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# Some Stochastic Differential Games in Spheres

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*Abstract*— **Some two person zero sum stochastic differential games are formulated and solved in the n-sphere for an arbitrary positive integer n. The payoff functional has a distance symmetry from the geometry of the n-sphere as a Riemannian symmetric space. Explicit solutions are obtained for the optimal control strategies of the two players in a direct way that does not require solutions of Hamilton-Jacobi-Isaacs equations. The optimal strategies are optimal for a large family of strategies.**

## I. Introduction

Two person zero sum stochastic differential games have been used to model many competitive situations. While these games are useful models, there are relatively few examples where the optimal control strategies for the two players have been explicitly obtained. In this work some stochastic games are formulated in the n-sphere for an arbitrary positive integer n where the payoff functional for a game has a distance symmetry property and the equation for the differential game is the sum of the differential of Brownian motion and two drift vectors for the control strategies of the two players in the tangent space of an arbitrary point in the sphere. The problem formulation and solution use some geometric properties of the n-sphere, $S^n$, which is described as a Riemannian symmetric space. The determination of the optimal control strategies for the two players does not require the solution of Hamilton-Jacobi-Isaacs equations ([4]) for this problem. Instead the solution method is direct which has its motivation from the well known completion of squares method from algebra. A special case of this approach considering a stochastic differential game in $S^2$ is given [2] and a corresponding control problem in $S^2$ is given in [1].

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## II. Problem Formulation

Initially some properties of the n-sphere $S^n$ are given that are used in this paper. The sphere $S^n$ is diffeomorphic to the rank one symmetric space $SO(n+1)/SO(n)$ where $SO(k)$ is the special orthogonal group in $\mathbb{R}^k$ for $k \in \mathbb{N}$ and this symmetric space is a simply connected, compact Riemannian manifold of constant positive sectional curvature ([5]). The Riemannian metric can be obtained using a positive constant times the negative of the Killing form. The Killing form metric on $S^n$ is obtained by multiplying the usual Riemannian structure (with curvature +1) by $-2n$ (p.175 [3]). The specific metric that is used here is obtained by restricting the usual Riemannian metric induced from $\mathbb{R}^{n+1}$ (e.g. [5]).

Choose an origin $o \in S^n$. The antipodal point of $o$ is the submanifold $A_o$ that is the distance $L$ from $o$. Let $T_oS^n$ be the tangent space to $S^n$ at $o$. The exponential mapping $exp_o : T_oS^n \to S^n$ is a diffeomorphism of the open ball $B_L(o) = \{x \in T_oS^n : |x| < L\}$ onto the open set $S^n \setminus A_o$. This diffeomorphism is explicitly given by the geodesic polar coordinates for $S^n$ at the origin as the map

$$exp_oY \to (r, \theta_1, ..., \theta_{n-1}) \tag{II.1}$$

where $Y \in B_L(o), r = |Y|$ and $(\theta_1, ..., \theta_{n-1})$ are the local coordinates of the unit vector $Y/|Y|$. The Riemannian structure is $ds^2 = dr^2 + sin^2(r)d\sigma^2$. In these geodesic polar coordinates the Laplace-Beltrami operator $\Delta_{S^n}$ for $S^n$ (e.g. p.169 [3]) is

$$\Delta = \frac{\partial^2}{\partial r^2} + (n-1)cot(r)\frac{\partial}{\partial r} + (sin(r))^{-2}\Delta_S \tag{II.2}$$

The maximal distance between any two points on $S^n$ using this metric is $L$ [3] where

$$L = \pi \tag{II.3}$$

The Laplace-Beltrami operator can be expressed more simply as

$$\Delta_{S^2} = \frac{\partial^2}{\partial r^2} + (n-1)cot(r)\frac{\partial}{\partial r} + \Delta_{S_r} \tag{II.4}$$

where $r \in (0, L)$ and $\Delta_{S_r}$ is the Laplace-Beltrami operator on the sphere of radius $r$ from the origin. The sum of the first two terms on the right hand side of (II.4) is called the radial part of the Laplace-Beltrami operator. It is unnecessary to describe $\Delta_{S_r}$ here.

The payoff function, $J_T$, is described as follows

$$\begin{aligned} J_T^0(U,V) &= \int_0^T (a\ sin^2\frac{|Y(t)|}{2} + U^2(t)cos^2\frac{|Y(t)|}{2} \\ &- V^2(t)cos^2\frac{|Y(t)|}{2})dt \qquad \text{(II.5)} \\ J_T(U,V) &= \mathbb{E}J_T^0(U,V) \qquad \text{(II.6)} \end{aligned}$$

Since the payoff only depends on the distance $X(t) = |Y(t)|$ from the origin $o$ where $|\cdot|$ is the Riemannian metric at $Y(t)$, the system description can be reduced to the radial part of the process $Y$, that is denoted $(X(t), t \in [0,T])$. Thus the stochastic differential game can be described by the following equation which describes the distance of the process, $(Y(t), t \in [0,T])$, from $o$.

$$\begin{aligned} dX(t) &= \frac{1}{2}(n-1)cotX(t)dt + bU(t)dt + cV(t)dt \\ &+ dB(t) \qquad \text{(II.7)} \\ X(0) &= X_0 \qquad \text{(II.8)} \end{aligned}$$

where $Y(t) \in S^2 \setminus A_o$, $X(t) = |Y(t)|$, $(B(t), t \in [0,T])$ is a real-valued standard Brownian motion for a fixed $T > 0$, and $X_0 \in (0, L)$ is a constant. The Brownian motion is defined on the complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and $(\mathscr{F}(t), t \in [0,T])$ is the filtration for the Brownian motion $B$. The terms $(b, c)$ are nonzero real numbers. An assumption on the relative size of these two real numbers is made subsequently. The family of admissible control strategies for $U$ is $\mathscr{U}$ and for $V$ is $\mathscr{V}$ that are defined as follows

$\mathscr{U} = \{U : U$ is an $\mathbb{R}^m$-valued process that is progressively measurable with respect to $(\mathscr{F}(t), t \in [0,T])$ such that $U \in L^2([0,T])$ a.s.$\}$

and

$\mathscr{V} = \{V : V$ is an $\mathbb{R}^p$-valued process that is progressively measurable with respect to $(\mathscr{F}(t), t \in [0,T])$ such that $V \in L^2([0,T])$ a.s.$\}$

If $U(t)$ and $V(t)$ are suitably smooth functions of $X(t)$, then $(X(t), t \in [0,T])$ is a Markov process with the infinitesimal generator

$$\frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{1}{2}(n-1)cot(r)\frac{\partial}{\partial r} + U(r)\frac{\partial}{\partial r} + V(r)\frac{\partial}{\partial r} \tag{II.9}$$

However the control strategies available to the two players are not restricted to be Markovian.

The payoff for the stochastic differential game with the control strategies $U$ and $V$ is denoted $J_T(U,V)$ that is described as follows

$$\begin{aligned} J_T^0(U,V) &= \int_0^T (a\ sin^2\frac{X(t)}{2} + U^2(t)cos^2\frac{X(t)}{2} \\ &- V^2(t)cos^2\frac{X(t)}{2})dt \qquad \text{(II.10)} \\ J_T(U,V) &= \mathbb{E}J_T^0(U,V) \qquad \text{(II.11)} \end{aligned}$$

The following scalar Riccati and linear equations are used in the solution of the game problem.

$$\begin{aligned} \frac{dg(t)}{dt} &= (n-1)g + g^2(b^2 - c^2) - a \qquad \text{(II.12)} \\ g(T) &= 0 \qquad \text{(II.13)} \\ \frac{dh(t)}{dt} &= -\frac{1}{2}(n-1)g \qquad \text{(II.14)} \\ h(T) &= 0 \qquad \text{(II.15)} \end{aligned}$$

If $b^2 > c^2$ then it is well known that the Riccati equation (II.12) has a unique positive solution on $[0,T]$ for any $T > 0$.

## III. Main Results

The following result describes optimal control strategies for the two players and the value of the game for the stochastic system (II.7) and the payoff functional (II.11).

*Theorem 1:* Let $b^2 > c^2 > 0$. The stochastic differential game described by (II.7) and (II.11) has optimal control strategies, $(U^*, V^*)$, for the two players that are given by

$$U^*(t) = -bg(t)tan\frac{X(t)}{2} \quad \text{(III.1)}$$

$$V^*(t) = cg(t)tan\frac{X(t)}{2} \quad \text{(III.2)}$$

where $t \in [0,T]$ and $g$ is the positive solution of (II.12). The value of the game is

$$J_T(U^*, V^*) = ag(0)sin^2\frac{X(0)}{2} + h(0) \quad \text{(III.3)}$$

where $h$ is given by (II.14).

*Proof:* (Sketch). Let $(Y(t), t \in [0,T])$ be the process given by

$$Y(t) = ag(t)sin^2\frac{X(t)}{2} + h(t) \quad \text{(III.4)}$$

where $g$ and $h$ are given by (II.12) and (II.14) respectively. Apply the change of variables of K. Ito to $(Y(t), t \in [0,T])$ to obtain

$$\begin{aligned} Y(T) - Y(0) = & \int_0^T agsin\frac{X}{2}cos\frac{X}{2}(\frac{1}{2}(n-1)cotXdt \\ & + bUdt + cVdt + dB(t)) \\ & + \frac{a}{4}cosX(t)dt + asin^2\frac{X}{2}((n-1)g \\ & + g^2(b^2 - c^2) - a)dt - \frac{1}{2}(n-1)gdt \end{aligned} \quad \text{(III.5)}$$

Add the terms in the payoff to both sides of the last equation. After some cancelations the RHS contains the difference of two quadratic terms with one depending on U and the other on V and a stochastic integral term. Taking expectation of both sides of the new equality and performing the separate min and max operations gives the optimal control strategies for the two players. ■

It can be shown that the process $(X(t), t \geq 0)$ with the optimal control strategies does not hit the origin $o$ or the antipodal point of $o$, the submanifold $A_0$.

## IV. Conclusions

This explicitly solvable two person stochastic differential game can be extended to other payoff functions by using other eigenfunctions of the radial part of the Laplace-Beltrami operator.

## References

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