APPLICATION OF A BUDAN-FOURIER ALGORITHM TO STABILITY ANALYSIS OF TIME-DELAY SYSTEMS.

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1. A Budan-Fourier algorithm for EPT functions. We consider the class of functions f that satisfy a homogeneous linear differential equation with constant coefficients in which the coefficient of f itself is nonzero, which can be called the Euler-d'Alembert class of functions after Euler and d'Alembert who were the first to give a general procedure to solve such differential equations. One way of characterizing this class is by noting a function f belongs to it iff f is infinitely differentiable (even entire) and has the property that the function f and its higher order derivatives $f^{(k)},\ k = 1, 2, \ldots$ span a finite dimensional vector space over $\mathbb{R}$ (or $\mathbb{C}$ in the complex case). Clearly if $f^{(n)}$ can be expressed as a linear combination of its predecessors $f^{(k)},\ k = 0, 1, \ldots, n-1$ then the same holds for all higher order derivatives $f^{(m)},\ m > n$. It is well-known that a function f is in the Euler-d'Alembert class if and only if it can be written in the form

$$f(s) = \sum_{k=1}^{K} p_k(s) e^{\mu_k s}$$

where p_k is a complex polynomial and μ_k is a complex number for each $k = 1, 2, \ldots, K$. It is clear from this representation that the class forms a ring under addition and multiplication.

In the case that f is real this can be rephrased to say that a real function f is in the Euler-d'Alembert case iff it can be written as

$$f(t) = \sum_{i=0}^{d} q_i(t) e^{\lambda_i t} \cos(\theta_i t + \phi_i),$$

where the q_i are real polynomials and λ_i, θ_i and ϕ_i are real numbers and $K \leq d \leq 2K$. For this reason we also call these functions exponential-polynomial-trigonometric or EPT. Examples of this class include the univariate polynomials, the exponential function and the trigonometric functions cos and sin. The class of EPT functions is invariant under scaling of the argument (or dilations).

As is well-known each higher order linear differential equation can be rewritten in the form of a first-order vector linear differential equation. This is the basis for the theory of state space realization of linear dynamical systems. Using this one can show in a straightforward manner that any EPT function f can also be written in the form

$$f(s) = \tilde{c} e^{\tilde{A}s} \tilde{b},$$

where $\tilde{A}$ is a square matrix, $\tilde{b}$ a column vector and $\tilde{c}$ a row vector. Also any such function is an Euler-d'Alembert function (as follows quickly by noting that $f^{(k)} = \tilde{c}\tilde{A}^k e^{\tilde{A}s}\tilde{b}$ and applying the Cayley-Hamilton theorem.) Using realization theory one can contruct a so-called minimal realization (A, b, c), A an $n \times n$ matrix, with the property

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that $f(s) = ce^{As}b$ and n is the minimal possible integer in any such realization (this number n is called the McMillan degree in the context of realization theory).

An algorithm was presented in [1] and summarized in [2], which allows one to determine all sign-changing zeros of any real EPT function on any given finite real interval, under some assumptions that relate to the possibility to determine the sign (positive, negative or zero) of the EPT functions used, in a number of points that are generated by the algorithm. Any EPT function which is not the zero function can only have isolated zeros, as an EPT function is an entire function (analytic function defined on all of $\mathbb{C}$). The term "sign-changing zero" relates to a point at which the EPT function is zero and which has the property that there is an open interval containing the point such that the function on that interval has opposite signs to the left and to the right of the point at which it is zero, so either goes from positive to negative or the other way round. The algorithm is based on the construction of a (finite) generalised Budan-Fourier sequence. The method is most easily explained in the case the spectrum of A in a minimal realization (A, b, c), $f(t) = ce^{At}b,\ t \in [0, T]$, has only real elements. Formally, the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A are all real. In the original Budan-Fourier theory, which just applies to (univariate) polynomials, the Budan-Fourier sequence is obtained by repeated differentiation until the zero function is produced. Obviously a finite sequence is obtained in this way as a polynomial only has a finite number of non-identically-zero higher order derivatives.

For EPT functions with $\lambda_1, \lambda_2, \ldots, \lambda_n$ all real, a generalised Budan-Fourier sequence is obtained by sequentially applying the linear differential operators $D-\lambda_1, D-\lambda_2, D-\lambda_3, \ldots, D-\lambda_n$ where D stands for the derivative with respect to the variable t. Each differential operator $D - \lambda_k,\ k = 1, 2, \ldots, n$ can be decomposed into three operators, namely: first multiply by $e^{-\lambda_k t}$ then apply the differentiation operator D and then multiply by $e^{\lambda_k t}$. Note that multiplication by either $e^{-\lambda_k t}$ or $e^{\lambda_k t}$ does not change the sign of a function at any point. Sequential application of the differential operators leads to the following Budan-Fourier sequence: $f_0(t) := f(t) = ce^{At}b; f_1(t) := c(A - \lambda_1 I)e^{At}b; f_2(t) := c(A - \lambda_1 I)(A - \lambda_2 I)e^{At}b; \ldots, f_k(t) := c(A - \lambda_1 I)(A - \lambda_2 I) \ldots (A - \lambda_k I)e^{At}b; \ldots; f_n(t) := c(A - \lambda_1 I)(A - \lambda_2 I) \ldots (A - \lambda_n I)e^{At}b$. This last function is identically zero as follows directly from the Cayley-Hamilton theorem. Actually even more can be said, as the McMillan degree of the elements in the BF sequence is decreasing by one at each step, i.e. the McMillan degree of f_k is equal to $n - k$ for each $k = 1, 2, \ldots, n$. It follows that the function f_{n-1} is either strictly positive or strictly negative and has no sign-changing zeros. This implies that $(0, T)$ is a simple interval for the function f_{n-2}, where an interval is called simple in relation to a function defined on it, if the function has at most one sign-changing zero on that interval. Note that if it is known that a given interval is simple for a given function, then it can be determined whether f has a zero on that interval by determining whether the function has opposite signs at (or near) the end points of the interval; if that is the case then the sign-changing zero can be determined by bisection. The main result is now that by determining the zeros on $[0, T]$ of the $k-th$ element in the BF sequence one obtains a partitioning of $[0, T]$ into intervals which are all simple for f_{k-1}. In this way one can recursively determine all the sign-changing zeros of each of the functions in the BF sequence on $[0, T]$ and hence also all the sign-changing zeros of the EPT function f on $[0, T]$. For the general case in which the spectrum of A can contain complex non-real elements, a BF sequence was developed as well. For details we refer to [1] (the results were first presented at a seminar at Brunel university).

In the presentation we intend to make some further remarks about the issue of the mathematically exact determination of the sign of an EPT function in a given point, based on the work of [3].

2. Stability of Time-Delay Systems. The stability of equilibria is a question of fundamental importance in the analysis of dynamical systems. In many practical situations, time-delays are an unavoidable part of a system description. A substantial body of work has been done on the analysis of linear time-invariant (LTI) delay systems of the form

$$\dot{x}(t) = Fx(t) + Gx(t-\tau),$$

where F and G are $m \times m$ real matrices. Nonetheless many authors continue to work on developing implementable tests for the stability of such systems[5, 6, 7, 8]. Some of these approaches rely on Lyapunov functionals, while others proceed by analysing the spectrum of the system: namely checking whether all solutions of the characteristic equation

$$f(s) := \det(sI - F - e^{-\tau s}G) = 0,$$

lie in the open left half of the complex plane. For systems with several time delays $\tau_1, \ldots, \tau_k$ one obtains a characteristic equation of the form

$$f(s) := \det(sI - F - \sum_{j=1}^{k} e^{-\tau_j s}G_j) = 0.$$

The function f can be written in the form

$$f(s) = s^m + \sum_{i=0}^{m-1} p_i(e^{-\tau_1 s}, \ldots, e^{-\tau_k s}).s^i,$$

where p_i is a real multivariable polynomial for each $i = 0, 1, 2, \ldots, m-1$. Clearly this is an EPT function and one with special properties. In a nice paper [4] has worked out how one can determine the number of right halfplane zeros of f from the relative location of the sign-changing zeros of the real and imaginary parts of the function f confined to a well-determined sufficiently large finite interval on the imaginary axis. If we define $h(\omega) = \mathrm{Re} f(i\omega)$ and $v(\omega) = \mathrm{Im} f(i\omega)$, then h and v are again EPT functions, and they are real for real values of ω. Therefore we can use the BF algorithm described above to determine all the sign-changing zeros of h and v on any given finite interval of the real line. This allows us to determine the number of right half plane zeros of f in a mathematically reliable way, at least if we assume that in each case encountered in the algorithm it is possible to determine the sign of an EPT function in a given point.

As is shown in [4] in case f has zeros in the right half plane, then there can only be finitely many of them. In that case the BF algorithm allows one to isolate all the right half plane zeros with arbitrary precision, by constructing for each right half plane zero a sequence of shrinking rectangles that contain it. This is possible because the BF algorithm can compute the winding number of f over each rectangle, by computing the zero locations of the real and imaginary parts of f along each side of such a rectangle. In each case the real and imaginary parts of f on such a line segment in the complex plane can be identified with real EPT functions on a finite interval on the real line, and hence the technique described above to locate the sign-changing zeros can be applied.

In the presentation remarks will be made about the value of the McMillan degree of the EPT function, as defined in the previous section (not to be confused with the dimension of the vector space containing the vector x in the state space description of the time-delay system) and the level of complexity of various computations that are required in the stability analysis.

We will also compare our method to recent approaches proposed in the literature and highlight some possible directions for future research.

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