

# Geometric-PBC Approach for Control of Circular Ball and Beam System\*

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**Abstract**—In this paper, a nonlinear control law is proposed for the curved ball and beam system with the ball at the upright position. The stabilization of such a system is challenging, as compared to conventional ball and (straight) beam system, due to the presence of two unstable equilibrium points and the gyroscopic forces appearing in its dynamics. Further, the system belongs to the interesting class of underactuated mechanical systems. We propose Geometric Passivity Based Control (PBC) methodology for synthesizing stabilizing control law for the system under consideration. The power flow between the controller and the unactuated dynamics is established by manipulating the mechanical connection. An additional dissipation term is included in the control law to deal with the gyroscopic forces. The main advantage of the proposed controller design strategy is that it does not involve solving any Partial Differential Equations (PDEs) or nonlinear transformations. The simulation results are presented to validate the control law. The system parameters used in simulation adheres to the physical model constraints.

**Keywords:** Control systems, stability, control of mechanical systems, circular ball and beam system.

## I. INTRODUCTION

The ball on curved beam system was originally developed in [1] from its original variant of straight beam and ball system. Although some control laws has been developed by linearizing the system, much work need to be done on nonlinear control law design for the ball on curved beam system.

On the linear control law design side, [1], [2] used the Jordan form of the linear model to extract the unstable part and obtain the control law with bounded inputs. In [3] the linear control law was developed using input and state constraints adopting Linear Matrix Inequality (LMI) based optimization. The cost function was minimized to result in a set of initial conditions for which the system trajectories satisfy the state constraint set. An experimental set-up was prepared in [4] by replacing straight beam by a curved beam mounted away from its center of rotation. The controller was designed using the Linear Quadratic Regulator (LQR) design approach. On nonlinear control, [5] obtains the control law using partial feedback linearization and backstepping. The nonlinear coordinate transformations are applied to obtain at a state-space representation resulting in standard quadratic non-triangular normal form. The backstepping methodology is then employed to design the control law.

\* We acknowledge World Bank funding under TEQIP Phase-II, Sub component 1.2.1.

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Controlled Lagrangian [6], [14], [15], [18], [19] forms another approach for nonlinear controller design of the class of underactuated mechanical systems. The method gives a control law that yields a closed-loop system in Lagrangian form under sufficient conditions, called matching conditions. The matching conditions ensure that the Euler Lagrange equations derived from the controlled Lagrangian are consistent with available control inputs. This approach is difficult to apply to the systems consisting of gyroscopic terms since, the simplified matching conditions are not satisfied which fails to give a stabilizing control law in the full state space. One such class of system with gyroscopic terms was described in a novel way in [10]. A control law was derived by converting the Furuta pendulum look like planer pendulum on a cart plus gyroscopic term using a suitable feedback. The planer pendulum is stabilized using a controlled Lagrangian approach [14] and an additional control is derived to handle these gyroscopic terms.

Passivity is a useful tool for the analysis of nonlinear systems [12], [13], [16] and is well related to the Lyapunov stability method. An equivalent method to control Lagrangian in Hamiltonian framework is Interconnection and damping assignment passivity-based control (IDA-PBC) control methodology [7], [8], [9], [11], [17] and [20]. A passivity based controller design was done in [21], [23] for control of a 2D SpiderCrane and PVTOL system with underactuation degree one. A similar passivity based controller was designed using backstepping technique for the chaotic 2D SpiderCrane System in [22].

In this paper, we present a novel nonlinear design methodology for the control of ball on a curved beam system. The control design is based on the intrinsic geometry of the mechanical system being controlled and passivity based control. The symmetries of mechanical system decompose the tangent space into a vertical component along the external variables (actuated variables) and a horizontal component which projects onto the base space. The structure of the split tangent space is modified along the actuated direction in such a way that the power flow between the controller and the system is channelized both to the actuated and the unactuated subsystems. In the system under consideration, there exist gyroscopic forces which adds to the complexity of controller design. To handle these forces controller is split in two parts, energy shaping controller to attain required equilibrium and dissipative controller to maintain stability of the system even in the presence of gyroscopic terms. The passive outputs of the subsystems can be identified which leads to the passivity

based formulation for the control of the entire system. The ideas of kinetic and potential shaping are formulated in terms of the passive outputs and the desired controller is obtained by using passivity based techniques.

The paper is organised as follows: The Geometric-PBC approach is presented in Section II. The control problem for circular ball and beam system is described using a geometrical approach in Section III. The control law design and stability analysis conducted, based upon the shaped energy, is discussed in Section IV. Finally in Section V, numerical simulations results for the same are presented that verifies the proposed control law.

## II. GEOMETRIC-PBC APPROACH

The geometric theory which forms a background to Geometric-PBC approach for controlling a class of underactuated mechanical systems is briefly described in this section. Readers are directed to [6] for more elaborate discussion. Here we follow some notations as [6] and some known facts are repeated here to make the paper self-contained.

*The setting:* The system has configuration space  $Q$  and Lie group  $G$  that acts freely and properly on  $Q$ . In our case,  $Q = S \times G$  with the Lie group  $G$  acting only on the second factor by acting on left by group multiplication. The goal of Geometric-PBC theory is to control the variables lying in the shape space  $Q/G$  using actuation which directly acts on  $G$  only. For a large class of underactuated mechanical systems of interest the Lagrangian  $L : TQ \rightarrow R$  is invariant under the action of  $G$  on  $Q$ . This implies that the Lagrangian  $L$  is cyclic in  $G$  variable. A special case of the above is when, only the kinetic energy term of the Lagrangian has the property of being invariant under the action of  $G$ .

*Principal connection:* A principal connection on the principal bundle  $\pi : Q \rightarrow Q/G$  is a map  $A : TQ \rightarrow \mathfrak{g}$  (where,  $A$  is a  $\mathfrak{g}$ -valued one-form) that is linear on each tangent space and at each point  $q \in Q$  we have the decomposition of the tangent space  $T_qQ = \text{Hor}_q \oplus \text{Ver}_q$ . The horizontal space of the connection at  $q \in Q$  is the linear space,  $\text{Hor}_q = (v_q \in T_qQ \mid A(v_q) = 0)$ .

A connection is uniquely defined by the specification of its horizontal space. Given a connection  $A$  the vector  $v_q \in T_qQ$  is decomposed as  $v_q = \text{Hor} v_q + \text{Ver} v_q$  where,  $\text{Ver}_q v = [A(q, v)]_Q(q)$  and  $\text{Hor}_q v = v - \text{Ver}_q v$ . Here,  $[A(q, v)]_Q$  denotes the infinitesimal generator corresponding to the Lie algebra element  $[A(q, v)]$ . The principal connection is a special case of the vertical valued Ehresmann connection defined on fiber bundle.

The symmetry of the mechanical system under the action of Lie group  $G$ , i.e. invariance of the Lagrangian under  $G$ , gives rise to what is termed as mechanical connection on the principal bundle  $\pi : Q \rightarrow Q/G$ . The action of the mechanical connection can be described in terms of  $\tau_m$ , a Lie-algebra valued horizontal one form on  $Q$ .

*Definition :*  $\tau_m$  is a horizontal one form on  $Q$  with values in the Lie algebra  $\mathfrak{g}$  of  $G$  that annihilates the vertical vectors. If  $v$  is a vertical vector, i.e. a vector along the group direction then  $[\tau_m(v)]_Q$  is the zero vector field on  $Q$ . The  $\tau_m$  horizontal

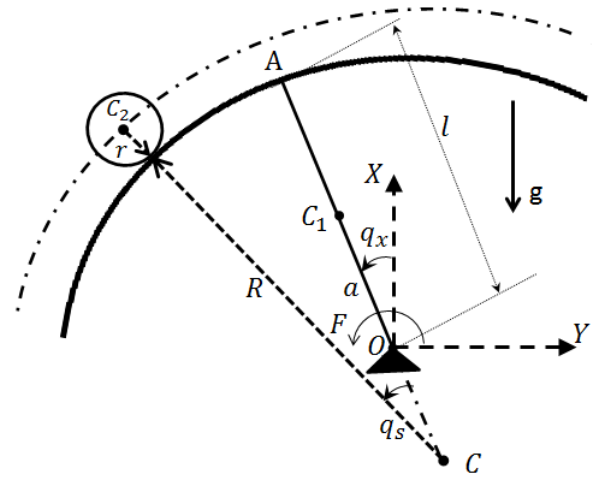


Fig. 1: Circular Ball and Beam System

space at  $q \in Q$  consists of tangent vectors at  $q \in Q$  of the form,  $\text{Hor}_{\tau_m} v_q = \text{Hor}(v_q) - [\tau_m(v)]_Q(q)$  and  $\text{Ver}_{\tau_m} v_q = \text{Ver}(v_q) + [\tau_m(v)]_Q(q)$ . It is obvious that  $v_q = \text{Hor}_{\tau_m}(v_q) + \text{Ver}_{\tau_m}(v_q)$ .

*Note:* Our definition of  $\text{Hor}(v_q)$  and  $\text{Ver}(v_q)$  with an example of circular Ball on a beam is explained in the next section, in terms of  $q_x$  and  $q_s$ .

## III. CIRCULAR BALL AND BEAM SYSTEM

Consider the circular ball and beam system presented in [1], where the beam is an arc with center  $C$ , radius  $R$  and a ball on top of it with center  $C_2$  and radius  $r$  as shown in Fig. 1. Beam is actuated through a motor shaft with torque  $F$  at  $O$ .  $C_1$  is the center of mass of the beam with its holder  $OA$ . Let,  $m_1$ ,  $\rho_1$  and  $m_2$ ,  $\rho_2$  be the mass and radius of inertia of the beam with its motor shaft and ball respectively. Let,  $l$  be the length of the shaft,  $OC_1 = a$  and  $g$  be the acceleration due to gravity. The beam angle i.e., angle from the vertical is denoted by  $q_x$  and the ball position on the curved beam is given by  $Rq_s$  where,  $q_s$  is the angle from the center  $C$  as marked in Fig. 1.

The curved ball and beam system has two degree of freedom and the configuration space  $Q \in \mathbb{R}^2$  is given by  $Q = S \times G$ , with the configuration variables  $(q_x, q_s)$ . The tangent space  $T_qQ$  has coordinates  $(q, \dot{q})$ . For the system considered no frictional forces are taken under account. The equation of motion for this system is given by [1], [3],

$$m_{xx}(q_s)\ddot{q}_x + m_{xs}(q_s)\ddot{q}_s + h_x(q, \dot{q}) + c_x(q, \dot{q}) = F \quad (1)$$

$$m_{sx}(q_s)\ddot{q}_x + m_{ss}(q_s)\ddot{q}_s + h_s(q, \dot{q}) + c_s(q, \dot{q}) = 0 \quad (2)$$

where,

$$\begin{aligned}
m_{xx}(q_s) &= a_1 + a_2 - 2a_3 \cos q_s \\
m_{xs}(q_s) &= m_{sx}(q_s) = a_2 - a_3 \cos q_s \\
m_{ss}(q_s) &= a_2 + a_4 \\
h_x(q, \dot{q}) &= a_3 \dot{q}_x \dot{q}_s \sin q_s + a_3 \dot{q}_s^2 \sin q_s - b_1 \sin q_x \\
&\quad - b_2 \sin(q_x + q_s) \\
h_s(q, \dot{q}) &= -b_2 \sin(q_x + q_s) \\
c_x(q, \dot{q}) &= a_3 \dot{q}_x \dot{q}_s \sin q_s \\
c_s(q, \dot{q}) &= -a_3 \dot{q}_x^2 \sin q_s \\
a_1 &= m_1 \rho_1^2 + m_2 (R-l)^2 \\
a_2 &= m_2 (R+r)^2 \\
a_3 &= m_2 (R+r)(R-l) \\
a_4 &= m_2 \left( \frac{\rho_2 R}{r} \right)^2 \\
b_1 &= (m_1 a - m_2 (R-l))g \\
b_2 &= m_2 (R+r)g
\end{aligned}$$

The control objective is to maintain position of the curved beam and the ball to the vertical upward position i.e.  $q_s = 0$  and  $\dot{q}_s = 0$ . In this system, two unstable modes are present, which makes it more challenging from the control theoretic perspective [1], [2], [3]. From the dynamics of the system, we observe that,

- (i) if the beam angle increases more than  $(-\frac{\pi}{2}, \frac{\pi}{2})$  it becomes difficult to maintain the ball in upward position
- (ii) if the ball moves more than some limit with beam angle between  $(-\frac{\pi}{2}, \frac{\pi}{2})$ , it will fall down due to curved nature of the beam.

In addition to these unstable modes, the gyroscopic forces are also present in the dynamics of this system. These factors makes the nonlinear control design for ball on circular beam system more complicated than the conventional ball on a straight beam system. This forms the motivation to design a nonlinear controller for the ball on a circular beam system.

The control designing is divided into two parts, energy shaping control and dissipative control. The circular ball and beam system equations consist of the terms  $c_x(q, \dot{q})$  and  $c_s(q, \dot{q})$  representing the gyroscopic forces present in the dynamics. It can be observed that  $c_x(q, \dot{q})\dot{q}_x + c_s(q, \dot{q})\dot{q}_s = 0$ , i.e., gyroscopic forces have no effect on the power flow or the rate of change of energy of the system. Thus, while forming an energy shaping controller gyroscopic forces are not considered and these are accounted using the dissipative control term. This is explained in Section IV.

As mentioned above, the beam angle  $q_x$  is bounded in the interval  $(-\frac{\pi}{2}, \frac{\pi}{2})$  to maintain the ball position in upward configuration and the ball position defined by  $Rq_s$  should be between  $(-\frac{\pi}{4}R, \frac{\pi}{4}R)$ . Thus, the configuration variables  $q = (q_x, q_s)$  are constrained in the set  $S_q = \{(-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{4}, \frac{\pi}{4})\}$ . The control objective is the full state-space stabilization for any initial condition belonging to  $S_q$ . In the next subsection a geometrical insight into the intrinsic structure of this system is studied.

### A. Geometrical Approach For Circular Ball and Beam

The role of geometry in formulating a control law for the stabilization of circular ball and beam system can be explained in terms of decomposition of  $T_q Q$  given a mechanical connection. In circular ball and beam system  $q_x$  forms the external variable along the direction of fiber bundle and  $q_s$  is the shape variable along the base space  $Q/G$ . The torque,  $F$  is applied in the direction of fiber, thus  $q_x$  is actuated under the influence of  $F$  where as  $q_s$  remains unactuated since no force is applied along base space. Thus,  $q_x$  is also known as actuated variable and  $q_s$  is the unactuated variable.

For the configuration space  $Q$ , a vector  $q \in Q$  defined as  $(q_x, q_s)$  and corresponding tangent vector  $v_q \in T_q Q$  is  $(\dot{q}_x, \dot{q}_s)$ . The tangent vector  $v_q$  can be decomposed into vertical and horizontal component,

$$v_q = (\dot{q}_x, \dot{q}_s) = \text{Ver}_{\tau_m}(v_q) + \text{Hor}_{\tau_m}(v_q) \quad (3)$$

The fundamental vector field of horizontal one form of (1) and (2) is given by  $[\tau_m(\dot{q}_s)]_Q(q) = m_{xx}^{-1}(q_s)m_{xs}(q_s)\dot{q}_s$ . The tangent vector  $v_q$  is projected onto the vertical component,  $\text{Ver}_{\tau_m}(v_q) = (\dot{q}_x + m_{xx}^{-1}(q_s)m_{xs}(q_s)\dot{q}_s, 0)$  and the horizontal component,  $\text{Hor}_{\tau_m}(v_q) = (-m_{xx}^{-1}(q_s)m_{xs}(q_s)\dot{q}_s, \dot{q}_s)$ . Since, the actuation is along the group direction  $G$ , the power flow between controller and system is,

$$\begin{aligned}
P &= [\dot{q}_x \quad \dot{q}_s] \begin{bmatrix} 1 \\ 0 \end{bmatrix} F \\
&= [\dot{q}_x + m_{xx}^{-1}(q_s)m_{xs}(q_s)\dot{q}_s \quad 0] \begin{bmatrix} 1 \\ 0 \end{bmatrix} F \\
&\quad + [-m_{xx}^{-1}(q_s)m_{xs}(q_s)\dot{q}_s \quad \dot{q}_s] \begin{bmatrix} 1 \\ 0 \end{bmatrix} F \\
&= \dot{q}_x F
\end{aligned} \quad (4)$$

Thus,  $F$  has no control on  $q_s$  i.e. on the ball position ( $Rq_s$ ). Therefore to control the ball position, a new control action is designed so that the power flow is established between controller and unactuated part i.e. the ball subsystem. This can be obtained if  $\text{Ver}_{\tau_m}(v_q)$  is changed to  $\tilde{\text{Ver}}_{\tau_m}(\dot{q})$ , by suitably modifying the Euler Lagrange equation along the actuated direction using some control force  $u$ , which is explained in the following subsection.

### B. Modified Euler Lagrange equation

As discussed in the above subsection the Euler Lagrange equation (1) and (2) can be modified by selecting

$$F = m_{xs}\ddot{q}_s + h_x(q, \dot{q}) + c_x(q, \dot{q}) + m_{xx}(q_s)u \quad (5)$$

The resulting modified dynamics is given by,

$$\ddot{q}_x = u \quad (6)$$

$$m_{sx}(q_s)\ddot{q}_x + m_{ss}(q_s)\ddot{q}_s + h_s(q, \dot{q}) + c_s(q, \dot{q}) = 0 \quad (7)$$

where  $u$  is some nonlinear feedback control. Now, for these modified equations the fundamental vector field of horizontal one form is given by  $[\tilde{\tau}_m(\dot{q}_s)]_Q(q) = m_{xs}(q_s)\dot{q}_s$ . The  $\text{Ver}_{\tau_m}(v_q)$  is changed to  $\tilde{\text{Ver}}_{\tau_m}(v_q) = (\dot{q}_x, 0)$  and  $\text{Hor}_{\tilde{\tau}_m}(v_q)$  remains

unchanged since, the unactuated subsystem part (7) is unchanged.

For this modified system, the power flow is given by,

$$P_m = \begin{bmatrix} \dot{q}_x & 0 \\ 0 & \dot{q}_s \end{bmatrix} u + \begin{bmatrix} -m_{xs}(q_s)\dot{q}_s & \dot{q}_s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ = (\dot{q}_x + \{-m_{xs}(q_s)\dot{q}_s\})u \quad (8)$$

For modified dynamics (6) under control law  $u$ ,  $\text{Ver}_{\tau_m}$  is changed to  $\tilde{\text{Ver}}_{\tilde{\tau}_m}$  then the power flow is established between the beam (controlled or actuated) and the ball (unactuated) subsystems. Thus, a control law can be designed for the stabilization of the circular ball and beam system. The passivity theory and energy shaping concept is used in the design of control law for which  $y_x = \dot{q}_x$  and  $y_s = -m_{xs}(q_s)\dot{q}_s$  in (8) are defined as passivating outputs of actuated and unactuated subsystems respectively. This yields,

$$P_m = (y_x + y_s)u \quad (9)$$

#### IV. CONTROLLER DESIGNING

Motivated by [9], [10], we propose controller design in two parts

- i) energy shaping control ( $u_{es}$ ) to shape the total energy of the modified system (6) and (7) by not considering gyroscopic and dissipative forces, to attain the desired equilibrium point
- ii) and a dissipative control ( $u_{diss}$ ) is included to account the effect of gyroscopic and dissipative forces. It also serves to improve the system stability.

This yields,  $u = u_{es} + u_{diss}$ . Thus one can rearrange the modified system equations (6) and (7) as

$$\ddot{q}_x = u_{es} + u_{diss} \quad (10)$$

$$m_{sx}(q_s)(u_{es} + u_{diss}) + m_{ss}(q_s)\ddot{q}_s + h_s(q, \dot{q}) = -c_s(q, \dot{q}) \quad (11)$$

The right hand side of the above equations consist of input control and gyroscopic terms. Rearranging (2),  $\dot{q}_s$  can be expressed as

$$\dot{q}_s = m_{ss}^{-1}(q_s)[-m_{sx}(q_s)u - h_s(q, \dot{q}) - c_s(q, \dot{q})] \quad (12)$$

##### A. Energy Shaping Control

For the energy shaping control the total energy of the modified system forms the storage function. This storage function can be found by considering the energy of the actuated and unactuated subsystems. Consider, the actuated subsystem (6) is at rest ( $\dot{q}_x = 0$ ) while calculating total energy of the unactuated subsystem. This causes the gyroscopic term in (7),  $c_s(q, \dot{q}) = 0$ . Also, consider  $u_{diss} = 0$  since gyroscopic terms are not accounted. Thus, modified system equation (10) and (11) can be expressed as

$$\ddot{q}_x = u_{es} \quad (13)$$

$$m_{sx}(q_s)u_{es} + m_{ss}(q_s)\ddot{q}_s + h_s(q, \dot{q}) = 0 \quad (14)$$

Therefore total energy of the unactuated subsystem (14) is,

$$E_s = \frac{1}{2}m_{ss}\dot{q}_s^2 + b_2[\cos(q_x + q_s) - 1] \\ \dot{E}_s = -m_{xs}(q_s)\dot{q}_s u_{es} = y_s u_{es} \quad (15)$$

where,  $y_s = -m_{xs}\dot{q}_s$ . The potential energy of the unactuated subsystem is considered as  $b_2[\cos(q_x + q_s) - 1]$  by using change of coordinate frame so that the system will have zero potential energy at the stabilizing point, i.e.,  $(q_x, q_s, \dot{q}_x, \dot{q}_s) = (0, 0, 0, 0)$ . Due to this term there will not be any change in the Euler Lagrange equations (1) and (2). Energy of the actuated subsystem (13) is given by,

$$E_x = \frac{1}{2}\dot{q}_x^2 \\ \dot{E}_x = \dot{q}_x \ddot{q}_x = y_x u_{es} \quad (16)$$

where,  $y_x = \dot{q}_x$ . The storage function is determined using  $E_x$  and  $E_s$  as,

$$S = E_x + E_s \\ \dot{S} = \dot{E}_x + \dot{E}_s = P_m = (y_x + y_s)u_{es} \quad (17)$$

which is similar to (9).

Let,  $V$  denote the energy of the modified system (13) and (14) including the energy shaping terms which also forms the Lyapunov function for system stability is expressed as,

$$V = k_e(E_x - kE_s) + \frac{1}{2}k_v(y_x - ky_s)^2 \\ + \frac{1}{2}k_p \left( \int y_x dt - k \int y_s dt \right)^2 \quad (18)$$

The gains  $k_e > 0$ ,  $k_v > 0$ ,  $k_p > 0$ ,  $k > 0$  and  $k_v \geq k_e$  are chosen such that  $V$  should be positive and  $V = 0$  at  $(0, 0, 0, 0)$ , i.e. equilibrium point. The second term in  $V$  is the shaped kinetic energy of the system so that the ball subsystem should stabilise at  $q_s = 0$ , relative equilibria. For circular ball and beam full state space stabilization is required. Thus, third term in  $V$ , potential energy shaping is included to achieve  $q_x = 0$ . We have,

$$\dot{V} = k_e [\dot{E}_x - k\dot{E}_s] + k_v [y_x - ky_s] [\dot{y}_x - k\dot{y}_s] \\ + k_p \left[ \int y_x dt - k \int y_s dt \right] [y_x - ky_s] \quad (19)$$

The differentiation and integration of passivating outputs  $y_x$  and  $y_s$  with respect to time are as follows,

$$\left. \begin{aligned} \dot{y}_x &= \ddot{q}_x = u_{es} \\ \dot{y}_s &= -a_3 \sin q_2 \dot{q}_2^2 + m_{xs}(q_s)\ddot{q}_s \\ \int y_x dt &= q_s \\ \int y_s dt &= -a_2 q_s + a_3 \sin q_s \end{aligned} \right\} \quad (20)$$

Substituting (12), (15), (16) and (20) into (19) and making some proper arrangements yield,

$$\dot{V} = [u_{es}(k_e + k_v - k_v k \frac{m_{xs}^2(q_s)}{m_{ss}(q_s)}) + k_v k \frac{m_{xs}(q_s)}{m_{ss}(q_s)} \\ (a_3 \dot{q}_x^2 \sin q_s + b_2 \sin(q_x + q_s)) + k_v k a_3 \dot{q}_s^2 \sin q_s \\ + k_p (q_x - k(-a_2 q_s + a_3 \sin q_s))] (y_x - ky_s)$$

Choosing  $u_{es}$  as,

$$u_{es} = \frac{1}{k_e + k_v - k_v k \frac{m_{xs}^2(q_s)}{m_{ss}(q_s)}} \left[ \begin{array}{l} -k_v k \frac{m_{xs}(q_s)}{m_{ss}(q_s)} (a_3 \dot{q}_x^2 \sin q_s + b_2 \sin(q_x + q_s)) \\ -k_v k a_3 \dot{q}_s^2 \sin q_s - k_p (q_x - k(-a_2 q_s + a_3 \sin q_s)) \end{array} \right] \quad (21)$$

will lead to,

$$\dot{V} = 0$$

The system (13) and (14) is stabilized under the control  $u_{es}$ .

### B. Dissipative Control

The stability of (10) and (11) under  $u_{diss} = 0$  is not assured under the control  $u_{es}$  due to the gyroscopic forces. For this case equation (10) and (11) can be expressed as

$$\ddot{q}_x = u_{es} \quad (22)$$

$$m_{sx}(q_s)u_{es} + m_{ss}(q_s)\ddot{q}_s + h_s(q, \dot{q}) = -c_s(q, \dot{q}) \quad (23)$$

Thus, for  $u_{diss} = 0$  the gyroscopic force will make it unstable. The  $u_{diss}$  is included to account the effect of the gyroscopic forces. So, (22) and (23) is modified to

$$\ddot{q}_x = u_{es} + u_{diss} \quad (24)$$

$$m_{sx}(q_s)u_{es} + m_{ss}(q_s)\ddot{q}_s + h_s(q, \dot{q}) = -c_s(q, \dot{q}) - m_{sx}(q_s)u_{diss} \quad (25)$$

Therefore, the effect of the gyroscopic forces is suppressed using  $u_{diss}$  which is defined as

$$u_{diss} = \frac{u_d}{k_e + k_v - k_v k \frac{m_{xs}^2(q_s)}{m_{ss}(q_s)}} \quad (26)$$

where,  $u_d = -k_d(k_1 y_x - k y_s)$ . The total control law  $u$  is

$$\begin{aligned} u &= u_{es} + u_{diss} \\ &= \frac{1}{k_e + k_v - k_v k \frac{m_{xs}^2(q_s)}{m_{ss}(q_s)}} \\ &\quad \left[ -k_v k \frac{m_{xs}(q_s)}{m_{ss}(q_s)} (a_3 \dot{q}_x^2 \sin q_s + b_2 \sin(q_x + q_s)) \right. \\ &\quad \left. - k_v k a_3 \dot{q}_s^2 \sin q_s - k_p (q_x - k(-a_2 q_s + a_3 \sin q_s)) \right] \\ &\quad + \frac{u_d}{k_e + k_v - k_v k \frac{m_{xs}^2(q_s)}{m_{ss}(q_s)}} \end{aligned} \quad (27)$$

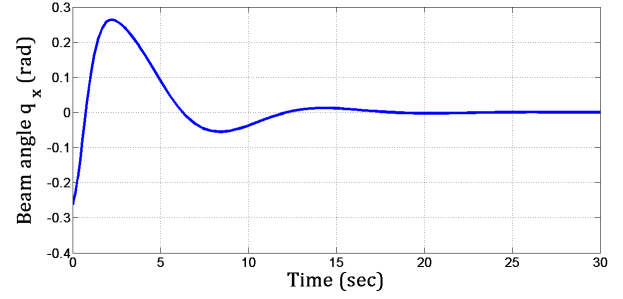
Then, for the system (24) and (25), the expression for  $\dot{V}$  is

$$\dot{V} = u_d(y_x - k y_s)$$

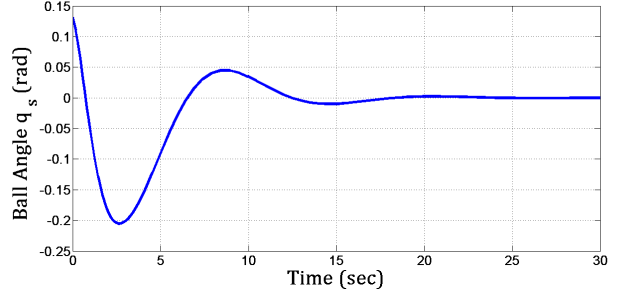
Therefore,  $\dot{V} \leq 0$  by selecting  $k_d > 0$ . For  $k_1 = 1$ ,  $\dot{V} = -k_d(y_x - k y_s)^2$  which guaranties  $\dot{V} \leq 0$ . However, to suppress the destabilizing effect of the gyroscopic forces, we panellize  $y_x$  in the dissipation term. From (5) the final complete control law is

$$F = m_{xx}(q_s)(u_{es} + u_{diss}) + m_{xs}(q_s)\ddot{q}_s + h_x(q, \dot{q}) + c_x(q, \dot{q}) \quad (28)$$

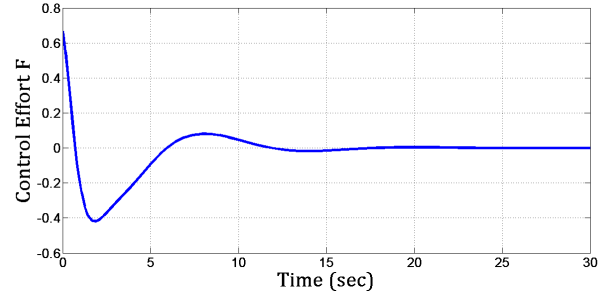
The acceleration term  $\ddot{q}_s$  can be substituted from (12).



(a) Beam Angle ( $q_x$ )



(b) Ball Angle ( $q_s$ )



(c) Control Effort ( $\tau$ )

Fig. 2: Simulation results for stabilization of circular ball and beam system

## V. SIMULATION RESULTS

In this section, MATLAB simulation results for the circular ball and beam system with initial condition  $(q_x, q_s, \dot{q}_x, \dot{q}_s) = (-\frac{\pi}{12}, \frac{\pi}{24}, 0.1, -0.1)$  are presented. The system parameters chosen for the simulation are  $m_1 = 1.0$  kg,  $\rho_1 = 0.2646$  m,  $R = 0.8$  m,  $m_2 = 0.2$  kg,  $\rho_2 = 0.1414$  m,  $r = 0.05$  m,  $l = 0.6$  m,  $a = 0.15$  m,  $g = 9.81$  m/sec<sup>2</sup>. The gains in the control (27) were set to  $k_e = 1$ ,  $k_v = 4$ ,  $k_p = 1$ ,  $k_d = 4$  and  $k = 45$ . Fig. 2 shows the time response of the closed loop system under the the control law (28) applied to the nonlinear system. The system is stabilized to the desired equilibrium position in the presence of the gyroscopic forces within 15 sec. Note that configuration variables are bounded to the set  $S_q$ .

## VI. CONCLUSION

The stabilization of the circular ball and beam system is difficult due to the presence of the two unstable modes and the gyroscopic terms. Thus, configuration variables are

bounded to set  $S_q$  and a control law is derived to shape the energy of the modified system without considering the gyroscopic terms for stability. The dissipative term is included in this control law to nullify the effect of gyroscopic term. The main advantage of proposed nonlinear control methodology is that there are no matching conditions needed to be satisfied. Hence, PDE's are not required to derive the control law.

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