(t,\xi) = 0 \qquad \text{(III.3)}$$

for $(t,\xi) \in \mathbb{R}_+ \times \partial D$, $k = 0,1,\ldots,m-1$, where $D \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary, $\frac{\partial}{\partial \nu}$ stands for conormal derivative, $x \in L^2(D)$ and L_{2m} is a $2m$th order uniformly elliptic operator of the form

$$L_{2m} = \sum_{|\alpha| \le 2m} a_\alpha(\xi) D^\alpha$$

with $a_\alpha \in C_b^\infty(D)$. Furthermore, $u \in \mathscr{U}$ where the set $\mathscr{U}$ of admissible controls is defined in Section 2 with an arbitrary Hilbert space U, and $B \in \mathscr{L}(U, L^2(D))$. Finally, η^H formally describes a space-dependent fractional noise with the Hurst parameter $H > \frac{1}{2}$. As usual, the formal system (III.1)–(III.3) is rewritten in the form (1)–(2) where $V = L^2(D)$, $A = L_{2m}$,

$$\mathrm{Dom}(A) = \{\varphi \in H^{2m}(D) | \frac{\partial^k}{\partial \nu^k}\varphi = 0 \text{ on } \partial D \; ; \; k = 0,\ldots,m-1\}$$

and $(B_H(t), t > 0)$ is an $L^2(D)$-valued fractional Brownian motion with a trace class incremental covariance $\widetilde{Q}$ (formally $\eta^H(t,\cdot) = \frac{\partial}{\partial t} B_H(t)$) and $H \in (\frac{1}{2}, 1)$. It is well known that the operator A generates a strongly continuous (and analytic) semigroup $(S(t), t \ge 0)$. The cost functional takes the form (I.6) where $L \in \mathscr{L}(L^2(D))$ and $R \in \mathscr{L}(U)$ is self-adjoint and invertible.

In order to verify applicability of Theorem 1 to the above formulated control problem it is necessary to examine the detectability and the stabilizability conditions (A1) and (A2). By [9], Theorem 3.3 (A1) and (A2) follow from exact null controllability of the pairs (A^*, L^*) and (A, B), respectively, that is, the following conditions must be satisfied

$$\mathrm{Range}(S^*(T)) \subset \mathrm{Range}((\int_0^T S^*(r)L^*LS(r)dr)^{1/2}) \qquad \text{(III.4)}$$

and

$$\mathrm{Range}(S(T)) \subset \mathrm{Range}((\int_0^T S(r)BB^*S^*(r)dr)^{1/2}) \qquad \text{(III.5)}$$

for some $T > 0$. Conditions of the type (III.4), (III.5) have been widely studied. For example, they are satisfied if $L^*L \ge \alpha I$, $BB^* \ge \alpha I$ for some $\alpha > 0$ (see e.g. [2], Remark B.9). Furthermore (III.5) is equivalent to the strong Feller property of solutions to the auxiliary stochastic linear equation

$$dY(t) = AY(t)dt + BdW(t), \; t > 0, \qquad \text{(III.6)}$$

where $(W(t), t \ge 0)$ is an arbitrary cylindrical Wiener process on U (and similarly for (III.4)). The strong Feller property of processes defined by (III.6) has also been extensively studied (see e.g. [8] for a general result of this type).

Example 2 (stochastic wave equation): General results of Section 2 may also be applied to stochastic controlled hyperbolic equations. Only a simple example following Example 6.14 in [1] is given. Consider the controlled stochastic wave equation

$$\frac{\partial^2 w}{\partial t^2}(t,\xi) = \frac{\partial^2 w}{\partial \xi^2}(t,\xi) + u_t(\xi) + \eta^H(t,\xi) \qquad \text{(III.7)}$$

for $(t,\xi) \in \mathbb{R}_+ \times (0,1)$ with the boundary condition

$$w(t,0) = w(t,1) = 0, \; t > 0, \qquad \text{(III.8)}$$

and initial condition

$$w(0,\xi) = x_1(\xi), \; \frac{\partial w}{\partial t}(0,\xi) = x_2(\xi), \; \xi \in (0,1),$$

where $u_t \in L^2(0,1)$ and η^H is a fractional noise on $L^2(0,1)$. The above system is rewritten in the abstract form (1)–(2) in the space $V = \mathrm{Dom}(A_0^{1/2}) \times L^2(0,1)$ with

$$A_0 = \frac{\partial^2}{\partial \xi^2}, \quad \mathrm{Dom}(A_0) = \{h \in L^2(0,1); h, h' \text{ absolutely continuous}, \; h'' \in L^2(0,1), h(0) = h(1) = 0\}.$$

Then the operator

$$A := \begin{pmatrix} 0 & I \\ A_0 & 0 \end{pmatrix},\ \mathrm{Dom}(A) = \mathrm{Dom}(A_0) \times \mathrm{Dom}(A_0^{1/2})$$

generates a strongly continuous semigroup in V and let

$$B = \begin{pmatrix} 0 \\ I \end{pmatrix} \in \mathscr{L}(L^2(0,1),V),\ U = L^2(0,1).$$

For the noise term choose an arbitrary fractional Brownian motion $(B_H(t), t \geq 0)$ in V with $H \in (\frac{1}{2}, 1)$ and an incremental covariance of the form

$$\widetilde{Q} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{Q}_1 \end{pmatrix} \in \mathscr{L}(V)$$

where $\widetilde{Q}_1$ is a nonnegative, self-adjoint and trace class operator on $L^2(0,1)$. The cost functional takes the form (I.6) where

$$J_T(x,u) = \int_0^T \int_0^1 |X_2(t)|^2 + |u_t(\xi)|^2 d\xi dt,$$
$$x = (x_1, x_2),\ X(t) = (X_1(t), X_2(t)).$$

In this case the detectability and stabilizability conditions (A1), (A2) are satisfied ([1], p. 310) and Theorem 1 is applicable.

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