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Impedance-conservative overdetermined multidimensional systems

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I. THE CLASSICAL CASE: 1D SYSTEMS

A continuous time scattering conservative input/state/output linear system is given by

$$\begin{aligned} i\frac{dx}{dt} &= -Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{1}$$

Here $\mathcal{H}$, $\mathcal{E}$ and $\mathcal{E}_*$ are the state space, the input space and the output space of the system, respectively, and $A\colon \mathcal{H} \to \mathcal{H}$, $B\colon \mathcal{E} \to \mathcal{H}$, $C\colon \mathcal{E}_* \to \mathcal{H}$ are linear operators (all the spaces are assumed to be Hilbert spaces, and all the operators are assumed to be bounded). The conservation of energy is given by the energy balance equation

$$\frac{d\langle x, x\rangle}{dt} = \langle u, u\rangle - \langle y, y\rangle. \tag{2}$$

Assuming that $D = I$ (as we may without loss of generality for scattering conservative systems with bounded operators) this translates to

$$\frac{1}{i}(A - A^*) = BB^*,\ C = -iB^*. \tag{3}$$

In general, the transfer function of the system (1) is given by

$$S(z) = D + C(z - A)^{-1}B. \tag{4}$$

For a scattering conservative system (2)–(3), the transfer function is analytic and contractive in the open upper half–plane $\mathbb{C}_+$, and the kernel $\frac{I - S(z)S(w)^*}{-i(z - w^*)}$ is positive.

There is a related class of impedance conservative systems which is governed by the energy balance equation

$$\frac{d\langle x, x\rangle}{dt} = 2\Re\langle u, y\rangle \tag{5}$$

(we assume here that the input and output spaces coincide: $\mathcal{E} = \mathcal{E}_*$). In terms of the realisation, impedance conservative systems are characterized by the equations

$$A = A^*,\ C = iB^*,\ D = -D^*. \tag{6}$$

The transfer function has now a positive real part in the open upper half–plane $\mathbb{C}_+$, and the kernel $\frac{S(z) + S(w)^*}{-i(z - w^*)}$ is positive.

For impedance conservative systems, two functional models are available. The first one in the reproducing kernel Hilbert space with reproducing kernel $\frac{S(z) + S(w)^*}{-i(z - w^*)}$, and the second one in the space $L_2(d\mu)$, where $d\mu$ is the measure appearing in the Herglotz representation of the function S,

$$S(z) = ia - ibz + \frac{1}{i}\int_{\mathbb{R}} d\mu(t)\left(\frac{1}{t - z} - \frac{t}{t^2 + 1}\right) \tag{7}$$

($a \in \mathbb{R}$, $b \geq 0$). In the first model the operator A becomes the compressed multiplication operator by the independent variable, and in the second model it becomes the multiplication operator by the independent variable. The link between the two models is given by the Cauchy transform with respect to the measure $d\mu$.

Scattering conservative and impedance conservative systems are related by the so called diagonal transform

$$u' = \frac{1}{\sqrt{2}}(u + y), \quad y' = \frac{1}{\sqrt{2}}(u - y), \tag{8}$$

where unprimed variables refer to scattering conservative systems and primed variables refer to impedance conservative systems. The main operator of the impedance conservative system is $A' = \Re A$ and the transfer functions are related by the Moebius transformation

$$S'(z) = (I - S(z))(I + S(z))^{-1}. \tag{9}$$

We mention Livsic [8], Arov [1] and Willems [15], [16] as core references for scattering conservative and impedance conservative systems. For a recent account in a very general setting of possibly unbounded operators, see Staffans [13] (whose terminology we largely follow). The diagonal trasform is explicit in [8], but it is implicit already in the early work on characteristic functions for nonselfadjoint operators [6]; in a different setting, see Clark [7]. In the one variable setting impedance conservative systems and their transfer functions play an important role in the inverse scattering problem and operator models. See de Branges [5].

II. SCATTERING CONSERVATIVE OVERDETERMINED MULTIDIMENSIONAL SYSTEMS

We consider now an overdetermined 2D input/state/output linear system

$$\begin{aligned} i\frac{\partial x}{\partial t_1} &= -A_1 x + \widetilde{B}\sigma_1 u, \\ i\frac{\partial x}{\partial t_2} &= -A_2 x + \widetilde{B}\sigma_2 u, \\ y &= Cx + Du, \end{aligned} \tag{10}$$

where A_1 and A_2 commute: $A_1A_2 = A_2A_1$. This system is overdetermined hence we assume that it comes equipped

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with compatibility conditions

$$\sigma_2 \frac{\partial u}{\partial t_2} - \sigma_1 \frac{\partial u}{\partial t_1} + i\gamma u = 0, \tag{11}$$

$$\sigma_{2*} \frac{\partial y}{\partial t_2} - \sigma_{1*} \frac{\partial y}{\partial t_1} + i\gamma_* y = 0 \tag{12}$$

for the input and the output signals. Here

$$A_1, A_2 \colon \mathcal{H} \to \mathcal{H},\ \widetilde{B} \colon \mathcal{E} \to \mathcal{H},\ C \colon \mathcal{H} \to \mathcal{E}_*,$$
$$\sigma_1, \sigma_2, \gamma \colon \mathcal{E} \to \mathcal{E},\ \sigma_{1*}, \sigma_{2*}, \gamma_* \colon \mathcal{E}_* \to \mathcal{E}_*,\ D, \widetilde{D} \colon \mathcal{E} \to \mathcal{E}_*,$$

satisfying

$$A_2 \widetilde{B} \sigma_1 - A_1 \widetilde{B} \sigma_2 = \widetilde{B} \gamma, \tag{13}$$

(the input vessel condition),

$$\sigma_{1*} C A_2 - \sigma_{2*} C A_1 = \gamma_* C, \tag{14}$$

(the output vessel condition), and

$$\sigma_{1*} D = \widetilde{D} \sigma_1, \quad \sigma_{2*} D = \widetilde{D} \sigma_2,$$
$$\gamma_* D = \widetilde{D} \gamma - \sigma_{1*} C \widetilde{B} \sigma_2 + \sigma_{2*} C \widetilde{B} \sigma_1 \tag{15}$$

(the linkage equations). A collection of spaces and operators ($A_1 A_2 = A_2 A_1$) satisfying (13)–(15) is also called a (commutative two) operator vessel. We will always assume that $\dim \mathcal{E}, \dim \mathcal{E}_* < \infty$.

Quasi–hermitian, or scattering conservative vessels, have $\sigma_1, \sigma_2, \gamma$ and $\sigma_{1*}, \sigma_{2*}, \gamma_*$ selfadjoint and are characterized by the energy balance equations

$$\frac{\partial \langle x, x \rangle}{\partial t_1} = \langle \sigma_1 u, u \rangle - \langle \sigma_{1*} y, y \rangle, \tag{16}$$

$$\frac{\partial \langle x, x \rangle}{\partial t_2} = \langle \sigma_2 u, u \rangle - \langle \sigma_{2*} y, y \rangle. \tag{17}$$

We suppose without loss of generality $D = \widetilde{D} = I$; the energy balance is then translated into the equations:

$$\frac{1}{i}(A_1 - A_1^*) = \widetilde{B}^* \sigma_1 \widetilde{B},\ \frac{1}{i}(A_2 - A_2^*) = \widetilde{B}^* \sigma_2 \widetilde{B},\ C = -i\widetilde{B}^*. \tag{18}$$

We return to general (not necessarily quasihermitian) vessels. The polynomial

$$\mathbf{p}(\lambda_1, \lambda_2) = \det(\lambda_1 \sigma_2 - \lambda_2 \sigma_1 + \gamma)$$

is called the input discriminant polynomial of the vessel. The generalized Cayley–Hamilton theorem holds: for a controllable vessel ($\bigvee_{k_1, k_2 \in \mathbb{N}} \operatorname{im} A_1^{k_1} A_2^{k_2} \widetilde{B} = \mathcal{H}$), $\mathbf{p}(A_1, A_2) = 0$. We define also the homogeneous polynomial $\mathbf{P}(\mu_0, \mu_1, \mu_2) = \det(\mu_1 \sigma_2 - \mu_2 \sigma_1 + \mu_0 \gamma)$, and a family of vector spaces

$$\boldsymbol{\mathcal{E}}(\mu) = \ker\ (\mu_1 \sigma_2 - \mu_2 \sigma_1 + \mu_0 \gamma)$$

on the associated projective curve $P(\mu_0, \mu_1, \mu_2) = 0$. Similarly, the polynomial

$$\mathbf{p}_*(\lambda_1, \lambda_2) = \det(\lambda_1 \sigma_{2*} - \lambda_2 \sigma_{1*} + \gamma_*)$$

is called the output discriminant polynomial, and here too the generalized Cayley–Hamilton theorem holds: $\mathbf{p}_*(A_1^*, A_2^*) = 0$ for an observable vessel ($\bigvee_{k_1, k_2 \in \mathbb{N}} \operatorname{im} A_1^{k_1 *} A_2^{k_2 *} \widetilde{C}^* = \mathcal{H}$). We consider also the homogeneous polynomial $\mathbf{P}_*(\mu_0, \mu_1, \mu_2) = \det\ (\mu_1 \sigma_{2*} - \mu_2 \sigma_{1*} + \mu_0 \gamma_*)$ and the corresponding family of vector spaces

$$\boldsymbol{\mathcal{E}}_*(\mu) = \ker\ (\mu_1 \sigma_{2*} - \mu_2 \sigma_{1*} + \mu_0 \gamma_*)$$

on the projective curve $P_*(\mu_0, \mu_1, \mu_2) = 0$.

We are mostly interested in the case where $\dim \mathcal{E} = \dim \mathcal{E}_*$ and the polynomials $\mathbf{P}(\mu_0, \mu_1, \mu_2)$ and $\mathbf{P}_*(\mu_0, \mu_1, \mu_2)$ coincide, up to a multiplicative constant. (It happens in particular when the matrices D and $\widetilde{D}$ are invertible, or more generally — see below — when the the transfer function and the adjoint transfer function are both somewhere invertible on every component of the discriminant curve.) We then have two families of vector spaces $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{E}}_*$ on a common projective curve. This curve is called the discriminant curve of the vessel, and will be denoted by $\mathbf{C}$. The joint transfer function S of the vessel is a mapping from the family $\boldsymbol{\mathcal{E}}$ into $\boldsymbol{\mathcal{E}}_*$ defined by (it is independent of the particular choice of (ξ_1, ξ_2))

$$S(\mu) = D + C(\xi_1 \mu_1 + \xi_2 \mu_2 - \xi_1 \mu_0 A_1 - \xi_2 \mu_0 A_2)^{-1}$$
$$\times \widetilde{B}(\xi_1 \sigma_1 + \xi_2 \sigma_2)\big|_{\boldsymbol{\mathcal{E}}(\mu)}\ :\ \boldsymbol{\mathcal{E}}(\mu) \to \boldsymbol{\mathcal{E}}_*(\mu). \tag{19}$$

Assume for simplicity that the discriminant curve $\mathbf{C}$ irreducible: $\mathbf{P} = \mathbf{F}^r$ where $\mathbf{F}$ is an irreducible polynomial, and has only nodes as singularities. The determinantal representations $\det(\mu_1 \sigma_2 - \mu_2 \sigma_1 + \mu_0 \gamma)$ and $\det(\mu_1 \sigma_{2*} - \mu_2 \sigma_{1*} + \mu_0 \gamma_*)$ of $\mathbf{F}^r$ are said to be maximal if the dimensions of the fibers $\boldsymbol{\mathcal{E}}(\mu)$ and $\boldsymbol{\mathcal{E}}_*(\mu)$ reach for all $\mu \in \mathbf{C}$ the maximal possible value sr where s is the multiplicity of μ on $\mathbf{C}$. $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{E}}_*$ can then be lifted to vector bundles of rank r, which we still denote by $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{E}}_*$, on the desingularising Riemann surface X of $\mathbf{C}$ (and the joint characteristic function S is a bundle map between these bundles). Furthermore,

$$\boldsymbol{\mathcal{E}} \otimes \mathcal{O}(m-2)(-D) \cong \mathbf{V}_\chi \otimes \Delta \quad \text{and}$$
$$\boldsymbol{\mathcal{E}}_* \otimes \mathcal{O}(m-2)(-D) \cong \mathbf{V}_{\chi_*} \otimes \Delta \tag{20}$$

where m is the degree of the curve, D is the divisor of singularities, $\mathbf{V}_\chi \otimes \Delta$ and $\mathbf{V}_{\chi_*} \otimes \Delta$ are bundles of multiplicative differentials of order $1/2$, that is, $\Delta \otimes \Delta \cong K_X$ (K_X is the canonical bundle of X) and $\mathbf{V}_\chi$ and $\mathbf{V}_{\chi_*}$ are flat bundles associated to representations χ and χ_* of $\pi_1(X)$ in $GL(r, \mathbb{C})$. Moreover, these bundles of multiplicative differentials have no global holomorphic sections: $h^0(\mathbf{V}_\chi \otimes \Delta) = h^0(\mathbf{V}_\chi \otimes \Delta) = 0$.

The isomorphisms (20) are given explicitly by matrices of normalized sections $\mathbf{u}^\times$ and $\mathbf{u}_*^\times$ for $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{E}}_*$ [3], [4]. Under these isomorphisms the joint transfer function $S \colon \boldsymbol{\mathcal{E}} \to \boldsymbol{\mathcal{E}}_*$ becomes the normalized joint transfer function $T \colon \mathbf{V}_\chi \otimes \Delta \to \mathbf{V}_{\chi *} \otimes \Delta$; explicitly,

$$S(p)\mathbf{u}^\times(\widetilde{p}) = \mathbf{u}_*^\times(\widetilde{p}) T(\widetilde{p}), \tag{21}$$

where $\widetilde{p}$ is a point on the universal covering $\widetilde{X}$ of X above $p \in X$. Notice that concretely, T is simply a multiplicative meromorphic $r \times r$ matrix valued function on $\widetilde{X}$ with multipliers given by the representations χ and χ_* of $\pi_1(X)$.

We return now to the quasihermitian case. In this case $\mathbf{C}$ is a real algebraic curve so that X is a compact real Riemann surface, i.e., a compact Riemann surface with an antiholomorphic involution τ induced by the antiholomorhic involution $\mu \mapsto \overline{\mu}$ on $\mathbf{C}$. We denote by $X_{\mathbb{R}}$ the set of real points of X, i.e., the fixed points of τ. The selfadjointness of the input determinantal representation translates into the flat vector bundle $\mathbf{V}_\chi$ being parahermitian, i.e., there is a parahermitian nondegenerate pairing between $\mathbf{V}_\chi$ and $\mathbf{V}_{\chi^\tau}$ (in other words, $\mathbf{V}_{\chi^\tau}^\vee \cong \mathbf{V}_\chi$ where we use $^\vee$ to denote the dual bundle); similarly, of course, for the output bundle. The normalized joint transfer function T is now $(\mathbf{V}_\chi,\mathbf{V}_{\chi_*})$-contractive in the sense that the kernel

$$iK(\chi_*;\widetilde{p},\widetilde{q}^\tau) - iT(p)K(\chi;\widetilde{p},\widetilde{q}^\tau)T(q)^* \tag{22}$$

is positive. Here $K(\chi;\widetilde{p},\widetilde{q})$ (and similarly for χ_*) denotes the Cauchy kernel for $\mathbf{V}_\chi \otimes \Delta$ [3], [4] — the unique meromorphic section of $\mathbf{V}_\chi \otimes \Delta$ in p and of $\mathbf{V}_\chi^\vee \otimes \Delta$ in q which is holomorphic except for a pole along the diagonal $p = q$ with residue I.

In the case when the real Riemann surface X is dividing (meaning that $X \setminus X_{\mathbb{R}}$ consists of two connected components X_+ and X_- interchanged by τ) and the parahermitian bundles $\mathbf{V}_\chi$ and $\mathbf{V}_{\chi_*}$ are positive (meaning essentially that the parahermitian pairings are positive at real points), the positivity of the kernel (22) means simply that $iK(\chi_*;\widetilde{p},\widetilde{p}^\tau) - iT(p)K(\chi;\widetilde{p},\widetilde{p}^\tau)T(p)^* \geq 0$. If furthermore the vector bundles are of the form $L^{\oplus r}$ for a line bundle L, this says simply that T is contractive on X_+. The positive dividing case always occurs if A_1 and A_2 are dissipative operators, so that $\sigma_1, \sigma_2 > 0$, but it can also take place in other situations.

Overdetermined systems, both in the general case and in the scattering conservative case, were actively studied by several mathematicians over the last decade [12], [3], [14], [10], [2]. The original motivation for the study of these systems came from operator theory (spectral analysis of tuples of nonselfadjoint and nonunitary operators) and function theory (study of meromorphic matrix valued functions on a Riemann surface, especially the analogues of the classical interpolation problems). They also arise naturally in a variety of applications: the wave-particle duality in quantum mechanics [11], the study of DNA chains in molecular biology [9], and most recently, in signal processing.

III. IMPEDANCE CONSERVATIVE OVERDETERMINED SYSTEMS

We consider an overdetermined 2D system (10) with the compatibility conditions (11)–(12), i.e., a vessel. We assume that the input and output spaces coincide: $\mathcal{E} = \mathcal{E}_*$, that the input and output compatibility equations are adjoint to each other: $\sigma_{1*} = \sigma_1^*$, $\sigma_{2*} = \sigma_2^*$, $\gamma_* = \gamma^*$, and that all the trajectories of the system satisfy impedance energy balance

$$\frac{\partial \langle x, x\rangle}{\partial t_1} = 2\Re\langle \sigma_1 u, y\rangle, \tag{23}$$

$$\frac{\partial \langle x, x\rangle}{\partial t_2} = 2\Re\langle \sigma_2 u, y\rangle. \tag{24}$$

This is (essentially) equivalent to

$$A_1^* = A_1,\ A_2^* = A_2,\ \ C = i\widetilde{B}^*,\ \ \widetilde{D} = -D^*. \tag{25}$$

A slightly different way to arrive at (25) is to require that for each system trajectory (u,x,y), $(-u,x,y)$ is a trajectory of the adjoint system, and conversely, for each trajectory (u_*,x_*,y_*) of the adjoint system, $(-u_*,x_*,y_*)$ is a trajectory of the original system (this also explains why the input and output compatibility equations are required to be adjoint).

A *selfadjoint vessel* is a vessel satisfying (25). It thus involves a pair of commuting selfadjoint operators A_1, A_2 on $\mathcal{H}$; notice that the input (13) and the output (14) vessel conditions now coincide,

$$A_2\widetilde{B}\sigma_1 - A_1\widetilde{B}\sigma_2 = \widetilde{B}\gamma, \tag{26}$$

and the linkage conditions become

$$\begin{aligned} &\sigma_1^* D = -D^*\sigma_1,\ \sigma_2^* D = -D^*\sigma_2,\\ &\gamma^* D = -D^*\gamma - i\sigma_1^*\widetilde{B}^*\widetilde{B}\sigma_2 + i\sigma_2^*\widetilde{B}^*\widetilde{B}\sigma_1. \end{aligned} \tag{27}$$

We will always assume that the input and the output discriminant polynomials of the selfadjoint vessel coincide. The discriminant curve $\mathbf{C}$ is thus real, but in contrast to the quasihermitian case it comes now equipped with a pair of determinantal representations that are adjoint to each other, rather than with a pair of selfadjoint determinantal representations. Of course the situation when the input and the output determinantal representations coincide and are selfadjoint is an important special case.

For the case when the (input hence also the output) determinantal representation is maximal we can proceed now to construct two different functional models for a minimal selfadjoint vessel (i.e., $\widetilde{B}(\mathcal{E})$ is cyclic for A_1, A_2; notice that it follows that $\mathbf{p}(A_1, A_2) = 0$). For notational simplicity we assume that the discriminant curve $\mathbf{C}$ is irreducible and we let as before X be the desingularizing Riemann surface and $\boldsymbol{\mathcal{E}} \otimes \mathcal{O}(m-2)(-D) \cong \mathbf{V}_\chi \otimes \Delta$, where $\mathbf{V}_\chi$ is a rank r flat vector bundle on X with $h^0(\mathbf{V}_\chi \otimes \Delta) = 0$ and $\boldsymbol{\mathcal{E}}$ is the kernel bundle of the input determinantal representation. Notice that for the output determinantal representation we have

$$\boldsymbol{\mathcal{E}}_*(\lambda) = \ker(\lambda_1\sigma_2^* - \lambda_2\sigma_1^* + \gamma^*) = \boldsymbol{\mathcal{E}}_\ell(\overline{\lambda})^*,$$

where $\boldsymbol{\mathcal{E}}_\ell$ denotes the left kernel bundle, hence (by the duality theory for kernel bundles [3]) $\boldsymbol{\mathcal{E}}_* \otimes \mathcal{O}(m-2)(-D) \cong \mathbf{V}_{\chi^\tau}^\vee \otimes \Delta$, and the joint transfer function $T\colon \mathbf{V}_\chi \otimes \Delta \to \mathbf{V}_{\chi^\tau}^\vee \otimes \Delta$.

The first functional model is the reproducing kernel Hilbert space with the (positive) reproducing kernel

$$iT(\widetilde{q})K(\chi;\widetilde{q},\widetilde{r}^\tau) + iK(\chi^{\tau\vee};\widetilde{q},\widetilde{r}^\tau)T(\widetilde{r})^*, \tag{28}$$

the operators A_1 and A_2 becoming in the model compressed multiplication operators by the coordinate functions λ_1 and λ_2.

The second functional model is a L^2 space, $L^2(\mathbf{V}_\chi \otimes \Delta, X_{\mathbb{R}}, d\mu)$, of sections of $\mathbf{V}_\chi \otimes \Delta$ over $X_{\mathbb{R}}$ with respect to an appropriate measure $d\mu$. $d\mu$ is essentially the compression of the joint spectral measure of the commuting selfadjoint operators A_1, A_2 to the input/output space $\mathcal{E}$ via $\widetilde{B}\colon \mathcal{E} \to \mathcal{H}$ — because of the vessel condition (26) this compression is supported on (the affine part of) the set $\mathbf{C}_{\mathbb{R}}$ of real points of the discriminant curve $\mathbf{C}$, and it "lives" in the kernel bundle. The operators A_1 and A_2 become in this model multiplication operators by the coordinate functions λ_1 and λ_2.

The mapping from the L^2 model to the reproducing kernel Hilbert space model is given by a Cauchy transform with respect to the measure $d\mu$,

$$f \mapsto -\int_{X_{\mathbb{R}}} K(\chi^{\tau\vee}; \cdot, \widetilde{p}) \frac{d\mu(\widetilde{p})}{\omega(p)} f(\widetilde{p}), \tag{29}$$

and we have the following identity

$$\begin{aligned} &\int_{X_{\mathbb{R}}} K(\chi^{\tau\vee}; \widetilde{q}, \widetilde{p}) \frac{d\mu(\widetilde{p})}{\omega(p)} K(\chi; \widetilde{p}, \widetilde{r}^{\tau}) \\ &\qquad = iT(\widetilde{q}) K(\chi; \widetilde{q}, \widetilde{r}^{\tau}) + iK(\chi^{\tau\vee}; \widetilde{q}, \widetilde{r}^{\tau}) T(\widetilde{r})^* \end{aligned} \tag{30}$$

for all $\widetilde{q}, \widetilde{r} \in X \setminus X_{\mathbb{R}}$.

If A_1, A_2 are the main operators of a quasihermitian vessel, then as shown in [12, Section 4.7] $\frac{1}{i}[\Re A_1, \Re A_2] = \frac{1}{4}\widetilde{B}(\gamma - \gamma_*)\widetilde{B}^*$; therefore $A_1' = \Re A_1$ and $A_2' = \Re A_2$ commute (essentially) if and only if $\gamma_* = \gamma$, i.e., the input and output determinantal representations coincide. So do then the input and output compatibility conditions and the usual diagonal transform can be applied to the quasihermitian vessel yielding a selfadjoint vessel (with the same (selfadjoint) determinantal representation, both at the input and at the output).

However, one can modify the diagonal transform to handle quasihermitian vessels with *equivalent* rather than *equal* determinantal representations at the input and at the output, i.e., arbitrary quasihermitian vessels with isomorphic input and output bundles. If

$$\rho^*\sigma_1\rho = \sigma_1,\ \rho^*\sigma_2\rho = \sigma_2,\ \rho^*\gamma_*\rho = \gamma,$$

then we define the generalized diagonal transform by

$$u' = \frac{1}{\sqrt{2}}(\rho u + y), \quad y' = \frac{1}{\sqrt{2}}(\rho u - y). \tag{31}$$

(u, x, y) is a trajectory of the given scattering conservative system if and only if (u', x, y') is a trajectory of an impedance conservative system with (we use the prime decoration for this new impedance conservative system)

$$\begin{aligned} A_k' &= A_k - i\widetilde{B}\sigma_k(\rho + I)^{-1}\widetilde{B}^* \qquad (32) \\ &= \Re A_k + \frac{i}{2}\widetilde{B}\sigma_k(\rho - I)(\rho + I)^{-1}\widetilde{B}^* \quad (k = 1, 2), \\ (\widetilde{B}')^* &= \sqrt{2}(I + \rho^{-1})^{-1}\widetilde{B}^*, \qquad (33) \\ D' &= \sqrt{2}(\rho - I)(\rho + I)^{-1}, \qquad (34) \end{aligned}$$

and with (both input and output) determinantal representation $\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma_*$. Conversely, given a selfadjoint vessel with equal or equivalent determinantal representations at the input and at the output, we can apply to it a similar diagonal transform to get a quasihermitian vessel with equivalent input and output determinantal representations. The normalized joint transfer functions are related by

$$T' = (I - T)(I + T)^{-1}. \tag{35}$$

Thus instead of considering real parts of a pair of commuting nonselfadjoint operators — which in general do not commute — we consider in (32) their *adjusted* real parts.

IV. CONCLUSIONS

In the classical 1D case, impedance conservative systems play both an important role by themselves — from a mathematical point of view as well as for engineering applications, and as a tool for the study of scattering conservative systems. We expect this to be the case also in the overdetermined multidimensional setting. Much work here remains to be done, especially as regards characterizing precisely the class of transfer functions, and the generalized diagonal transform.

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