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An Algebra for Cascade Effects

Elie M. Adam, Munther A. Dahleh and Asuman Ozdaglar

Abstract— **We introduce a new class of (dynamical) systems that inherently capture cascading effects (viewed as consequential effects) and are naturally amenable to combinations. We develop an axiomatic general theory around those systems, and guide the endeavor towards an understanding of cascading failure. The theory evolves as an interplay of lattices and fixed points, and its results may be instantiated to commonly studied *models* of cascade effects.**

We characterize the systems through their fixed points, and equip them with two operators. We uncover properties of the operators, and express *global* systems through combinations of *local* systems. We enhance the theory with a notion of failure, and understand the class of shocks inducing a system to failure. We develop a notion of μ-rank to capture the energy of a system, and understand the minimal amount of *effort* required to fail a system, termed *resilience*. We deduce a dual notion of *fragility* and show that the combination of systems sets a limit on the amount of fragility inherited.

Index Terms— **cascade effects, lattice, fixed point, dynamical system, systemic failure.**

AMS Subject Classification— **06B99, 06A15, 93A99.**

EXTENDED ABSTRACT

Cascade effects refer to situations where the expected behavior governing a certain (sub)system appears to be *enhanced* as this component is embedded into a greater system. The effects of change in a subsystem may pass through *interconnections* and enforce an indirect change on the state of any remote subsystem. As such effects are pervasive — appearing in various scenarios of ecological systems, communication infrastructures, financial networks, power grids and societal networks — there is an interest (and rather a need) to understand them. Models are continually proposed to capture instances of cascading behavior, yet the *universal* properties of this phenomenon remain untouched. Our goal is to capture some essence of cascade effects, and develop an axiomatic theory around it.

A reflection on such a phenomenon reveals two informal aspects of it. The first aspect uncovers a notion of *consequence* relation that seemingly drives the phenomenon. Capturing *chains of events* seems to be inescapably necessary. The second aspect projects cascade effects onto a theory of subsystems, combinations and interaction. We should not expect any cascading behavior to occur in *isolation*.

The line of research will be pursued within the context of systemic failure, and set along a guiding informal question. When handed a system of interlinked subsystems, when would a *small* perturbation in some subsystems induce the system to failure? The phenomenon of cascade effects (envisioned in this abstract) restricts the possible systems to those satisfying posed axioms. The analysis of cascade effects shall be perceived through an analysis on these systems.

We introduce a new class of (dynamical) systems that inherently capture cascading effects (viewed as *consequential* effects) and are naturally amenable to combinations. We develop a general theory around those systems, and guide the endeavor towards an understanding of cascading failure. The theory evolves as an interplay of lattices and fixed points, and its results may be instantiated to commonly studied *models* of cascade effects.

Our Systems

The systems will be motivated through an elementary example. This example is labeled M.0 and further referred to throughout the abstract.

Let $G(V, A)$ be a digraph, and define $N(S) \subset V$ to be the set of nodes j with $(j, i) \in A$ and $i \in S$. A vertex is of one of two colors, either black or white. The vertices are initially colored, and X_0 denotes the set of black colored nodes. The system evolves through discrete time to yield $X_1, X_2, \cdots$ sets of black colored nodes. Node i is colored black at step $m+1$ if any of its neighbors in $N(i)$ is black at step m. Once a node is black it remains black forever.

Our systems will consist of a collection of states along with internal dynamics. The collection of states is a finite set P. The dynamics dictate the evolution of the system through the states and are *governed* by a class of maps $P \to P$. The state space in M.0 is the set 2^V where each $S \subset 2^V$ identifies a subset of *black* colored nodes; the dynamics are dictated by $g : X \mapsto X \cup N(X)$ as $X_{m+1} = gX_m$.

We intuitively consider some states to be *worse* or *less desirable* than others. The color *black* may be undesirable in M.0, representing a *failed* state of a node. State S is then considered to be worse than state T if it includes T. We formalize this notion by equipping P with a partial order $\leq$. The order is only partial as not every pair of state may be comparable. It is natural to read $a \leq b$ in this abstract as b is a worse (or less desirable) state than a. The state space 2^V in M.0 is ordered by set inclusion $\subset$.

We expect two properties from the dynamics driving the systems. We require the dynamics to be *progressive*. The system may only evolve to a state considered less desirable than its initial state. We also require *undesirability* to be preserved during an evolution. The less desirable the initial state of a system is, the less desirable the final state the system evolves to will be. We force each map $f : P \to P$ governing the dynamics to satisfy two axioms:

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All authors are with the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA 02139. (emails: eadam@mit.edu, dahleh@mit.edu, asuman@mit.edu)

A.1 If $a \in P$, then $a \leq fa$.

A.2 If $a, b \in P$ and $a \leq b$, then $fa \leq fb$.

The map $X \mapsto X \cup N(X)$ in M.0 satisfies both A.1 and A.2 as $S \subset S \cup N(S)$, and $S \cup N(S) \subset S' \cup N(S')$ if $S \subset S'$.

Our interest lies in the limiting outcome of the dynamics, and the understanding we wish to develop may be solely based on the *asymptotic* behavior of the system. In M.0, we are interested in the set X_m for m large enough as a function of X_0. As V is finite, it follows that $X_m = X_{|V|}$ for $m \geq |V|$. We are thus interested in the map $g^{|V|} : X_0 \mapsto X_{|V|}$. More generally, as iterative composition of a map satisfying A.1 and A.2 eventually yield idempotent maps, we equip the self-maps f on P with a third axiom:

A.3 If $a \in P$, then $ffa = fa$.

Our class of interest is the (self-)maps (on P) satisfying axioms A.1, A.2 and A.3. Each system will be identified with one such map. The system generated from an instances of M.0 corresponds to the map $X_0 \mapsto X_{|V|}$. We further restrict our attention to posets $(P, \leq)$ that are lattices, as opposed to arbitrary posets.

Models and Examples

The axioms A.1 and A.2 hold for typical "models" adopted for cascade effects. We present three models (in addition to M.0 provided in Section I) supported on the Boolean lattice, two of which — M.1 and M.3 — are standard examples (see [3], [6] and [7]). It can be helpful to identify a set 2^S with the set of all $black$ and $white$ coloring on the objects of S. A subset of S then denotes the objects colored $black$. The model M.1 generalizes M.0 by assigning *thresholds* to nodes in the graph. Node i is colored $black$ when the number of neighbors colored $black$ surpasses its threshold. The model M.2 is *noncomparable* to M.0 and M.1, and the model M.3 generalizes all of M.0, M.1 and M.2.

M.1 Given a digraph over a set S or equivalently a map $N : S \to 2^S$, a map $k : S \to \mathbb{N}$ and a subset X_0 of S, let $X_1, X_2, \cdots$ be subsets of S recursively defined such that $i \in X_{m+1}$ if, and only if, either $|N_i \cap X_m| \geq k_i$ or $i \in X_m$.

M.2 Given a collection $\mathcal{C} \in 2^S$ for some set S, a map $k : \mathcal{C} \to \mathbb{N}$ and a subset X_0 of S, let $X_1, X_2, \cdots$ be subsets of S recursively defined such that $i \in X_{m+1}$ if, and only if, either there is a $C \in \mathcal{C}$ such that $|C \cap X_m| \geq k_c$ or $i \in X_m$.

M.3 Given a collection $\phi_1, \cdots, \phi_n$ of monotone maps from 2^S into $\mathbb{F}_2$ for some set S and a subset X_0 of S, let $X_1, X_2, \cdots$ be subsets of S recursively defined such that $i \in X_{m+1}$ if, and only if, either $\phi_i(X_m) = 1$ or $i \in X_m$.

We necessarily have $X_{|S|} = X_{|S|+1}$ in the three cases above, and the map $X_0 \mapsto X_{|S|}$ then satisfies A.1, A.2 and A.3. The dynamics depicted above may be captured in a more general form.

M.4 Given a finite lattice L, an order-preserving map $h : L \to L$, and $x_0 \in L$, let $x_1, x_2, \cdots \in L$ be recursively defined such that $x_{m+1} = x_m \vee h(x_m)$.

We have $x_{|L|} = x_{|L|+1}$ and the map $x_0 \mapsto x_{|L|}$ satisfies A.1, A.2 and A.3.

The axioms allow greater variability if the state space is modified or augmented accordingly. We nevertheless restrict our attention to systems of the above form.

The axioms A.1, A.2 and A.3 naturally permeate a number of areas of mathematics and logic. Within metamathematics and (universal) logic, Tarski introduced these three axioms (along with supplementary axioms) and launched his theory of consequence operator (see [10] and [12]). He aimed to provide a general characterization of the notion of deduction. As such, if S represents a set of statements taken to be true (i.e. premises), and $Cn(S)$ denotes the set of statements that can be *deduced* to be true from S, then Cn (as an operator) obeys A.1, A.2 and A.3. In formal languages, we consider a set A, called the *alphabet*, whose elements are called *symbols*. A finite string of symbols is called a *word*. Let A^* denote the set of all words, a subset of L of A^* is called a *language*. The Kleene star $*$ sends each language L to L^*, the language formed by *concatenating* words of L, e.g. $\{a, ba\}^* = \{a, ba, aa, aba, baa, baba, aaa, aaba, \cdots\}$. The operator $*$ satisfies the axioms when languages are ordered by inclusion. Many familiar maps also adhere to the axioms. As examples, we may consider the function that maps (i) a subset of a topological space to its topological closure, (ii) a subset of a vector space to its linear span, (iii) a subset of an algebra (e.g. group) to the subalgebra (e.g. subgroup) it generates, (iv) a subset of a Euclidean n-space to its convex hull. Such functions may be referred to as *closure operators* (see e.g. [1], [2], [8] and [13]), and are typically objects of study in *universal algebra*.

Outline of the Contribution

We illustrate the contribution through M.0. We define f and g to be the systems derived from two instances (V, A) and (V, A') of M.0.

We establish that our systems are uniquely identified with their set of fixed points. We can reconstruct f knowing only the sets S containing $N(S)$ (i.e. the fixed points of f) with no further information on (V, A). We further provide a complete characterization of the systems through the fixed points. The characterization yields a remarkable conceptual and analytical simplification in the study.

We equip the systems with a lattice structure, uncover operators ($+$ and $\cdot$) and express *complex* systems through formulas built from *simpler* systems. The $+$ operator *combines* the effect of systems, possibly derived from different models. The system $f + g$ is derived from $(V, A \cup A')$. The $\cdot$ operator *projects* systems onto each other allowing, for instance, the recovery of *local* evolution rule. We fundamentally aim to extract properties of $f + g$ and $f \cdot g$ through properties of f and g separately. We show that $+$ and $\cdot$ lend themselve to well behaved operations when systems are represented through their fixed-points.

We realize the systems as interlinked components and formalize the notion of *cascade effects*. Nodes in V are identified with maps $e_1, \cdots, e_{|V|}$. The system $f \cdot e_i$ then defines the evolution of i as a function of the system state, and is identified with the set of nodes that *reach* i in (V, A).

We draw a connection between shocks and systems, and enhance the theory with a notion of failure. We show that minimal shocks (that fail a system h) exhibit a unique property that uncovers *complement* subsystems in h, termed *weaknesses*. A system is shown to be *injectively* decomposed into its *weaknesses*, and any weakness in $h+h'$ cannot result but from the combination of weaknesses in h and h'.

We introduce a notion of μ-rank of a system—akin to the (analytic) notion of a "norm" as used to capture the energy of a system—and show that such a notion is unique should it adhere to natural principles. The μ-rank is tied to the number of connected components in (V, A) when A is symmetric.

We finally set to understand the minimal amount of *effort* required to fail a system, termed *resilience*. Weaknesses reveals a dual (equivalent) quantity, termed *fragility*, and further puts resilience and μ-rank on comparable grounds. The fragility is tied to the size of the largest connected component in (V, A) when A is symmetric. It is thus possible to formally define *high ranked* systems that are not necessarily *fragile*. The combination of systems sets a limit on the amount of fragility the new system inherits. Combining two subsystems cannot form a fragile system, unless one of the subsystems is initially fragile.

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