

# On balanced truncation for symmetric nonlinear systems

Kenji Fujimoto<sup>1</sup> and Jacquelin M. A. Scherpen<sup>2</sup>

**Abstract**—This paper is concerned with model order reduction based on balanced realization for symmetric nonlinear systems. A new notion of symmetry for nonlinear systems was characterized recently. It plays an important role in linear systems theory and is expected to provide new insights to nonlinear systems. In this paper, we provide a novel framework of balanced realization for this class of systems and apply it for model order reduction preserving symmetry.

## I. INTRODUCTION

Model order reduction is one of the key tools in control systems theory. In particular for linear systems, several practical techniques are proposed so far, see e.g. [1], [13] for detail. For nonlinear systems, on the other hand, model order reduction is a difficult task because the analysis of approximation error becomes difficult due to the nonlinearity of the system, so nonlinear model order reduction is an on-going research topic. Balanced realization and the corresponding model order reduction technique for nonlinear systems is proposed in [15] and its variations are investigated by many authors such as natural extension of observability function [9], time-varying balancing [17], and nonlinear singular value analysis [7], [8]. There are other approaches such as nonlinear moment matching [2], empirical Gramians [10], model order reduction for closed loop systems [14], and so on.

Symmetric systems attract the attention of researchers since it can describe important class of systems such as gradient systems and passive ones. In fact, many physical systems intrinsically have such properties and there are many model order reduction techniques to preserve special properties of the original system. For example, passivity [6] and dissipativity [11] are important properties to be preserved. Symmetry of dynamical systems are also an attractive research topic so far and some results are reported such as analysis with cross Gramians [5], on symmetric linear systems [4], characterization via Sylvester equations [16], nonlinear gradient systems [3], etc. The authors also worked on nonlinear symmetric systems and provide a new characterization using nonlinear Sylvester equations recently [12]. The present paper provides a new framework of nonlinear model order reduction to preserve the symmetry defined in the authors' previous result.

In this paper, we pay attention to a special nonlinear map called cross map which is a nonlinear generalization

<sup>1</sup>Kenji Fujimoto is with Department of Aeronautics and Astronautics, Kyoto University, 615-8540, Kyoto, Japan k.fujimoto@ieee.org

<sup>2</sup>Jacquelin M. A. Scherpen is with Faculty of Mathematics and Natural Sciences, University of Groningen, 9747 AG, Groningen, The Netherlands J.M.A.Scherpen@rugl.nl

of the cross Gramian originally given in [5] and plays an important role in model order reduction in the linear case. By introducing a balanced representation of the cross map, we provide a model order reduction procedure preserving the symmetry of the original nonlinear system in addition to stability, controllability and observability. Although the current version of the paper only contains the main result of the paper, the final version will take care of an additional discussion such as the existence of the solutions as well.

## II. PRELIMINARIES

### A. Balanced realization for nonlinear systems

This section refers to preliminary results on nonlinear balanced realization in [15], [7], [8]. Consider an input-affine, time invariant, asymptotically stable nonlinear system

$$\Sigma: \begin{cases} \dot{x} &= f(x) + g(x) u \\ y &= h(x) \end{cases} \quad (1)$$

with  $f(0) = 0$  where  $x(t) \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^m$ , and  $y(t) \in \mathbb{R}^r$ . Its controllability function  $L_c(x)$  and observability function  $L_o(x)$  are defined by

$$L_c(x^0) \equiv \inf_{\substack{u \in L_2^- \\ x(-\infty)=0, x(0)=x^0}} \frac{1}{2} \|u\|_{L_2}^2$$

$$L_o(x^0) \equiv \frac{1}{2} \|y\|_{L_2}, \quad x(0) = x^0, \quad u(t) \equiv 0.$$

In the linear case,

$$L_c(x) = \frac{1}{2} x^T W^{-1} x, \quad L_o(x) = \frac{1}{2} x^T M x$$

hold with the controllability and observability Gramians  $W$  and  $M$ , respectively. The functions  $L_c(x)$  and  $L_o(x)$  fulfill the following Hamilton-Jacobi equations.

$$\frac{\partial L_c(x)}{\partial x} f(x) + \frac{1}{2} \frac{\partial L_c(x)}{\partial x} g(x) g(x)^T \frac{\partial L_c}{\partial x}^T = 0$$

$$\frac{\partial L_o(x)}{\partial x} f(x) + \frac{1}{2} h(x)^T h(x) = 0$$

Here  $\dot{x} = -f - gg^T(\partial L_c(x)/\partial x)^T$  is asymptotically stable in a neighborhood of the origin.

Here we review the nonlinear balanced realization.

*Theorem 1:* [8] Suppose that  $L_c(x)$  and  $L_o(x)$  exist and that Hankel singular values of the Jacobian linearization of  $\Sigma$  are nonzero and distinct.

- 1) Then there exist a neighborhood  $X \subset \mathbb{R}^n$  of the origin and a coordinate transformation  $x \mapsto \tilde{x}$  on  $X$  converting

the system into the following form

$$\tilde{L}_c(\tilde{x}) = \frac{1}{2} \sum_{i=1}^n \frac{\tilde{x}_i^2}{\sigma_i(\tilde{x}_i)} \quad (2)$$

$$\tilde{L}_o(\tilde{x}) = \frac{1}{2} \sum_{i=1}^n \tilde{x}_i^2 \sigma_i(\tilde{x}_i) \quad (3)$$

where  $\tilde{L}_c(\tilde{x})$  and  $\tilde{L}_o(\tilde{x})$  are the functions  $L_c(x)$  and  $L_o(x)$  in the new coordinate  $\tilde{x}$ , respectively.

- 2) There also exist weaker versions of balanced realizations satisfying the following properties.

$$\tilde{x}_i = 0 \iff \frac{\partial \tilde{L}_c(\tilde{x})}{\partial \tilde{x}_i} = 0 \iff \frac{\partial \tilde{L}_o(\tilde{x})}{\partial \tilde{x}_i} = 0$$

$$\sigma_i(\tilde{x}_i)^2 = \frac{\tilde{L}_o(0, \dots, 0, \tilde{x}_i, 0, \dots, 0)}{\tilde{L}_c(0, \dots, 0, \tilde{x}_i, 0, \dots, 0)}$$

Here the singular value functions  $\sigma_i(s)$  are in order such as

$$\min_{s=\pm c} \sigma_i(s) > \max_{s=\pm c} \sigma_{i+1}(s), \quad i = 1, 2, \dots, n-1$$

for any sufficiently small positive number  $c$ .

The Hankel singular value functions  $\sigma_i(s)$ 's represent the importance of the state variables  $\tilde{x}_i$ 's with respect to the input-output behavior of the system. Therefore we can get a reduced order model by removing the less important states. This technique is called *balanced truncation*. Let us now suppose that the system  $\Sigma$  as in Eq. (1) is balanced in the sense of Theorem 1 and that

$$\min_{s=\pm c} \sigma_l(s) \gg \max_{s=\pm c} \sigma_{l+1}(s)$$

holds. Divide the state space accordingly as

$$x = \begin{pmatrix} x^a \\ x^b \end{pmatrix}, \quad x^a \equiv \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix}, \quad x^b \equiv \begin{pmatrix} x_{l+1} \\ \vdots \\ x_n \end{pmatrix}$$

and

$$\begin{pmatrix} \dot{x}^a \\ \dot{x}^b \end{pmatrix} = \begin{pmatrix} f^a(x^a, x^b) \\ f^b(x^a, x^b) \end{pmatrix} + \begin{pmatrix} g^a(x^a, x^b) \\ g^b(x^a, x^b) \end{pmatrix} u$$

$$y = h(x^a, x^b).$$

Then the balanced truncation with respect to the important state  $x^a$  is given by

$$\Sigma^a : \begin{cases} \dot{x}^a &= f^a(x^a, 0) + g^a(x^a, 0)u \\ y &= h(x^a, 0) \end{cases} \quad (4)$$

The reduced order system thus obtained satisfies the following properties.

*Theorem 2:* [8] The controllability and observability functions  $L_c^a(x^a)$  and  $L_o^a(x^a)$  of the system  $\Sigma^a$  satisfy

$$L_c^a(x^a) \equiv L_c(x^a, 0)$$

$$L_o^a(x^a) \equiv L_o(x^a, 0)$$

Furthermore, Hankel singular values  $\sigma_i^a(x_i^a)$  of  $\Sigma^a$  satisfy

$$\sigma_i^a(x_i^a) = \sigma_i(x_i), \quad i = 1, \dots, l.$$

This theorem implies that several important properties of  $\Sigma$  are preserved such as controllability, observability and Lyapunov stability. Local asymptotic stability is also preserved.

## B. Symmetric nonlinear systems

This section briefly refers to the authors former result on symmetric nonlinear systems [12] which is a nonlinear generalization of the linear case result [5], [4], [16]. Symmetric systems form an important class of systems in the linear case. They correspond to passive and/or gradient systems. Some of their properties are recalled in this subsection.

First of all, the difference between linear and nonlinear symmetric systems are summarized in TABLE I.

TABLE I  
LINEAR V.S. NONLINEAR SYMMETRIC SYSTEMS

Name	Nonlinear case	Linear case
system parameter	$f(x)$	$Ax$
system parameter	$g(x)$	$B$
system parameter	$h(x)$	$Cx$
Controllability func.	$L_c(x)$	$x^T W^{-1} x$
Observability func.	$L_o(x)$	$x^T M x$
Cross map	$\Phi(x)$	$Xx = WTx = T^{-1}Mx$
Pseudo-metric map	$\chi(x)$	$Tx$
state of $\mathcal{O}$	$x$	$x$
state of $\mathcal{O}^\dagger$	$\xi = \Phi(x)$	$\xi = Xx = WTx$
state of $(d\mathcal{O}(\mathcal{O}^\dagger)(\xi))^*$	$p = \chi(x) = \nabla L_c(\xi)$	$p = Tx = W^{-1}\xi$

Consider the nonlinear system  $\Sigma$  as in Eq. (1) with  $r = m$ , that is, the dimensions of input and output are the same. For this class of systems we can define a cross map as follows.

*Definition 1:* The cross map<sup>1</sup>  $\Phi$  of  $\Sigma$  is defined by

$$\Phi(x) \equiv \mathcal{O} \circ \mathcal{O}(x).$$

Here  $\mathcal{O}$  and  $\mathcal{O}$  are controllability and observability operators of the system  $\Sigma$ , respectively, defined by

$$\mathcal{O} : u \in L_{2-} \mapsto x(0) : \dot{x} = f(x) + g(x)u, \quad x(-\infty) = 0$$

$$\mathcal{O} : x(0) \mapsto y \in L_{2+} : \begin{cases} \dot{x} &= f(x) \\ y &= h(x) \end{cases}$$

The cross map in the linear case becomes

$$\Phi(x) = Xx$$

with the cross Gramian  $X$ . It satisfies the following property.

*Lemma 1 (Nonlinear Sylvester equation [12]):* The cross map  $\Phi(x)$  satisfies

$$\frac{\partial \Phi(x)}{\partial x} f(x) + f(\Phi(x)) + g(\Phi(x))h(x) = 0. \quad (5)$$

The Sylvester equation in the linear case is as follows.

$$XA + AX + BC = 0$$

The definition of the symmetry of nonlinear systems is as follows.

*Definition 2:* The system  $\Sigma$  is said to be *symmetric* if

$$\text{Im } \mathcal{O}^\dagger = \text{Im } \mathcal{O}.$$

A symmetric system has the following property.

*Lemma 2:* [12] For a symmetric system, the following property holds.

$$L_o(x) = L_c(\Phi(x))$$

<sup>1</sup>This map is called *cross operator* in [12].

We have a necessary and sufficient condition for the symmetry.

*Lemma 3:* [12] A nonlinear system is symmetric if and only if the cross map  $\Phi(x)$  is invertible and satisfies

$$h(x) = g(\xi)^T \nabla L_c(\xi) \Big|_{\xi=\Phi(x)} \quad (6)$$

and one of the following two equations

$$\frac{\partial \Phi(x)}{\partial x} f(x) = (-f(\xi) - g(\xi)g(\xi)^T \nabla L_c(\xi)) \Big|_{\xi=\Phi(x)} \quad (7)$$

$$\frac{\partial \chi(x)}{\partial x} f(x) = \frac{\partial (f(\xi) + g(\xi)g(\xi)^T \chi(x))}{\partial \xi} \Big|_{\xi=\Phi(x)} \chi(x). \quad (8)$$

Eq. (6) corresponds to

$$Cx = B^T W^{-1} Xx = B^T T x$$

in the linear case. If the system is symmetric, the three dynamics  $\mathcal{O}(x)$ ,  $\mathcal{C}^\dagger(\xi)$ , and  $d\mathcal{C}(\mathcal{C}^\dagger(\xi))^*(p)$  are equivalent. This equivalence is characterized by Eqs. (7) and (8) in Lemma 3 which correspond to

$$\begin{aligned} WTAx &= -AWTx - BB^T T x \\ TAx &= A^T T x \end{aligned}$$

in the linear case. Further, the latter reduces to

$$TAx = -Px = -\frac{\partial}{\partial x} \left( \frac{1}{2} x^T P x \right)^T$$

which implies that the system is a gradient system with a potential function  $(1/2)x^T P x$ . See [12] for more detail on the relationship between symmetric systems and gradient ones.

Although some properties of symmetric nonlinear systems are clarified in [12], balanced realization and the corresponding model order reduction technique for those systems are not considered so far.

### III. MAIN RESULT

#### A. Balanced realization for symmetric systems

This section gives a novel characterization of balanced realization for symmetric nonlinear systems and clarifies some of their properties. Consider a nonlinear symmetric system  $\Sigma$  of the form (1) and let us define the balanced realization for them as follows.

*Definition 3:* 1) The controllability function  $L_c(x)$  is said to be *balanced* if it is separable with respect to the states such as

$$L_c(x) = \sum_{i=1}^n l_{c,i}(x_i).$$

2) The observability function  $L_o(x)$  is said to be *balanced* if it is separable with respect to the states such as

$$L_o(x) = \sum_{i=1}^n l_{o,i}(x_i).$$

3) The cross map  $\Phi(x)$  is said to be *balanced* if it satisfies

$$\Phi(x) = \begin{pmatrix} \phi_1(x_1) \\ \phi_2(x_2) \\ \vdots \\ \phi_n(x_n) \end{pmatrix}.$$

4) A symmetric system is said to be *balanced* if its observability function  $L_o(x)$  and the cross map  $\Phi(x)$  are balanced.

We also have its weaker version as follows.

*Definition 4:* 1) The controllability function  $L_c(x)$  is said to be *weakly balanced* if it satisfies

$$x_i = 0 \iff \frac{\partial L_c(x)}{\partial x_i} = 0, \quad i = 1, 2, \dots, n.$$

2) The observability function  $L_o(x)$  is said to be *weakly balanced* if it satisfies

$$x_i = 0 \iff \frac{\partial L_o(x)}{\partial x_i} = 0, \quad i = 1, 2, \dots, n.$$

3) A symmetric system is said to be *weakly balanced* if its observability function  $L_o(x)$  is weakly balanced and if the cross map  $\Phi(x)$  is balanced.

Note that the balancing and weak balancing defined above correspond to the balanced realizations 1) and 2) in Theorem 1, respectively. We will prove some properties using those balanced realizations.

In the above definitions, observability requirement for symmetry can be replaced by controllability as follows.

*Proposition 1:* 1) A symmetric system is balanced if and only if the controllability function  $L_c(\xi)$  and the cross map  $\Phi(x)$  are balanced.

2) A symmetric system is weakly balanced if and only if the controllability function  $L_c(\xi)$  is weakly balanced and the cross map  $\Phi(x)$  is balanced.

*Proof:*

1) Sufficiency: Since the system is symmetric, it follows from Lemma 2 and Definition 3 that

$$L_o(x) = L_c(\Phi(x)) = \sum_{i=1}^n l_{c,i}(\xi_i) \Big|_{\xi=\Phi(x)} = \sum_{i=1}^n l_{c,i}(\phi_i(x_i))$$

which implies that  $L_o(x)$  is balanced.

Necessity:

$$\begin{aligned} L_c(\xi) &= L_o(\Phi^{-1}(\xi)) = \sum_{i=1}^n l_{o,i}(x_i) \Big|_{x=\Phi^{-1}(\xi)} \\ &= \sum_{i=1}^n l_{o,i}(\phi_i^{-1}(\xi_i)) \end{aligned}$$

which implies that  $L_c(\xi)$  is balanced.

2) This part can be proved in a similar way to part 1).

Proof is completed.  $\blacksquare$

The pseudo-metric map  $p = \chi(x)$  satisfies the following property.

*Lemma 4:* If a symmetric system is balanced, then its pseudo-metric map  $\chi(x)$  is also balanced such as

$$\chi(x) = \begin{pmatrix} \chi_1(x_1) \\ \chi_2(x_2) \\ \vdots \\ \chi_n(x_n) \end{pmatrix}.$$

*Proof:* The statement can be readily obtained by

$$\begin{aligned} \chi(x) &= \nabla L_c(\xi)|_{\xi=\Phi(x)} = (l'_{c,1}(\xi_1), \dots, l'_{c,n}(\xi_n))^T \Big|_{\xi=\Phi(x)} \\ &= (l'_{c,1}(\phi_1(x_1)), \dots, l'_{c,n}(\phi_n(x_n)))^T \\ &\equiv (\chi_1(x_1), \dots, \chi_n(x_n))^T \end{aligned}$$

which completes the proof.  $\blacksquare$

If the system is symmetric and balanced, we can obtain the following input-normal realization which shows the intrinsic nature of the symmetric system.

*Theorem 3:* Consider a symmetric system  $\Sigma$  of the form (1). Suppose that the assumptions of Theorem 1 hold and that  $\Sigma$  is balanced in the sense of Definition 3. Then there exists another balanced realization satisfying the following properties.

$$\Phi(x) = \chi(x) = \begin{pmatrix} \pm x_1 \sigma_1(x_1) \\ \vdots \\ \pm x_n \sigma_n(x_n) \end{pmatrix} \quad (9)$$

$$L_c(\xi) = \frac{1}{2} \xi^T \xi \quad (10)$$

$$L_o(x) = \frac{1}{2} \sum_{i=1}^n x_i^2 \sigma_i(x_i)^2 \quad (11)$$

*Proof:* First of all, once we obtain a realization balanced in the sense of Definition 3, its form is invariant for any coordinate transformation of the form

$$x_i \mapsto \tilde{x}_i(x_i), \quad i = 1, 2, \dots, n \quad (12)$$

that is, expansion with respect to the coordinate axis. Applying the proof of Theorem 1-1) to this realization, then the input-normal balanced realization satisfying Eqs. (10) and (11) can be obtained by a coordinate transformation of the form (12). Then the result follows immediately (e.g., using Lemma 4).  $\blacksquare$

In this way, the cross map  $\xi = \Phi(x)$  and the pseudo-metric map  $p = \chi(x)$  play important roles in characterizing balanced realizations of nonlinear symmetric systems as in the linear case. The following subsection discusses on model order reduction preserving symmetric property of the original system.

### B. Model order reduction

Consider a symmetric nonlinear system  $\Sigma$  as in Eq. (1) and suppose that it is balanced in the sense of Definition 3 or 4. Let us apply the model order reduction procedure explained in Section II-A.

We divide the state  $\xi$  and the cross map  $\xi = \Phi(x)$  according to the division of the state  $x$  as follows.

$$\begin{aligned} \xi &= \begin{pmatrix} \xi^a \\ \xi^b \end{pmatrix}, \quad \xi^a \equiv \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_l \end{pmatrix}, \quad \xi^b \equiv \begin{pmatrix} \xi_{l+1} \\ \vdots \\ \xi_n \end{pmatrix} \\ \Phi(x^a, x^b) &\equiv \begin{pmatrix} \Phi^a(x^a) \\ \Phi^b(x^b) \end{pmatrix} \end{aligned}$$

Note that the properties of balanced truncation proved in Theorem 2 is valid for the systems (weakly) balanced in the sense of Definitions 3 and 4. That is, the statement of the following proposition is true.

*Proposition 2:* Consider a symmetric system  $\Sigma$  in (1). Suppose that the assumptions of Theorem 1 hold and the system is weakly balanced in the sense of Definition 4. Then the reduced order system  $\Sigma^a$  in (4) obtained by balanced truncation satisfies the properties proved in Theorem 2.

*Proof:* Proof is directly obtained by that of Theorem 2 ([8]).  $\blacksquare$

In addition to this property, we can prove the following result showing that balanced truncation for a symmetric system preserves its symmetry as well.

*Theorem 4:* 1) Suppose that the original system  $\Sigma$  in (1) is symmetric and balanced. Then the truncated reduced order system  $\Sigma^a$  in (4) is also symmetric and balanced.

2) Suppose that the original system  $\Sigma$  is symmetric and weakly balanced. Then the truncated reduced order system  $\Sigma^a$  is also symmetric and weakly balanced.

*Proof:* First of all, we prove that the reduced order system is symmetric if the conditions of the part 2) are satisfied which are also necessary condition for the part 1). Substituting  $x^b = 0$  for Eq. (6) in Lemma 3, we have

$$\begin{aligned} h(x^a, 0) &= g(\xi)^T \nabla L_c(\xi)|_{\xi=\Phi(x^a, 0)} \\ &= \left( g^a(\xi^a, 0)^T, g^b(\xi^a, 0)^T \right) \begin{pmatrix} \nabla L_c^a(\xi^a) \\ 0 \end{pmatrix} \Big|_{\xi^a=\Phi^a(x^a)} \\ &= g^a(\xi^a, 0)^T \nabla L_c^a(\xi^a)|_{\xi^a=\Phi^a(x^a)} \end{aligned}$$

which is equivalent to the condition (6) for the reduced order system  $\Sigma^a$  where  $L_c^a(\xi^a) \equiv L_c(\xi^a, 0)$ . Furthermore, substituting  $x^b = 0$  into (7),

$$\begin{aligned} &\left( \frac{\partial \Phi(x^a, 0)}{\partial x^a}, \frac{\partial \Phi(x^a, 0)}{\partial x^b} \right) \begin{pmatrix} f^a(x^a, 0) \\ f^b(x^a, 0) \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial \Phi^a(x^a)}{\partial x^a} f^a(x^a, 0) \\ 0 \end{pmatrix} \\ &= (-f(\xi) - g(\xi)g(\xi)^T \nabla L_c(\xi))|_{\xi=\Phi(x^a, 0)} \\ &= - \begin{pmatrix} f^a(\xi^a, 0) \\ f^b(\xi^a, 0) \end{pmatrix} - \begin{pmatrix} g^a(\xi^a, 0) \\ g^b(\xi^a, 0) \end{pmatrix} g^a(\xi^a, 0)^T \nabla L_c^a(\xi^a)|_{\xi^a=\Phi^a(x^a)} \end{aligned}$$

reducing to

$$\begin{aligned} &\frac{\partial \Phi^a(x^a)}{\partial x^a} f^a(x^a, 0) \\ &= -f^a(\xi^a, 0) - g^a(\xi^a, 0)g^a(\xi^a, 0)^T \nabla L_c^a(\xi^a)|_{\xi^a=\Phi^a(x^a)} \end{aligned}$$

which is equivalent to the condition (7) for the reduced system  $\Sigma^a$ . What we need to prove next is that  $\xi^a = \Phi^a(x^a)$  is the cross map of the reduced order system  $\Sigma^a$ . Substituting  $x^b = 0$  into Eq. (5) in Lemma 1,

$$\begin{pmatrix} \frac{\partial \Phi^a(x^a)}{\partial x^a} f^a(x^a, 0) \\ 0 \end{pmatrix} + \begin{pmatrix} f^a(\Phi^a(x^a), 0) \\ f^b(\Phi^a(x^a), 0) \end{pmatrix} + \begin{pmatrix} g^a(\Phi^a(x^a), 0) \\ g^b(\Phi^a(x^a), 0) \end{pmatrix} h(x^a, 0) = 0$$

which reduces to the Sylvester equation for the reduced order system  $\Sigma^a$  as

$$\frac{\partial \Phi^a(x^a)}{\partial x^a} f^a(x^a, 0) + f^a(\Phi^a(x^a), 0) + g^a(\Phi^a(x^a), 0)h(x^a, 0) = 0.$$

This proves that the reduced order system  $\Sigma^a$  is symmetric. The other balancing properties are proved as follows.

- 1) Proposition 1 implies that both  $L_c(\xi)$  and  $\Phi(x)$  are balanced.

$$L_c(\xi) \equiv \underbrace{\sum_{i=1}^l l_{c,i}(\xi_i)}_{L_c^a(\xi^a)} + \underbrace{\sum_{i=l+1}^n l_{c,i}(\xi_i)}_{L_c^b(\xi^b)}.$$

Since the controllability function of the reduced order system  $\Sigma^a$  is  $L_c^a(\xi^a)$ , it is balanced. The cross map  $\xi^a = \Phi^a(x^a)$  is also proved to be balanced which completes the proof of this part.

- 2) The controllability function of the reduced order system  $\Sigma^a$  is described by

$$L_c^a(\xi^a) \equiv L_c(\xi^a, 0)$$

which implies the weak balancing property

$$\xi_i = 0 \iff \frac{\partial L_c^a(\xi^a)}{\partial \xi_i} = 0, \quad i = 1, 2, \dots, l.$$

This completes the proof. ■

This new result in Theorem 4 allows us to preserve the symmetric property of the original system in addition to the stability, controllability and observability as the outcome of Proposition 2.

#### IV. CONCLUSION

This paper discussed some properties of balanced realization of symmetric nonlinear systems and provides the corresponding model order reduction procedure preserving symmetry of the original system. This has been done by extending the well-known notions for linear system such as gross Gramians and Sylvester equations. It is expected to provide a new insight to nonlinear systems theory.

The future work would take care of the existence of the solutions, the computational issue, and physical application and so on.

#### REFERENCES

- [1] A. C. Antoulas. *Approximation of Large-Scale Dynamical Systems*. SIAM, Philadelphia, 2005.
- [2] A. Astolfi. Model reduction by moment matching for linear and nonlinear systems. *IEEE Trans. Autom. Contr.*, 50(10):235–258, 2010.
- [3] J. Cortes, A. J. van der Schaft, and P. E. Crouch. Characterization of gradient control systems. *SIAM Journal on Control and Optimization*, 44:1192–1214, 2006.
- [4] F. Fagnani and J. C. Willems. Representation of symmetric linear dynamical systems. *SIAM Journal on Control and Optimization*, 31(5):1267–1293, 1993.
- [5] K. V. Fernando and H. Nicholson. On the cross-Gramian for symmetric MIMO systems. *IEEE Trans. Circuits Syst.*, CAS-32(5):487–489, 1985.
- [6] K. Fujimoto. Balanced realization and model order reduction for port-Hamiltonian systems. *Journal of System Design and Dynamics*, 2(3):694–702, 2008.
- [7] K. Fujimoto and J. M. A. Scherpen. Nonlinear input-normal realizations based on the differential eigenstructure of Hankel operators. *IEEE Trans. Autom. Contr.*, 50(1):2–18, 2005.
- [8] K. Fujimoto and J. M. A. Scherpen. Balanced realization and model order reduction for nonlinear systems based on singular value analysis. *SIAM Journal on Control and Optimization*, 48(7):4591–4623, 2010.
- [9] W. S. Gray and J. P. Mesko. Observability functions for linear and nonlinear systems. *Systems & Control Letters*, 38:99–113, 1999.
- [10] J. Hahn and T. F. Edgar. An improved method for nonlinear model reduction using balancing of empirical gramians. *Comp. Chem. Eng.*, 26(10):1379–1397, 2002.
- [11] T. Ionescu, K. Fujimoto, and J. M. A. Scherpen. Dissipativity preserving balancing for nonlinear systems: A Hankel operator approach. *Systems & Control Letters*, 59(3-4):180–194, 2010.
- [12] T. Ionescu, K. Fujimoto, and J. M. A. Scherpen. Singular value analysis of nonlinear symmetric systems. *IEEE Trans. Autom. Contr.*, 56(9):2073–2086, 2011.
- [13] G. Obinata and B. D. O. Anderson. *Model Reduction for Control System Design*. Springer-Verlag, London, 2001.
- [14] H. Sandberg and R. M. Murray. Model reduction of interconnected linear systems. *Optimal control applications and methods*, 30:225–245, 2009.
- [15] J. M. A. Scherpen. Balancing for nonlinear systems. *Systems & Control Letters*, 21:143–153, 1993.
- [16] D. C. Sorenen and A. C. Antoulas. The Sylvester equation and approximate balanced truncation. *Lin. Algebra Appl.*, 351-352:671–700, 2002.
- [17] E. I. Verriest. Time varying balanced realizations and nonlinear balancing. In W. H. A. Schilders, H. A. van der Vorst, and J. Rommes, editors, *Model Order Reduction: Theory, Research Aspects and Applications*, pages 213–250. Springer-Verlag, 2008.