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On Robustness of Sampled-Data Systems with Nonuniform Samplings

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Abstract— **This manuscript is concerned with stability analysis of linear sampled-data systems with non-uniform sampling patterns. The stability problem is tackled from a continuous-time point of view, via the so-called "input delay approach". The system is viewed as feedback interconnection of a stable linear time-varying system and the "average delay-difference" operator, which is utilized to model the "aperiodic sampling operation". Characterization based on integral quadratic constrains (IQC) is identified for the "average delay-difference" operator, and the IQC theory is applied to derive convex stability criteria. The link between the IQC approach and the the Lyapunov-Krasovskii approach are made, and it is shown that one of the proposed criteria is equivalent to the main result of [1], which generalizes many previous results. Moreover, a new criterion which is proven to be less conservative is proposed.**

I. INTRODUCTION

This manuscript concerns the linear sampled-data systems of the following form

$$\begin{aligned} &\dot{x}(t) = Ax(t) + B(u(t) + f(t)),\ x(0) = 0 \\ &u(t) = Fx(t_k),\ \forall t \in [t_k, t_{k+1}] \end{aligned} \tag{1}$$

where x and u respectively denote the state and the control input taking values in $\mathbb{R}^n$ and $\mathbb{R}^m$, $f \in \mathcal{L}_2$ is a finite-energy disturbance, and the sampling sequences $\{t_k\}_{k=0}^{\infty} := \mathcal{S}$ satisfy $0 = t_0 < t_1 < \cdots < t_k < \cdots$, with $\lim_{k\to\infty} t_k = \infty$, and $h_\ell \leq t_{k+1} - t_k \leq h_u$. By representing the sampled-data input u as the result of a continuous time function subject to a time-varying delay, robustness of (1) against the nonuniform and uncertain sampling pattern is addressed by an operator-theoretic approach known as Integral Quadratic Constraint (IQC) analysis.

The technical contributions of this manuscript is twofolds. First, we prove that one criterion derived by the proposed IQC analysis is equivalent to the LMI stability condition proposed in [1], which is fairly recent and to our knowledge generalizes many previous results such as [2], [3], [4]. The link between IQCs of the time-varying delay operator and pieces of Lyapunov functionals commonly seen in the time-delay systems literature is established. Secondly, a novel IQC is proposed, which apparently leads to a new criterion which is more flexible and less conservative than the one proposed in [1].

Notation and Terminology

The notation we adopt here is commonly seen in the control theory literature. Symbols I_n and $0_{n\times m}$ is used to denote n-dimensional identity matrix and $n \times m$ zero matrices. The subscripts and superscripts are dropped when the dimension is evident from the context. Given a matrix M, the transposition and the conjugate transposition are denoted by M' and M^*, respectively. Symbol $\mathcal{L}_2$ denotes the space of $\mathbb{R}^n$-valued, square integral functions defined on positive real axis.

Given a bounded operator G on the $\mathcal{L}_2$ space, we use $\|G\|_{\mathcal{L}_2}$ to denote the $\mathcal{L}_2$ induced norm of G. The adjoint of G is denoted by G^*. When G is linear time-invariant, G is represented in the frequency domain by a transfer matrix, which is denoted by the same symbol.

Given a Hermitian matrix M (i.e., $M = M^*$), the notation $M > 0$ ("$\geq$","$<$", and "$\leq$", respectively) is used to denote positive definiteness (positive semi-definiteness, negative definiteness, and negative semi-definiteness, respectively). The same notations are applied to self-adjoint operators on $\mathcal{L}_2$.

With a bounded self-adjoint operator Π, we say operator $\Delta : \mathcal{L}_2 \to \mathcal{L}_2$ satisfies the integral quadratic constraint (IQC) defined by Π if the integral quadratic inequality

$$\sigma_\Pi(v, \Delta v) := \left\langle \begin{bmatrix} v \\ \Delta v \end{bmatrix}, \Pi \begin{bmatrix} v \\ \Delta v \end{bmatrix} \right\rangle \geq 0$$

holds for all $v \in \mathcal{L}_2$. The operator Π is referred to as the multiplier of the quadratic form σ_Π, which is often block partitioned according to the dimensions of v and Δv. Furthermore, the multiplier Π is often parameterized by certain parameter, say X, and we sometimes write $\Pi(X)$ to indicate this dependency.

II. MAIN RESULTS

To tackle the problem of certifying robust stability of system (1) against nonuniform sampling, we view (1) as feedback interconnection of a nominal linear time-varying system G

$$\begin{aligned} &\dot{x}(t) = (A + BF)x(t) - BF\tau(t)w(t) + Bf(t),\ x(0) = 0 \\ &v(t) = \dot{x}(t) = (A + BF)x(t) - BF\tau(t)w(t) \end{aligned} \tag{2}$$

and an "average delay-difference" operator $\Upsilon : v \mapsto w$

$$\begin{aligned} (\Upsilon v)(t) &:= \frac{1}{\tau(t)} \int_{t_k}^{t} v(s)ds \\ &= \frac{1}{\tau(t)}(x(t) - x(t_k)), \quad t \in [t_k, t_{k+1}) \end{aligned}$$

where $\tau(t)$ is the sawtooth function: $\tau(t) = t - t_k,\ t \in [t_k, t_{k+1})$. Let the function x^s be defined as $x^s(t) \equiv x(t_k)$ for $t \in [t_k, t_{k+1})$; i.e., x^s is obtained by subjecting x to the

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time-varying delay τ. Then $(\Upsilon v)(t) = (x(t) - x^s(t))/\tau(t)$ and hence the name "average delay-difference" for Υ.

To facilitate the development, let us denote the integral operation by $\mathcal{I}$; i.e. $(\mathcal{I}v)(t) := \int_0^t v(s)ds$. Moreover, let ϕ be $\phi(t) = t_{k+1} - t,\ t \in [t_k, t_{k+1})$, which is another sawtooth function that satisfies $\phi(t) + \tau(t) = t_{k+1} - t_k$ for all t. In the followings, we will slightly abuse the notation by using $\frac{1}{\tau}$ to denote the time-domain multiplication operator $v(t) \mapsto \frac{1}{\tau(t)} v(t)$ for all t. In the followings, the same notation will be applied to multiplication with scalar functions involving τ and ϕ.

IQC characterization for Υ, which will be utilized to derive robust stability criteria for (1), are summarized in the following proposition.

Proposition 1: Υ satisfies integral quadratic constraints defined by each and any conic combination[1] of the multipliers $\Pi_1(X_1, X_2)$, $\Pi_2(X_3, X_4)$, $\Pi_3(X_5, X_6)$ defined in (3) to (5), where X_1 is any positive semi-definite matrix, X_2, X_3, X_5 are any symmetric matrices, and X_4, X_6 are any real matrices.

$$\Pi_1(X_1, X_2) = \begin{bmatrix} \phi X_1 & \phi\tau X_2 \\ \phi\tau X_2 & -\tau X_1 - \tau^2 X_2 \end{bmatrix}, \tag{3}$$

$$\begin{aligned} &\Pi_2(X_3, X_4) = \\ &\begin{bmatrix} \mathcal{I} & 0 \\ \mathcal{I} & -\tau I \end{bmatrix}^* \begin{bmatrix} 0 & 0 \\ \phi X_4' & 0 \end{bmatrix} + \begin{bmatrix} 0 & \phi X_4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{I} & 0 \\ \mathcal{I} & -\tau I \end{bmatrix} + \\ &\begin{bmatrix} \mathcal{I} & 0 \\ \mathcal{I} & -\tau I \end{bmatrix}^* \begin{bmatrix} 0 & -X_4 \\ -X_4' & (\phi - \tau)X_3 + X_4 + X_4' \end{bmatrix} \begin{bmatrix} \mathcal{I} & 0 \\ \mathcal{I} & -\tau I \end{bmatrix} \end{aligned} \tag{4}$$

$$\begin{aligned} &\Pi_3(X_5, X_6) = \\ &\begin{bmatrix} \mathcal{I} & 0 \\ \mathcal{I} & -\tau I \end{bmatrix}^* \begin{bmatrix} (\phi - \tau)X_5 & (\phi - \tau)X_6 \\ (\phi - \tau)X_6' & 0 \end{bmatrix} \begin{bmatrix} \mathcal{I} & 0 \\ \mathcal{I} & -\tau I \end{bmatrix} + \\ &\begin{bmatrix} \mathcal{I} & 0 \\ \mathcal{I} & -\tau I \end{bmatrix}^* \begin{bmatrix} \phi\tau X_5 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \phi\tau X_5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{I} & 0 \\ \mathcal{I} & -\tau I \end{bmatrix} + \\ &\begin{bmatrix} I & 0 \\ \mathcal{I} & -\tau I \end{bmatrix}^* \begin{bmatrix} 0 & \phi\tau X_6 \\ \phi\tau X_6' & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathcal{I} & -\tau I \end{bmatrix} \end{aligned} \tag{5}$$

Some of these IQCs are shown in [5]. With the IQCs described in Proposition 1, the IQC theorem [6] can be applied to (2) to derive the following stability criterion for aperiodic sampled-data system (1).

Proposition 2: The aperiodic sampled-data system (1) is robustly stable for all sampling sequences satisfying $h_k := t_{k+1} - t_k \in [h_\ell, h_u]$ if there exist $X_1 = X_1' > 0$, $X_2 = X_2'$, $X_3 = X_3'$, X_4, $X_5 = X_5'$, X_6, $\varepsilon > 0$ such that

$$\begin{bmatrix} G \\ I \end{bmatrix}^* \Pi_{\mathrm{c}}(X_1, \cdots, X_6) \begin{bmatrix} G \\ I \end{bmatrix} \leq -\varepsilon I \tag{6}$$

where G has the state-space representation as defined in (2), and $\Pi_{\mathrm{c}} = \Pi_1 + \Pi_2 + \Pi_3$.

The operator inequality (6) gives rise to an integral quadratic inequality in time-domain. It can be shown that such inequality holds if certain parameter-dependent linear matrix inequalities w.r.t. X_1 to X_6 and some auxiliary matrix variables are satisfied. The result is stated in the following proposition.

Proposition 3: Condition (6) holds if there exists $P = P' > 0$ and $\varepsilon > 0$ such that[2]

$$(\star)'\Psi(t) \begin{bmatrix} A + BF & -BF \\ I & 0 \\ 0 & I \\ 0 & \frac{1}{\tau(t)} I \end{bmatrix} \leq -\varepsilon I,\ \forall t \in \mathbb{R}_+, \tag{7}$$

where $\Psi(t)$ is equal to

$$\begin{bmatrix} \Psi_{11}(t) & \Psi_{12}(t) & \Psi_{13}(t) & 0 \\ (\star) & \Psi_{22}(t) & \Psi_{23}(t) & 0 \\ (\star) & (\star) & \Psi_{33}(t) & 0 \\ (\star) & (\star) & (\star) & -\tau(t)X_1 \end{bmatrix} \tag{8}$$

and

$$\begin{aligned} \Psi_{11}(t) &= \phi(t)X_1, \\ \Psi_{12}(t) &= \phi(t)X_4 + \phi(t)\tau(t)(X_6 + X_5) + P, \\ \Psi_{13}(t) &= \phi(t)(X_2 - X_4) - \phi(t)\tau(t)X_6, \\ \Psi_{22}(t) &= (\phi(t) - \tau(t))(X_3 + X_5 + X_6 + X_6'), \\ \Psi_{23}(t) &= -(\phi(t) - \tau(t))(X_3 + X_6) - X_4', \\ \Psi_{33}(t) &= -X_2 + (\phi(t) - \tau(t))X_3 + (X_4 + X_4') \end{aligned}$$

Condition (7) is infinite-dimensional in that the inequality has to hold for all $t \in \mathbb{R}_+$. Note that the time-dependency is due to $\phi(t)$ and $\tau(t)$, which by their definitions range within the convex polyhedron: $\mathcal{R}_{\phi\tau} := \{(\phi, \tau) : h_\ell \leq \phi + \tau \leq h_u, \phi \geq 0, \tau \geq 0\}$. Thus, one can verify the condition *approximately* by evaluating the left-hand-side of (7) on a dense grid of $\mathcal{R}_{\phi\tau} \backslash \{(\phi, 0) : h_\ell \leq \phi \leq h_u\}$. It is also possible to verify condition (7) *exactly* by applying the so-called robust semi-definite programming technique. To this end, we require the following lemma.

Lemma 1 (Finsler's lemma): Given matrices $A = A' \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times n}$, the following two statements are equivalent

(a) $x'Ax < 0$ for all $x \neq 0$ such that $Bx = 0$;
(b) There exists $Y \in \mathbb{R}^{n \times m}$ such that $A + YB + B'Y' < 0$.

To facilitate the development, from now on we will drop the time-dependency of Ψ but write $\Psi(\phi, \tau, \theta)$ to emphasize the dependency of Ψ on ϕ, τ, and $\theta := \phi\tau$. One can readily verify that the right-hand-side of (7) is equal to

$$\begin{aligned} &(\star)' \begin{bmatrix} \tilde{\Psi}_{11}(\mu, \tau, \theta) & \tilde{\Psi}_{12}(\mu, \tau, \theta) & 0 \\ \tilde{\Psi}_{12}(\mu, \tau, \theta)' & \tilde{\Psi}_{22}(\mu, \tau, \theta) & 0 \\ 0 & 0 & -\tau X_3 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \\ 0 & \frac{1}{\tau} I \end{bmatrix} \\ &:= \hat{\Psi}(\mu, \tau, \theta) \end{aligned}$$

where $\tilde{\Psi}_{11}$, $\tilde{\Psi}_{12}$, $\tilde{\Psi}_{22}$ are defined as follows

$$\begin{bmatrix} \tilde{\Psi}_{11}(\cdot,\cdot,\cdot) & \tilde{\Psi}_{12}(\cdot,\cdot,\cdot) \\ (\star) & \tilde{\Psi}_{22}(\cdot,\cdot,\cdot) \end{bmatrix} = (\star)'\Psi(\cdot,\cdot,\cdot) \begin{bmatrix} \tilde{A} & BF \\ I & 0 \\ 0 & -I \\ 0 & 0 \end{bmatrix} \tag{9}$$

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[1] Note that by definition, if Υ satisfies IQCs defined by Π_1 and Π_2, then it satisfies the IQC defined by $\chi_1\Pi_1 + \chi_2\Pi_2$, $\chi_1 \geq 0$, $\chi_2 \geq 0$.

[2] Throughout the manuscript, we use $(\star)$ to denote the part of a matrix that is necessary to make the matrix symmetric.

Here $\tilde{A} := A + BF$.

Thus it is clear that, for any given (ϕ, τ, θ) with nonzero τ, $\hat{\Psi} < 0$ if and only if $\tilde{y}'\tilde{\Psi}(\phi, \tau, \theta)\tilde{y} < 0$ for all $\tilde{y} \neq 0$ such that $B(\tau)\tilde{y} = 0$, where $B(\tau) = \begin{bmatrix} 0 & I & \tau I \end{bmatrix}$ and

$$\tilde{\Psi}(\cdot,\cdot,\cdot) := \begin{bmatrix} \tilde{\Psi}_{11}(\cdot,\cdot,\cdot) & \tilde{\Psi}_{12}(\cdot,\cdot,\cdot) & 0 \\ \tilde{\Psi}_{12}(\cdot,\cdot,\cdot)' & \tilde{\Psi}_{22}(\cdot,\cdot,\cdot) & 0 \\ 0 & 0 & -\tau X_1 \end{bmatrix}$$

By applying Lemma 1, $\hat{\Psi}(\phi, \tau, \theta) < 0$ if and only if

$$\tilde{\Psi}(\phi,\tau,\theta) + \begin{bmatrix} 0 & Y_1 & \tau Y_1 \\ (\star)' & Y_2 + Y_2' & \tau Y_2 + Y_3' \\ (\star)' & (\star)' & \tau(Y_3 + Y_3') \end{bmatrix} < 0 \qquad (10)$$

for some $Y_i \in \mathbb{R}^{n \times n}$, $i = 1, 2, 3$. This in turn implies that condition (7) holds (for all $t \in \mathbb{R}_+$) if and only if inequality (10) holds for all $(\phi, \tau, \theta) \in \{(\phi, \tau, \phi\tau) : h_\ell \leq \phi + \tau \leq h_u, \phi \geq 0, \tau > 0\} := \mathcal{R}_{\phi\tau\theta}$.

A sufficient condition for inequality (10) to hold for all $(\phi, \tau, \theta) \in \mathcal{R}_{\phi\tau\theta}$, which involves only a finite number of LMIs, can be derived as follows. First, it can be shown that $\mathcal{R}_{\phi\tau\theta}$ is a subset of the convex polyhedron with the following vertices: $(h_u^-, 0^+, 0)$, $(h_\ell^-, 0^+, 0)$, $(0, h_u, 0)$, $(0, h_\ell, 0)$, $(\frac{h_u}{4}, \frac{3h_u}{4}, \frac{h_u^2}{4})$, $(\frac{3h_u}{4}, \frac{h_u}{4}, \frac{h_u^2}{4})$, $(\frac{h_\ell}{4}, \frac{3h_\ell}{4}, \frac{h_\ell^2}{4})$, $(\frac{3h_\ell}{4}, \frac{h_\ell}{4}, \frac{h_\ell^2}{4})$ (where, $(h_u^-, 0^+, 0)$ $(h_\ell^-, 0^+, 0)$ respectively denote the points $(h_u - \epsilon, \epsilon, 0)$, $(h_\ell - \epsilon, \epsilon, 0)$, $\epsilon > 0$, and $\epsilon \to 0$). Second, we note that the left-hand-side of (10) is linear w.r.t. (ϕ, τ, θ). Thus, to ensure inequality (10) holds on every point of the aforementioned convex polyhedron, it is necessary and sufficient to verify the inequality on the vertices. This leads to the following theorem.

Theorem 1: The aperiodic sampled-data system (1) is robustly stable for all sampling sequences satisfying $h_k := t_{k+1} - t_k \in [h_\ell, h_u]$ if there exist $P = P' > 0$, $X_1 = X_1' > 0$, $X_2 = X_2'$, $X_3 = X_3'$, X_4, $X_5 = X_5'$, X_6, Y_1, and Y_2 such that

$$\begin{bmatrix} \tilde{\Psi}_{11}(h_\ell, 0, 0) & \tilde{\Psi}_{12}(h_\ell, 0, 0) + Y_1 \\ (\star) & \tilde{\Psi}_{22}(h_\ell, 0, 0) + Y_2 + Y_2' \end{bmatrix} < 0 \qquad (11)$$

$$\begin{bmatrix} \tilde{\Psi}_{11}(h_u, 0, 0) & \tilde{\Psi}_{12}(h_u, 0, 0) + Y_1 \\ (\star) & \tilde{\Psi}_{22}(h_u, 0, 0) + Y_2 + Y_2' \end{bmatrix} < 0 \qquad (12)$$

and

$$\begin{bmatrix} \tilde{\Psi}_{11}(\phi, \tau, \theta) & \tilde{\Psi}_{12}(\phi, \tau, \theta) + Y_1 & \tau Y_1 \\ (\star) & \tilde{\Psi}_{22}(\phi, \tau, \theta) + Y_2 + Y_2' & \tau Y_2 \\ (\star) & (\star) & -\tau X_1 \end{bmatrix} < 0 \qquad (13)$$

for $(\phi, \tau, \theta) = (0, h_u, 0)$, $(0, h_\ell, 0)$, $(h_u/4, 3h_u/4, h_u^2/4)$, $(3h_u/4, h_u/4, h_u^2/4)$, $(h_\ell/4, 3h_\ell/4, h_\ell^2/4)$, $(3h_\ell/4, h_\ell/4, h_\ell^2/4)$.

The above theorem is obtained based on the integral quadratic constraint defined by $\Pi_1 + \Pi_2 + \Pi_3$. If Π_3 is neglected and we only take $\Pi_1 + \Pi_2$ in condition (6), the corresponding stability criterion leads to the following theorem.

Theorem 2: The aperiodic sampled-data system (1) is robustly stable for all sampling sequences satisfying $h_k := t_{k+1} - t_k \in [h_\ell, h_u]$ if there exist $P = P' > 0$, $X_1 = X_1' > 0$, $X_2 = X_2'$, $X_3 = X_3'$, X_4, Y_1, and Y_2 such that

$$\begin{bmatrix} \tilde{\Psi}_{11}^r(h_\ell, 0) & \tilde{\Psi}_{12}^r(h_\ell, 0) + Y_1 \\ (\star) & \tilde{\Psi}_{22}^r(h_\ell, 0) + Y_2 + Y_2' \end{bmatrix} < 0 \qquad (14)$$

$$\begin{bmatrix} \tilde{\Psi}_{11}^r(h_u, 0) & \tilde{\Psi}_{12}^r(h_u, 0) + Y_1 \\ (\star) & \tilde{\Psi}_{22}^r(h_u, 0) + Y_2 + Y_2' \end{bmatrix} < 0 \qquad (15)$$

$$\begin{bmatrix} \tilde{\Psi}_{11}^r(0, h_\ell) & \tilde{\Psi}_{12}^r(0, h_\ell) + Y_1 & h_\ell Y_1 \\ (\star) & \tilde{\Psi}_{22}^r(0, h_\ell) + Y_2 + Y_2' & h_\ell Y_2 \\ (\star) & (\star) & -h_\ell X_1 \end{bmatrix} < 0 \qquad (16)$$

$$\begin{bmatrix} \tilde{\Psi}_{11}^r(0, h_u) & \tilde{\Psi}_{12}^r(0, h_u) + Y_1 & h_u Y_1 \\ (\star) & \tilde{\Psi}_{22}^r(0, h_u) + Y_2 + Y_2' & h_u Y_2 \\ (\star) & (\star) & -h_u X_1 \end{bmatrix} < 0 \qquad (17)$$

where $\tilde{\Psi}_{ij}^r(\phi, \tau)$ are obtained by removing matrices X_5 and X_6 from $\tilde{\Psi}_{ij}(\phi, \tau, \theta)$ (i.e., setting X_5 and X_6 to zero). Note that by removing X_5 and X_6, the resulting $\tilde{\Psi}_{ij}^r$ no longer depends on the term $\theta := \phi\tau$.

In the case where $h_\ell \to 0$, Theorems 1 and 2 are simplified to the following Corollaries 1 and 2, respectively.

Corollary 1: The aperiodic sampled-data system (1) is robustly stable for all sampling sequences satisfying $h_k := t_{k+1} - t_k \in (0, h_u]$ if there exist $P = P' > 0$, $X_1 = X_1' > 0$, $X_2 = X_2'$, $X_3 = X_3'$, X_4, $X_5 = X_5'$, X_6, Y_1, and Y_2 such that

$$\begin{bmatrix} \tilde{\Psi}_{11}(0, 0, 0) & \tilde{\Psi}_{12}(0, 0, 0) + Y_1 \\ (\star) & \tilde{\Psi}_{22}(0, 0, 0) + Y_2 + Y_2' \end{bmatrix} < 0 \qquad (18)$$

$$\begin{bmatrix} \tilde{\Psi}_{11}(h_u, 0, 0) & \tilde{\Psi}_{12}(h_u, 0, 0) + Y_1 \\ (\star) & \tilde{\Psi}_{22}(h_u, 0, 0) + Y_2 + Y_2' \end{bmatrix} < 0 \qquad (19)$$

and

$$\begin{bmatrix} \tilde{\Psi}_{11}(\phi, \tau, \theta) & \tilde{\Psi}_{12}(\phi, \tau, \theta) + Y_1 & \tau Y_1 \\ (\star) & \tilde{\Psi}_{22}(\phi, \tau, \theta) + Y_2 + Y_2' & \tau Y_2 \\ (\star) & (\star) & -\tau X_1 \end{bmatrix} < 0 \qquad (20)$$

for $(\phi, \tau, \theta) = (0, h_u, 0)$, $(h_u/4, 3h_u/4, h_u^2/4)$, $(3h_u/4, h_u/4, h_u^2/4)$.

Corollary 2: The aperiodic sampled-data system (1) is robustly stable for all sampling sequences satisfying $h_k := t_{k+1} - t_k \in (0, h_u]$ if there exist $P = P' > 0$, $X_1 = X_1' > 0$, $X_2 = X_2'$, $X_3 = X_3'$, X_4, Y_1, and Y_2 such that

$$\begin{bmatrix} \tilde{\Psi}_{11}^r(h_\ell, 0) & \tilde{\Psi}_{12}^r(h_\ell, 0) + Y_1 \\ (\star) & \tilde{\Psi}_{22}^r(h_\ell, 0) + Y_2 + Y_2' \end{bmatrix} < 0 \qquad (21)$$

$$\begin{bmatrix} \tilde{\Psi}_{11}^r(h_u, 0) & \tilde{\Psi}_{12}^r(h_u, 0) + Y_1 \\ (\star) & \tilde{\Psi}_{22}^r(h_u, 0) + Y_2 + Y_2' \end{bmatrix} < 0 \qquad (22)$$

$$\begin{bmatrix} \tilde{\Psi}_{11}^r(0, h_u) & \tilde{\Psi}_{12}^r(0, h_u) + Y_1 & h_u Y_1 \\ (\star) & \tilde{\Psi}_{22}^r(0, h_u) + Y_2 + Y_2' & h_u Y_2 \\ (\star) & (\star) & -h_u X_1 \end{bmatrix} < 0 \qquad (23)$$

where $\tilde{\Psi}_{ij}^r(\phi, \tau)$ are defined as in Theorem 2.

III. LINK BETWEEN THE LYAPUNOV-KRASOVSKII APPROACH AND THE IQC APPROACH

In this section, the link between several robust stability criteria derived based on the Lyapunov-Krasovskii approach for system (1) and those by the IQC approach are made. Specifically, we show that the main theorem (Theorem 2) of a recent article [1], which is restated as Proposition 4 in the followings, is equivalent to Theorem 2. As Theorem 2 of [1] surpasses (to the author's knowledge) previous results by the Lyapunov-Krasovskii approach, such as those proposed in [3] and [2], one can infer that those proposed in [3] and [2] are special cases of Theorem 2 of this paper.

Proposition 4 (Seuret [1]): The aperiodic sampled-data system (1) is robustly stable for all sampling sequences satisfying $h_k := t_{k+1} - t_k \in [h_\ell, h_u]$ if there exist $P_s > 0$, $R > 0$, $S_1 = S_1'$, $S_3 = S_3'$, $S_2 \in \mathbb{R}^{n\times n}$, and $N \in \mathbb{R}^{2n\times n}$ such that the following LMIs is feasible.

$$\Omega_1 + \alpha(\Omega_2 + \Omega_3) < 0, \quad \begin{bmatrix} \Omega_1 - \alpha\Omega_3 & \alpha N \\ (\star) & -\alpha R \end{bmatrix} < 0, \qquad (24)$$

$$\text{for } \alpha \in \{h_\ell, h_u\}$$

where

$$\Omega_1 = (\star)' + \left\{ M_1' P_s M_0 - M_{12}' \left(\frac{1}{2} S_1 M_{12} + S_2 M_2 + N' \right) \right\},$$

$$\Omega_2 = M_0' R M_0 + M_0'(S_1 M_{12} + S_2 M_2) + (S_1 M_{12} + S_2 M_2)' M_0,$$

$$\Omega_3 = M_2' S_3 M_2,$$

with $M_0 = \begin{bmatrix} A & BF \end{bmatrix}$, $M_1 = \begin{bmatrix} I & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & I \end{bmatrix}$ and $M_{12} = M_1 - M_2$.

The link between Proposition 4 and Theorem 2 is stated as follows.

Theorem 3: There exist $P = P' > 0$, $X_1 = X_1' > 0$, $X_2 = X_2'$, $X_3 = X_3'$, X_4, Y_1, and Y_2 such that LMIs (14) to (17) hold if and only if there exist $P_s > 0$, $R > 0$, $S_1 = S_1'$ and $S_3 = S_3'$, $S_2 \in \mathbb{R}^{n\times n}$, and $N \in \mathbb{R}^{2n\times n}$ such that LMIs stated in (24) hold. More specifically, LMIs (14) and (15) corresponds to LMIs $\Omega_1 + h_\ell(\Omega_2 + \Omega_3) < 0$ and $\Omega_1 + h_u(\Omega_2 + \Omega_3) < 0$ in (24), respectively, while LMIs (16) and (17) corresponds to the other two LMIs in (24).

Proof: Pre- and post-multiplying $\Omega_1 + h_\ell(\Omega_2 + \Omega_3) < 0$ with $\begin{bmatrix} I & 0 \\ I & I \end{bmatrix}'$ and $\begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$ and replacing P_s by P, R by X_1, S_1 by X_2, S_2 by X_4, S_3 by X_3, and N by $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$, we obtain LMI (14). By exactly the same arguments, we see the equivalence between $\Omega_1 + h_u(\Omega_2 + \Omega_3) < 0$ and LMI (15). To show the equivalence between $\begin{bmatrix} \Omega_1 - h_\ell\Omega_3 & h_\ell N \\ (\star) & -h_\ell R \end{bmatrix} < 0$ and LMI (16), $\begin{bmatrix} \Omega_1 - h_u\Omega_3 & h_u N \\ (\star) & -h_u R \end{bmatrix} < 0$ and LMI (17) one applies similar arguments, but this time, Pre- and post-multiplying the other two LMIs stated in (24) with $\begin{bmatrix} I & 0 & 0 \\ I & I & 0 \\ 0 & 0 & I \end{bmatrix}'$ and $\begin{bmatrix} I & 0 & 0 \\ I & I & 0 \\ 0 & 0 & I \end{bmatrix}$. ■

Proposition 4 was obtained based on the Lyapunov-Krasovskii functional of the following form

$$V(t) = x(t)'Px(t) + (t_{k+1} - t) \int_{t_k}^{t} \dot{x}(s)' R \dot{x}(s) ds +$$

$$(t_{k+1} - t) \begin{bmatrix} x(t) - x(t_k) \\ x(t_k) \end{bmatrix}' \begin{bmatrix} S_1 & S_2 \\ S_2' & (t - t_k) S_3 \end{bmatrix} \begin{bmatrix} x(t) - x(t_k) \\ x(t_k) \end{bmatrix}$$

$$\text{for } t \in [t_k, t_{k+1})$$

By the equivalence between Proposition 4 and Theorem 2, one observes the correspondence between the IQC defined by $\Pi_1 + \Pi_2$ and the above-mentioned Lyapunov-Krasovskii functional. A simplified criterion based on the IQC defined by Π_1 would correspond to that by a simplified version of V, where one sets S_2 and S_3 to zero. Likewise, the criterion based on the IQC defined by Π_1 would correspond to that with R and S_1 set to zero.

Moreover, the criteria proposed in [2] and [3] are based on simplified versions of V. where one sets R and S_3 to zero. For the criterion proposed in [2], the integral term and the term $(t_{k+1} - t)(t - t_k)x(t_k)'S_3x(t_k)$ are not considered. For the criterion proposed in [3], the term $(t_{k+1} - t)(t - t_k)x(t_k)'S_3x(t_k)$ is not considered. Therefore, one can infer that the criteria proposed in [2] and [3] are special cases of Theorem 2, by setting some matrix variables to zero. A formal proof for this claim will be presented in the future version of this manuscript. Finally, since Theorem 2 is a special case of Theorem 1, by setting X_5 and X_6 in Theorem 1 to zero and removing some redundant LMIs, one concludes that the criterion stated in Theorem 1 is the least conservative compared to those proposed in [2], [3], [1].

IV. CONCLUDING REMARKS

IQC analysis is applied to (2) to tackle the problem of robust stability analysis of aperiodic sampled-data systems defined in (1). Criteria posed as convex feasibility programs are derived. It is shown that one of the proposed criteria is equivalent to the main result of [1], derived based on a certain Lyapunov-Krasovskii functional. The link between the proposed IQCs for the "average delay-difference" operator and the Lyapunov-Krasovskii functional is established. Also established is that the results derived based on simplified versions of this Lyapunov-Krasovskii functional are special cases of the criterion proposed in this manuscript.

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