21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

A graphic condition for the stability of dynamical distribution networks with flow constraints

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Abstract— **We consider a basic model of a dynamical distribution network, modeled as a directed graph with storage variables corresponding to every vertex and flow inputs corresponding to every edge, subject to unknown but constant inflows and outflows. In [1] we showed how a distributed proportional-integral controller structure, associating with every edge of the graph a controller state, regulates the state variables of the vertices, irrespective of the unknown constant inflows and outflows, in the sense that the storage variables converge to the same value (load balancing or consensus). In many practical cases, the flows on the edges are constrained. The main result of [1] is a sufficient and necessary condition, which only depend on the structure of the network, for load balancing for arbitrary constraint intervals of which the intersection has nonempty interior. In this paper, we will consider the stability of the same model as in [1] with given network structure and constraint intervals. We will derive a graphic condition, which is sufficient and necessary, for load balancing. This will be proved by a Lyapunov function and the analysis the kernel of incidence matrix of the network. Furthermore, we will show that by modified PI controller, the storage variable on the nodes can be driven to an arbitrary point of admissible set.**

I. INTRODUCTION

Production-distribution systems form a very important class of systems that have a large number of applications. In this paper, we pursue an approach similar to that proposed in [1]. Given the network which is depicted as a directed graph, we assign a state variable with every vertex of the graph, and a control input corresponding to flow with every edge, which is constrained in a given closed interval. Furthermore, the system is open to environments by some ports, namely some of the vertices serves as terminals, where an unknown-but-constant flow may enter or leave the network in such a way that the total sum of inflows and outflows is equal to zero.

There are many relevant references on this topic. In [2], a class of cooperative control algorithms is proposed in the context of distribution network under time-varying exogenous in/outflows. The author dealt with constraint for control input. However, the constraint intervals are all symmetric with respect to the origin. In [3], the main problem is the joint presence of buffer/flow capacity and of the unknown in/outflows. A discontinuous control strategy is proposed to drive the system states, whose components are also storage in nodes, into consensus for all possible unknown in/outflows by using control input that are subject to hard bounds. Similarly, in [4] a continuous control strategy which is saturated proportional controller, is employed to achieve practical stability and optimality at steady-states. In [1], a sufficient and necessary condition is derived that with PI controller and arbitrary constraint intervals whose intersection has nonempty interior, the state variables corresponding to all vertices converge to consensus if and only if the directed graph is strongly connected and balanced. In [5], [6], a similar model with constraint is considered from physical perspective.

The control problem to be studied here is to derive a criteria, which only depends on the structure of network and flow constraints, to decide whether or not the system with a distributed control structure (the control input corresponding to each edge only depending on the difference of state variables of its two endpoints) will be stable and describe the corresponding steady-states for given constraint intervals and constant unknown in/outflows. Notice that this control strategy is decentralized. In [7], a sufficient condition with respect to consensus, i.e the state variables associate to all vertices converge to the same value, is given.

The organization of this paper is as follows. Some preliminaries and notations will be given in Section 2. In Section 3, the class of systems and a basic assumption about flow constraints under study will be introduced.

The main contribution of this paper resides in Section 4, 5. In Section 4, it will be shown that the state variables associated to all the vertices converge to consensus, if and only if the network and flow constraints satisfy IPC (interior point condition) which will be defined later. Similar to [3], if the information of desirable state is available, we can modify the PI controller such that the storage converge to the desirable point. This will be explained in Section 5. Finally, Section 6 contains the conclusion.

II. PRELIMINARIES AND NOTATIONS

First we recall some standard definitions regarding directed graphs, as can be found e.g. in [8]. A *directed graph* $\mathcal{G}$ consists of a finite set $\mathcal{V}$ of *vertices* and a finite set $\mathcal{E}$ of *edges*, together with a mapping from $\mathcal{E}$ to the set of ordered pairs of $\mathcal{V}$, where no self-loops are allowed. Thus to any edge $e \in \mathcal{E}$ there corresponds an ordered pair $(v, w) \in \mathcal{V} \times \mathcal{V}$ (with $v \neq w$), representing the tail vertex v and the head vertex w of this edge. An undirected graph $\mathcal{G}^o$ is obtained from $\mathcal{G}$ by ignoring the orientation of the edges. A cycle in $\mathcal{G}^o$

*The work of the first author is supported by Chinese Scholarship Council.The research of the second author leading to these results has received funding from the EU 7th Framework Programme [FP7/2007-2013] under grant agreement no. 257462 HYCON2 Network of Excellence.

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ISBN: 978-90-367-6321-9

is a closed path in which the internal vertices are distinct. An oriented cycle in $\mathcal{G}$ is a cycle in $\mathcal{G}^o$ with an orientation assigned by an ordering of the vertices in the cycle. Given an oriented cycle $\mathcal{C}$, we define the vectorial representation of the cycle $\mathcal{C}$ as C whose component is given as

$$C_i = \begin{cases} 0 & e_i \notin \mathcal{C} \\ 1 & e_i \in \mathcal{C} \text{ and the orientations agree} \\ -1 & e_i \in \mathcal{C} \text{ and the orientations disagree,} \end{cases} \quad (1)$$

A oriented cycle of a directed graph $\mathcal{G}$ which has an orientation which agrees with the orientations in the graph is called positive circuit.

A directed graph is completely specified by its *incidence matrix* B, which is an $n \times m$ matrix, n being the number of vertices and m being the number of edges, with $(i, j)^{\text{th}}$ element equal to 1 if the j^{th} edge is towards vertex i, and equal to -1 if the j^{th} edge is originating from vertex i, and 0 otherwise. A directed graph is *strongly connected* if it is possible to reach any vertex starting from any other vertex by traversing edges following their directions. A directed graph $\mathcal{G}$ is called *weakly connected* if $\mathcal{G}^o$ is connected. A digraph is weakly connected if and only if $\ker B^T = \operatorname{span} \mathbb{1}_n$. Here $\mathbb{1}_n$ denotes the n-dimensional vector with all elements equal to 1. We omit the subscript if the dimension of the vector is unambiguous from the context. A digraph that is not weakly connected falls apart into a number of weakly connected subgraphs, called the weakly connected components. The number of weakly connected components is equal to $\dim \ker B^T$. A subgraph $\mathcal{T} \subseteq \mathcal{G}^o$ is a tree if it is connected and acyclic, and a spanning tree if it is a tree and contains all the vertices of $\mathcal{G}^o$.

Given a graph, we define its *vertex space* as the vector space of all functions from $\mathcal{V}$ to some linear space $\mathcal{R}$. In the rest of this paper we will take for simplicity $\mathcal{R} = \mathbb{R}$, in which case the vertex space can be identified with $\mathbb{R}^n$. Similarly, we define its *edge space* as the vector space of all functions from $\mathcal{E}$ to $\mathcal{R} = \mathbb{R}$, which can be identified with $\mathbb{R}^m$. In this way, the incidence matrix B of the graph can be also regarded as the matrix representation of a linear map from the edge space $\mathbb{R}^m$ to the vertex space $\mathbb{R}^n$.

A cone in $\mathbb{R}^n$ is a closed subset K such that $K \cap \{-K\} = \{0\}$ and $\alpha K + \beta K \subseteq K$ for all $\alpha, \beta \geq 0$. A cone is generated by a set of vectors in K if any $x \in K$ can be written as a linear combination of vectors in the set, using only nonnegative coefficients. The dimension of K is the number of elements in a minimal generating set.

Notation: For $a, b \in \mathbb{R}^m$ the notation $a \leqslant b$ will denote element-wise inequality $a_i \leq b_i$, $i = 1, \ldots, m$. For $a_i \leq b_i$, $i = 1, \ldots, m$ the multidimensional saturation function $\operatorname{sat}(x\,; a, b) : \mathbb{R}^m \to \mathbb{R}^m$ is defined as

$$\operatorname{sat}(x\,; a, b)_i = \begin{cases} a_i & \text{if } x_i < a_i, \\ x_i & \text{if } a_i \leq x_i \leq b_i, \quad i = 1, \ldots, m. \\ b_i & \text{if } x_i > b_i, \end{cases} \quad (2)$$

Its integral $S(x\,; a, b) : \mathbb{R}^m \to \mathbb{R}^m$ is defined as

$$S(x\,; a, b)_i = \int_0^{x_i} \operatorname{sat}(y\,; a_i, b_i) dy. \quad (3)$$

III. A DYNAMIC NETWORK MODEL WITH INPUT CONSTRAINTS

Let us consider the following dynamical system defined on the graph ([9], [10], [11])

$$\begin{aligned} \dot{x} &= Bu + Ed, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, \quad d \in \mathbb{R}^k \\ y &= B^T \tfrac{\partial H}{\partial x}(x), \quad y \in \mathbb{R}^m, \end{aligned} \quad (4)$$

where $H : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function, and $\frac{\partial H}{\partial x}(x)$ denotes the column vector of partial derivatives of H. Here the i^{th} element x_i of the state vector x is the state variable associated to the i^{th} vertex, while u_j is a flow input variable associated to the j^{th} edge of the graph. And E is an $n \times k$ matrix whose columns consist of exactly one entry equal to 1 (inflow) or -1 (outflow), while the rest of the elements is zero. Thus E specifies the k terminal vertices where flows can enter or leave the network ([12]). System (4) defines a port-Hamiltonian system ([12], [13]), satisfying the energy-balance

$$\frac{d}{dt} H = u^T y + \frac{\partial^T H}{\partial x}(x) E d. \quad (5)$$

As explained in [1], when $d \neq 0$, the proportional control will not be sufficient to reach load balancing. Hence we consider a proportional-integral (PI) controller given by the dynamic output feedback

$$\begin{aligned} \dot{x}_c &= R B^T \frac{\partial H}{\partial x}(x) \\ u &= -R B^T \frac{\partial H}{\partial x}(x) - R \frac{\partial H_c}{\partial x_c}(x_c) \end{aligned} \quad (6)$$

Then the closed-loop can be represented as the following port-Hamiltonian system

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} -BRB^T & -BR \\ RB^T & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x}(x) \\ \frac{\partial H_c}{\partial x_c}(x_c) \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} d, \quad (7)$$

with $H_{tot}(x, x_c) = H(x) + H_c(x_c)$.

Since $\mathbb{1}^T \dot{x} = \mathbb{1}^T E d$, the system has a steady state if and only if $\mathbb{1}^T E d = 0$. For any weakly connected graph with n vertices, it implies that $Ed \in \operatorname{im} B$, for all Ed such that $\mathbb{1}^T E d = 0$. Suppose now the constant disturbance $\bar{d}$ and satisfies the *matching condition*, i.e. there exists a controller state $\bar{x}_c$ such that

$$E\bar{d} = B \frac{\partial H_c}{\partial x_c}(\bar{x}_c). \quad (8)$$

In many practical cases, the elements of the vector of flow inputs $u \in \mathbb{R}^m$ corresponding to the edges of the graph will be *constrained*, that is

$$u \in \mathcal{U} := \{u \in \mathbb{R}^m \mid u^- \leqslant u \leqslant u^+\} \quad (9)$$

for certain vectors u^- and u^+ satisfying $u_i^- < u_i^+, i = 1, \ldots, m$. In our previous paper [1] we focused on the cases where $u_i^- \leqslant 0 < u_i^+, i = 1, 2, \ldots, m.$ or $0 \leq u_i^- <$

$u_i^+, i = 1, 2, \ldots, m$ of which the intersection has nonempty interior. In the present paper we consider *arbitrary* constraint intervals, necessitating a novel approach to the problem.

Thus we consider a general constrained version of the PI controller given as

$$\begin{aligned} \dot{x}_c &= Ry, \\ u &= \operatorname{sat}\big(-Ry - R\tfrac{\partial H_c}{\partial x_c}(x_c)\,; u^-, u^+\big) \end{aligned} \tag{10}$$

For simplicity of exposition we consider throughout the rest of this paper the identity gain matrix $R = I$. Furthermore we throughout assume that the Hessian matrix of Hamiltonian $H(x)$ is positive definite for any x and we only consider $H_c(x_c) = \frac{1}{2}\|x_c\|^2$. Then the system (4) with constraint PI controller (10) is given as

$$\begin{aligned} \dot{x} &= B \operatorname{sat}\big(-B^T \frac{\partial H}{\partial x}(x) - x_c\,; u^-, u^+\big) + E\bar{d}, \\ \dot{x}_c &= B^T \frac{\partial H}{\partial x}(x), \end{aligned} \tag{11}$$

Remark 1: For arbitrary diagonal positive definitive gain matrix R, we can use as Lyapunov function instead of (25) the expression

$$V(x, x_c) = \mathbb{1}^T R^{-1} S\big(-RB^T \frac{\partial H}{\partial x}(x) - Rx_c\,; u^-, u^+\big) + H(x) \tag{12}$$

as Lyapunov function and obtain the same conclusions.

The constrained system is different from the one in ([5]) where the saturation is added separately on the proportional and integral part of the controller.

In the rest of this section, we will first show how the disturbance can be *absorbed* into the constraint intervals. Indeed, for any $\eta \in \mathbb{R}^n$, we have the identity

$$\operatorname{sat}(x - \eta\,; u^-, u^+) + \eta = \operatorname{sat}(x\,; u^- + \eta, u^+ + \eta). \tag{13}$$

Therefore for an in/out flow $\bar{d}$ satisfying the matching condition, i.e., such that there exists $\bar{x}_c$ with $B\bar{x}_c = E\bar{d}$, we can rewrite system (11) as

$$\begin{aligned} \dot{x} &= B \operatorname{sat}(-B^T \frac{\partial H}{\partial x}(x) - x_c'\,; u^- + \bar{x}_c, u^+ + \bar{x}_c), \\ \dot{x}_c' &= B^T \frac{\partial H}{\partial x}(x), \end{aligned} \tag{14}$$

where $x_c' = x_c - \bar{x}_c$.

Next, we will show how the orientation can be made compatible with the flow constraints.

Any *bi-directional* edge whose constraint interval satisfies $u_i^- < 0 < u_i^+$, it can be divided into *two uni-directional* edges with constraint intervals $[u_i^-, 0], [0, u_i^+]$ respectively, and the same orientation. This follows from

$$\operatorname{sat}(u_i; u_i^-, u_i^+) = \operatorname{sat}(u_i; u_i^-, 0) + \operatorname{sat}(u_i; 0, u_i^+) \tag{15}$$

for any $u_i^- < 0 < u_i^+$.

Furthermore, we may change the *orientation* of some of the edges of the graph at will; replacing the corresponding columns b_i of the incidence matrix B by $-b_i$. By the identity

$$\operatorname{sat}(-x\,; u_i^-, u_i^+) = -\operatorname{sat}(x\,; -u_i^+, -u_i^-), \tag{16}$$

we may therefore assume *without loss of generality* that the orientation of the graph is chosen such that

$$u_i^+ > 0,\ i = 1, 2, \ldots, m. \tag{17}$$

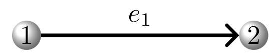


Fig. 1. Illustrative graph

Example 3.1: Consider the graph in Fig.1, where the constraint interval for edge e_1 is $[-2, -1]$. The network is equivalent to the network where the edge direction is reversed from v_2 to v_1 while the constraint interval is modified into $[1, 2]$.

By dividing bi-directional edges into uni-directional ones and changing orientations afterwards, we can therefore assume that

$$u_i^+ \geq u_i^- \geq 0,\ i = 1, 2, \ldots, m. \tag{18}$$

where the two equality signs do not hold at the same time.

Assumption (18) will be standing throughout the rest of the paper. In general, we will say that orientation of the the graph is *compatible with the flow constraints* if (18) hold.

IV. Convergence conditions for the closed-loop dynamics with general flow constraints

The following lemma is a cornerstone of this paper.

Lemma 2: ([14],Lemma 3.2.9 in [15],[16]) The set of minimal generators of the convex polyhedral cone $\ker B \cap \mathbb{R}^m_{\geq 0}$ is composed of positive circuits.

Definition 3: (Interior Point Condition) Given a directed graph with arbitrary constraints $[u^-, u^+]$ (maybe not compatible with the orientation), the network will be said to satisfy the interior point condition if there exists a vector $z \in [u^-, u^+]$ such that

$$B \operatorname{sat}(z; u^-, u^+) = Bz = 0, \tag{19}$$

and the set of edges along which the corresponding element of z is an interior point of the constraint interval contains a spanning tree.

Remark 4: The interior point condition is independent to the unknown in/outflows Ed, i.e. the system (11) satisfies interior point condition if and only if the system (14) does. Indeed, suppose z is the vector corresponding to $[u^-, u^+]$ in Definition 3, then for $[u^- + \bar{x}_c, u^+ + \bar{x}_c]$ the corresponding vector can be chosen as $z + \bar{x}_c$. It follows that, without loss of generality, we can restrict ourselves to the study of the closed-loop system

$$\begin{aligned} \dot{x} &= B \operatorname{sat}\big(-B^T \frac{\partial H}{\partial x}(x) - x_c\,; u^-, u^+\big), \\ \dot{x}_c &= B^T \frac{\partial H}{\partial x}(x). \end{aligned} \tag{20}$$

for general u^- and u^+ with $u_i^- \leq u_i^+, i = 1, \ldots, m$. The following results in this paper about (20) can be extended to general system (11) with compatible constraint intervals and in/outflows satisfying matching condition in a straight forward manner.

Remark 5: We will show that interior point condition is independent of the choice of $\beta \in \ker B$, i.e. a graph $\mathcal{G}$ with constraints $[u^-, u^+]$ satisfies the interior point condition, then $\mathcal{G}$ with constraints $[u^- + \beta, u^+ + \beta]$ also satisfies it for any $\beta \in \ker B$. This problem can be caused in (20), since for any $\beta \in \ker B$, system (20) can be rewritten with new constraints $[u^- + \beta, u^+ + \beta]$ and new edge states $x_c' = x_c - \beta$. Indeed, $z \in [u^-, u^+] \iff z + \beta \in [u^- + \beta, u^+ + \beta]$ and $z_i \in int[u_i^-, u_i^+] \iff z_i + \beta_i \in int[u_i^- + \beta_i, u_i^+ + \beta_i]$. That verifies our statement.

Remark 6: For any network with compatible orientation, we can assume in Definition 3 that $z \in \mathbb{R}^m_{\geq 0}$. This will be assumed throughout the rest of this paper.

Basing on the vector z from the interior point condition, we can divide the edge set $\mathcal{E}$ into the following four subsets

$$\begin{aligned} &\mathcal{E}_0(z; u^-, u^+) = \{e_i \mid e_i \in \mathcal{E}, z_i = 0\} \\ &\mathcal{E}_1(z; u^-, u^+) = \{e_i \mid e_i \in \mathcal{E}, z_i > 0\} \\ &\mathcal{E}_2(z; u^-, u^+) = \{e_i \mid e_i \in \mathcal{E}, z_i \in int[u_i^-, u_i^+]\} \\ &\mathcal{E}_3(z; u^-, u^+) = \{e_i \mid e_i \in \mathcal{E}, z_i = u_i^- \text{ or } z_i = u_i^+\} \end{aligned} \tag{21}$$

with $\mathcal{E}_0 \cup \mathcal{E}_1 = \mathcal{E}_2 \cup \mathcal{E}_3 = \mathcal{E}$. The subgraph $\mathcal{G}_i = \{\mathcal{V}, \mathcal{E}_i\}, i = 0, 1, 2, 3$, can be defined respectively.

Lemma 7: corollary Let $\mathcal{G}$ be a weakly connected directed graph with compatible constraint intervals $[u^-, u^+]$. Then $\mathcal{G}$ is strongly connected if it satisfies the interior point condition.

Proof: Let $z \in [u^-, u^+] \cap \mathbb{R}^m_{\geq 0}$ be such that $Bz = 0$. Then by Lemma 2, z can be represented as a positive linear combination of positive circuits. Furthermore, the set of edges along which z is a interior point of the constraint intervals contains a spanning tree, then $\mathcal{G}$ contains a strongly connected subgraph $\mathcal{G}_1 = \{\mathcal{V}, \mathcal{E}_1(z; u^-, u^+)\}$. In conclusion, $\mathcal{G}$ is strongly connected. ■

If $\mathcal{G}$ is strongly connected it must contain positive cycles, and it is easy to see that in this case any cycle that is not positive can be written as linear combination of positive circuits. Consequently, when $\mathcal{G}$ is strongly connected, positive circuits compose a basis of $\ker B$.

Let $z \in [u^-, u^+] \cap \mathbb{R}^m_{\geq 0}$ be the vector from the interior point condition. By Lemma 2, z can be represented as

$$z = \sum_{i=1}^{k} \alpha_i C_i \;\; \alpha_i > 0 \tag{22}$$

where $\mathcal{C}_i$ is a positive circuit of $\mathcal{G}_1$ and C_i is vectorial representation of $\mathcal{C}_i, i = 1, \ldots, k$. Denote the set of these k positive circuits as $\tilde{\mathcal{C}} = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_k\}$. By Lemma 7, the graph $\mathcal{G}_1(z; u^-, u^+)\}$ can be covered by $\tilde{\mathcal{C}}$.

Next we will explain the relation between $\tilde{\mathcal{C}}$ and $[u^-, u^+]$. Suppose an edge e_h is not overlapped in $\tilde{\mathcal{C}}$, i.e. there is only one positive circuit in $\mathcal{C}_i \in \tilde{\mathcal{C}}$ such that $e_h \in \mathcal{C}_i$, then clearly

$$\alpha_i \in [u_h^-, u_h^+] \tag{23}$$

where α_i is given as in (22). However, if an edge e_h is overlapped in $\tilde{\mathcal{C}}$, without loss of generality, say e_h belongs to $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_d$, then we have

$$\sum_{i=1}^{d} \alpha_i \in [u_h^-, u_h^+]. \tag{24}$$

Lemma 8: Consider the dynamical system (20) defined on the network satisfying the interior point condition. Then

(i) Along every trajectory $(x(t), x_c(t)), t \geqslant 0$, of (20), the function

$$\begin{aligned} V(x(t), x_c(t)) = \mathbb{1}^T S\big(- B^T \frac{\partial H}{\partial x}(x(t)) - x_c(t)\,; u^-, u^+\big) \\ + H(x(t)) \end{aligned} \tag{25}$$

is bounded from below,

(ii) The trajectory $(x(t), x_c(t)), t \geqslant 0$, is bounded,

(iii) $\lim_{t\to\infty} \dot{V}(x(t), x_c(t)) = 0$,

Proof: (i) Since $H(x)$ is positive definite, we only need to show that the components of $V_1(x, x_c) := S\big(- B^T \frac{\partial H}{\partial x}(x) - x_c\,; u^-, u^+\big)$ are bounded from below. Suppose the ith component of $V_1(x(t), x_c(t))$ converges to $-\infty$, by the property of integral of saturation function, this holds if and only if on the ith edge $\big(-B^T \frac{\partial H}{\partial x}(x(t)) - x_c(t)\big)_i \to -\infty$ and $u_i^- > 0$.

Let us define two subsets of $\mathcal{E}$

$$\begin{aligned} \mathcal{E}_{-\infty} = \{e_i \in \mathcal{E} \mid \big(- B^T \frac{\partial H}{\partial x}(x(t)) - x_c(t)\big)_i \to -\infty\}, \\ \mathcal{E}_{+\infty} = \{e_i \in \mathcal{E} \mid \big(- B^T \frac{\partial H}{\partial x}(x(t)) - x_c(t)\big)_i \to +\infty\}. \end{aligned} \tag{26}$$

When $\mathcal{E}_{-\infty} = \emptyset$, it is easy to see that $V(x(t), x_c(t))$ is bounded from below.

When $\mathcal{E}_{-\infty} \neq \emptyset$, for large enough t, on the edges of $\mathcal{E}_{-\infty}$, the summation of corresponding components of $V_1(x(t), x_c(t))$ is equal to

$$\sum_{e_i \in \mathcal{E}_{-\infty}} u_i^- \big(- B^T \frac{\partial H}{\partial x}(x(t)) - x_c(t)\big)_i. \tag{27}$$

up to constants which depend on the initial condition. Similarly, on the edges of $\mathcal{E}_{+\infty}$, the summation is equal to

$$\sum_{e_i \in \mathcal{E}_{+\infty}} u_i^+ \big(- B^T \frac{\partial H}{\partial x}(x(t)) - x_c(t)\big)_i. \tag{28}$$

Let $z \in \ker B \cap \mathbb{R}^m_{\geq 0}$ be the vector from the interior point condition, we have

$$\begin{aligned} &0 = z^T B^T \frac{\partial H}{\partial x}(x) \\ &\Rightarrow \sum_{e_i \in \mathcal{E}} z_i x_{c_i}(t) = \sum_{e_i \in \mathcal{E}} z_i x_{c_i}(0) \\ &\Rightarrow \sum_{e_i \in \mathcal{E}} z_i \big(-B^T \frac{\partial H}{\partial x}(x(t)) - x_c(t) \big)_i = -\sum_{e_i \in \mathcal{E}} z_i x_{c_i}(0), \\ &\forall t > 0 \end{aligned} \tag{29}$$

Then by (29) and the fact that $u_i^- \leq z_i \leq u_i^+, \forall e_i \in \mathcal{E}$, we have that function (25) is bounded from below.

(ii) Notice that $\dot{V} = -\dot{x}^T \frac{\partial^2 H}{\partial x^2} \dot{x} \leq 0$.

Suppose that $x(t), t \geq 0$, is not bounded, then there exists a sequence $\{t_k\}, t_k \geq 0$ such that

$$\lim_{k \to \infty} \|x(t_k)\| = +\infty. \tag{30}$$

Since $H(x)$ is unbounded, this implies

$$\lim_{k \to \infty} V(x(t_k), x_c(t_k)) = +\infty. \tag{31}$$

This is a contradiction with $\dot{V} \leq 0$.

Suppose x_c is unbounded, we first show that this can not happen only on the edges of $\mathcal{E}_3(z; u^-, u^+)$. Indeed, suppose x_c is unbounded on $e_i \in \mathcal{E}_3(z; u^-, u^+)$ where $e_i \sim (x_p, x_q)$, then by the property of dynamics of x_c in (20), along any positive path from x_q to x_p, there exists at least one edge, on which x_c is unbounded, belongs to $\mathcal{E}_3(z; u^-, u^+)$. Then $\{\mathcal{V}, \mathcal{E} \setminus \mathcal{E}_3(z; u^-, u^+)\} = \{\mathcal{V}, \mathcal{E}_2(z; u^-, u^+)\}$ is not weakly connected which is a contradiction with the fact that $\mathcal{E}_2(z; u^-, u^+)$ contains a spanning tree.

Then if x_c is unbounded, there must be some edges of $\mathcal{E}_2(z; u^-, u^+)$ on which x_c is unbounded. However, if on $e_i \in \mathcal{E}_2(z; u^-, u^+)$, there exists a sequence $\{t_k\}, t_k \geq 0$ such that

$$\lim_{k \to \infty} \|x_{c_i}(t_k)\| = +\infty, \tag{32}$$

then similar to (i), by using (29) and the fact that $u_i^- < z_i < u_i^+, \forall e_i \in \mathcal{E}_2(z; u^-, u^+)$, we have

$$\lim_{k \to \infty} V(x(t_k), x_c(t_k)) = +\infty. \tag{33}$$

This is a contradiction to $\dot{V} \leq 0$ again.

In conclusion, (x, x_c) is bounded.

(iii)From the dynamics (20) and (ii), it can be shown that $\frac{d}{dt}(-B^T \frac{\partial H}{\partial x}(x) - x_c)$ is bounded. Combining the facts that $V(x, x_c)$ is bounded from below with $\dot{V} \leqslant 0$, we have that $\lim_{t \to \infty} \dot{V}(x(t), x_c(t)) = 0$.

Indeed, suppose $\dot{V}(x(t), x_c(t))$ does not converge to zero. In other words, there exists a real $\delta > 0$ and a sequence $\{t_k\}$, satisfying $\lim_{k \to \infty} t_k = +\infty$, such that $\dot{V}(x(t_k), x_c(t_k)) < -\delta$. Since $\frac{d}{dt}(-B^T \frac{\partial H}{\partial x}(x) - x_c)$ is bounded, then for each $k = 1, 2, \ldots$, there exists a time interval I_k and an $\epsilon > 0$ such that $|I_k| > \epsilon, t_k \in I_k$, and $\forall t \in I_k, \dot{V}(x(t), x_c(t)) < -\frac{\delta}{2}$. This implies that

$$\lim_{t \to \infty} V(x(t), x_c(t)) = -\infty,$$

which is contradicted by (i). In conclusion, $\lim_{t \to \infty} \dot{V}(x(t), x_c(t)) = 0$. ■

We obtain our main theorem.

Theorem 9: Consider the dynamical system (20) defined on a weakly connected directed graph with compatible constraints $[u^-, u^+]$. Then the trajectories will converge into

$$\mathcal{E}_{\text{tot}} = \{(x, x_c) \mid \frac{\partial H}{\partial x}(x) = \alpha \mathbb{1}_n,\ B \operatorname{sat}(-x_c\,; u^-, u^+) = 0\}. \tag{34}$$

if and only if the network satisfies the interior point condition.

Proof: Sufficiency: Suppose the network satisfies interior point condition with vector z which has representation as (22). Consider the following function

$$V(x, x_c) = \mathbb{1}^T S\big(-B^T \frac{\partial H}{\partial x}(x) - x_c\,; u^-, u^+\big) + H(x) \tag{35}$$

as Lyapunov function. Notice that $\dot{V} = -\dot{x}^T \frac{\partial^2 H}{\partial x^2} \dot{x} \leq 0$.

Using Lemma 8 and LaSalle's principle, it follows that $(x(t), x_c(t))$ converges to the largest invariant set $\mathcal{I}$ contained in $\{(x, x_c) \mid \dot{V} = 0\}$. If a solution $(x(t), x_c(t)) \in \mathcal{I}$, then x is a constant vector, denoted as ν. Furthermore, $\mathcal{I}$ is given as

$$\begin{aligned} &\mathcal{I} = \{(\nu, x_c) \mid x_c = B^T \frac{\partial H}{\partial x}(\nu) t + x_c(0), \\ &B \operatorname{sat}\big(-B^T \frac{\partial H}{\partial x}(\nu) - B^T \frac{\partial H}{\partial x}(\nu) t - x_c(0)\,; u^-, u^+ \big) = 0, \\ &\forall t \geq 0\}. \end{aligned} \tag{36}$$

We will prove $B^T \frac{\partial H}{\partial x}(\nu) = 0$ by contradiction. Suppose we choose $(\nu, x_c(0)) \in \mathcal{I}$, according to the definition of invariant set, $V\big(\nu, B^T \frac{\partial H}{\partial x}(\nu) t + x_c(0)\big)$ is a constant along the trajectory. Furthermore,

$$V_1(t) := \mathbb{1}^T S\big(-B^T \frac{\partial H}{\partial x}(\nu) - B^T \frac{\partial H}{\partial x}(\nu) t - x_c(0)\,; u^-, u^+\big) \tag{37}$$

will be a constant. Notice that

$$\begin{aligned} \dot{V}_1 = &- \operatorname{sat}^T \big(-B^T \frac{\partial H}{\partial x}(\nu) - B^T \frac{\partial H}{\partial x}(\nu) t - x_c(0), u^-, u^+ \big) \\ &B^T \frac{\partial H}{\partial x}(\nu) \end{aligned} \tag{38}$$

We can assume that $\dot{V}_1(0) = 0$.

Suppose if on an edge e_i, $B_i^T \frac{\partial H}{\partial x}(\nu) > 0$, the for large enough t

$$\begin{aligned} u_i^- &= \operatorname{sat}\big(-B_i^T \frac{\partial H}{\partial x}(\nu) - B_i^T \frac{\partial H}{\partial x}(\nu) t - x_{c_i}(0), u_i^-, u_i^+ \big) \\ &\leq \operatorname{sat}\big(-B_i^T \frac{\partial H}{\partial x}(\nu) - x_{c_i}(0), u_i^-, u_i^+ \big), \end{aligned} \tag{39}$$

while if $B_i^T \frac{\partial H}{\partial x}(\nu) < 0$, the for large enough t

$$\begin{aligned} u_i^+ &= \text{sat}\big(-B_i^T \frac{\partial H}{\partial x}(\nu) - B_i^T \frac{\partial H}{\partial x}(\nu)t - x_{c_i}(0), u_i^-, u_i^+\big) \\ &\geq \text{sat}\big(-B_i^T \frac{\partial H}{\partial x}(\nu) - x_{c_i}(0), u_i^-, u_i^+\big). \end{aligned} \tag{40}$$

Furthermore, if the edge $e_i \in \mathcal{E}_2$, the above two inequalities hold strictly, which implies that $\dot{V}_1(t) > 0$ for large enough t. This is a contradiction. So along all the edges of $\mathcal{E}_2$, $B_i^T \frac{\partial H}{\partial x}(\nu) = 0$. Since $(\mathcal{V}, \mathcal{E}_2)$ contains a spanning tree, $B^T \frac{\partial H}{\partial x}(\nu) = 0$

Necessity First of all, if there does not exist z such that $B\,\text{sat}(z; u^-, u^+) = 0$, the system (20) is unstable. Suppose now the network does not satisfy interior point condition, i.e there exist a vector z such that

$$Bz = 0, z \in [u^-, u^+]. \tag{41}$$

however for any z such that (41) holds, the $\mathcal{E}_2(z; u^-, u^+)$ does not contain a spanning tree.

For this case, we will show that the dynamical system (20) will form a clustering by setting suitable initial condition $(x(0), x_c(0))$ with $B^T \frac{\partial H}{\partial x}(x(0)) \neq 0$ and $\dot{x}(t) = 0, t \geq 0$.

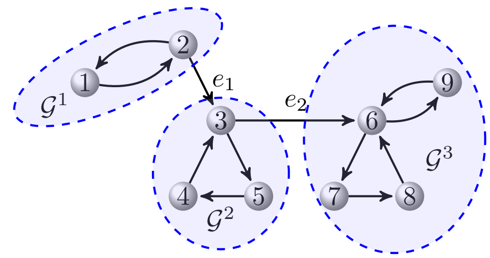
Since $\mathcal{E}_2(z; u^-, u^+) \cup \mathcal{E}_3(z; u^-, u^+) = \mathcal{E}, \mathcal{E}_2(z; u^-, u^+) \cap \mathcal{E}_3(z; u^-, u^+) = \emptyset$ and $\mathcal{E}_2(z; u^-, u^+)$ does not contain a spanning tree, then $\mathcal{E}_3(z; u^-, u^+)$ contains a cut set. Suppose the graph $\mathcal{G}_2(z)$ is not weakly connected and has k weakly connected components, denoted as $\mathcal{G}_2^1(z), \cdots, \mathcal{G}_2^k(z)$, we can introduce a reduced graph $\tilde{\mathcal{G}}(z)$ which has k vertices that each of them represents a component of $\mathcal{G}_2(z)$, $\mathcal{E}(\tilde{\mathcal{G}}(z)) \subseteq \mathcal{E}_3(z; u^-, u^+)$ is the set of edges connecting the components of $\mathcal{G}_2(z)$. With a slight abuse of notation, we denote the vertices of $\tilde{\mathcal{G}}(z)$ as $\mathcal{G}_2^1(z), \cdots, \mathcal{G}_2^k(z)$ too. This reduction is shown in Figure 2.

Next we will conduct the following algorithm on $\tilde{\mathcal{G}}(z)$.

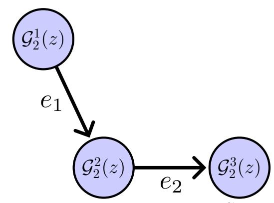
Algorithm 10: Initialization, find any $z^0 \in \mathbb{R}^m_{\geq 0}$ such that (41) holds. Let z^k denote the value of z from the previous iteration. For z^k, check if $\tilde{\mathcal{G}}(z^k)$ satisfies case 1 or case 2 which are given below. If does, we will derive a new z^{k+1} satisfying (41) and repeat this step for $\tilde{\mathcal{G}}(z^{k+1})$; if not, the algorithm stops. Notice that $\tilde{\mathcal{G}}(z^{k+1})$ has fewer vertices than $\tilde{\mathcal{G}}(z^k)$.

Case 1: Consider the subgraph of $\tilde{\mathcal{G}}(z^k)$ given as in Figure 3(left), where $\mathcal{G}_2^i(z^k), \mathcal{G}_2^j(z^k)$ are two nodes in $\tilde{\mathcal{G}}(z^k)$ which are connected by edges e_{l_1} and e_{l_2}. Suppose z^k on e_{l_1} and e_{l_2} reach different bounds, for instance, upper bound on e_{l_1} and lower bound on e_{l_2}, then we can modify z^k to z^{k+1} satisfying (41) such that z^{k+1} belongs to interior of constraint intervals on e_{l_1} and e_{l_2}. Besides $\mathcal{G}_2^i(z^k)$ and $\mathcal{G}_2^j(z^k)$ will merge into one node in $\tilde{\mathcal{G}}(z^{k+1})$. Indeed, in this case there exist a closed path of $\mathcal{G}$ composed by the edges in $\mathcal{G}_2^i(z^k), \mathcal{G}_2^j(z^k)$, e_{l_2} and reversed e_{l_1}, with incidence vector denoted as w, such that for small enough $\epsilon > 0$, $z_{l_i}^{k+1} \in int[u_{l_i}^-, u_{l_i}^+], i = 1, 2$ where $z^{k+1} = z^k + \epsilon w$.

Case 2: Suppose along any positive circuit in $\tilde{\mathcal{G}}(z^k)$, z^k reaches the same bounds, i.e. upper bounds simultaneously or lower bounds. An example is given as in Figure 3(right), where z^k on $e_{l_1}, e_{l_2}, e_{l_3}$ reaches lower bounds at the same time. Then similar to the previous case, there exists a closed path of $\mathcal{G}$ composed by the edges in $\mathcal{G}_2^h(z^k), \mathcal{G}_2^i(z^k), \mathcal{G}_2^j(z^k)$ and $e_{l_i}, i = 1, 2, 3$ with the incidence vector denoted as w, such that for small enough $\epsilon > 0$, $z_{l_i}^{k+1} \in int[u_{l_i}^-, u_{l_i}^+], i = 1, 2, 3$ where $z^{k+1} = z^k + \epsilon w$. For the case when z^k on $e_{l_1}, e_{l_2}, e_{l_3}$ reaches upper bounds, we could use the reversed closed path with incidence vector $-w$.



(a) The whole network $\mathcal{G}$ with its weakly connected components



(b) Simplified graph $\tilde{\mathcal{G}}(z)$

Fig. 2. For a given z, the pony-shape network, given as (a), falls into three weakly connected components after deleting $\mathcal{E}_3$ where $e_1, e_2 \in \mathcal{E}_3$. By denoting each weakly connected component as a node, we get the simplified graph $\tilde{\mathcal{G}}(z)$, given as in (b). In this case $\mathcal{G}_2^i(z), i = 1, 2, 3$ represent either a node in $\tilde{\mathcal{G}}(z)$ or a weakly component of $\mathcal{G}_2(z)$.

Since there exist only a finite number of vertices of $\mathcal{G}$, the algorithm will stop after finite steps. Let us denote the final value of z as z^*. If the network satisfies the interior point condition, then $\tilde{\mathcal{G}}(z^*)$ is a trivial graph with only one vertex. If not, the graph $\tilde{\mathcal{G}}(z^*)$ satisfies: first, z^* on the edges of $\tilde{\mathcal{G}}(z^*)$ with the same starting and ending nodes reaches the same bounds; second, along any positive circuit in $\tilde{\mathcal{G}}(z^*)$, z^* reaches upper and lower bounds simultaneously.

Finally we can set the suitable initial condition of system (20) on $\mathcal{G}$ such that $\dot{x} = 0, B^T \frac{\partial H}{\partial x} \neq 0, \forall t > 0$. Based on z^*, we can set $x_c(0) = -z^*$. For $\frac{\partial H}{\partial x}$, we can assign it the same value in each weakly connected component of $\mathcal{G}_2(z^*)$. In fact, we can set it on $\mathcal{G}_2^i(z^*)$ larger than it on $\mathcal{G}_2^j(z^*)$ if there is an edge from $\mathcal{G}_2^j(z^*)$ to $\mathcal{G}_2^i(z^*)$ on which z^* reaches lower bound. Similarly, $\frac{\partial H}{\partial x}$ on $\mathcal{G}_2^i(z^*)$ is assigned to be smaller than it on $\mathcal{G}_2^j(z^*)$ if z^* reaches upper bound on the edges from $\mathcal{G}_2^j(z^*)$ to $\mathcal{G}_2^i(z^*)$. We can verify that $\dot{x}(t) = 0$, but $B^T \frac{\partial H}{\partial x}(x(t)) \neq 0, \forall t > 0$. ■

Remark 11: For any final value z^* from the previous algorithm, notice that in any positive circuit of $\mathcal{G}$, there is either no edge of $\tilde{\mathcal{G}}(z^*)$ or at least two.

First, we will show that for a weakly connected network

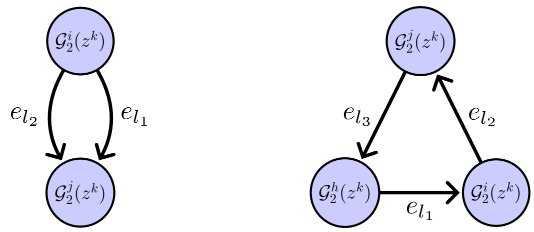


Fig. 3. Two cases which are studied in proof of Theorem 9.

satisfying Interior Point Condition, the algorithm will not end up with a graph $\tilde{\mathcal{G}}(z^*)$ with more than one node for any initial condition z^0. Indeed, suppose $\tilde{\mathcal{G}}(z^*)$ has more than one node, by Lemma 2, $z - z^*$ can be represented by a linear combination of positive circuits where z is a vector from Interior Point Condition. However, since $\tilde{\mathcal{G}}(z^*)$ does not satisfy Case 2, for any linear combination of positive circuits, denoted as w, $z^* + w \notin [u^-, u^+]$. This is contradicted to $z \in [u^-, u^+]$.

Next we will show that for a weakly connected network the algorithm will not produce different $\tilde{\mathcal{G}}(z_1^*)$ and $\tilde{\mathcal{G}}(z_2^*)$ by using different initial conditions z_1^0, z_2^0. Indeed suppose there exist two nodes x_i, x_j which belong to the same node of $\tilde{\mathcal{G}}(z_1^*)$ but two different nodes of $\tilde{\mathcal{G}}(z_2^*)$, denoted as $\mathcal{G}_2^i(z_2^*)$ and $\mathcal{G}_2^j(z_2^*)$. By Lemma 2, $z_1^* - z_2^*$ can be represented as linear combination of positive circuits. However for any linear combination of positive circuits, denoted as w, $z_2^* + w \notin [u^-, u^+]$. This is contradicted to $z_1^* \in [u^-, u^+]$.

In conclusion, Interior Point Condition is a property of the network which does not depend the initial condition of the algorithm.

From the necessity part of the proof for Theorem 9 and the following remark, we have the next corollary

Corollary 12: Consider dynamical system (20) defined on a weakly connected graph with compatible constraint interval $[u^-, u^+]$. If the network does not satisfy interior point condition, then $\frac{\partial H}{\partial x}$ will converge to a clustering along the partition introducing by deleting the edge set $\mathcal{E}_3(z^*, u^-, u^+)$ where z^* is given by Algorithm 10.

Example 4.1: Consider the dynamical system (20) with $H(x) = \frac{1}{2}\|x\|_2^2$ defined on network given as in Figure 5. The constraint intervals are $u^- = [0, 1, 2, 0, 0, 0, 0]^T$ and $u^+ = [1, 3, 3, 2, 2, 1, 2]^T$. In order to check interior point condition, we can take $z = [1, 2, 3, 1, 1, 1, 1]^T$ in which case $\mathcal{E}_2(z; u^-, u^+) = \{e_2, e_4, e_5, e_7\}$ which contains a spanning tree. As can be seen from Fig.4, all the states in vertices converge to consensus.

V. Controlling to arbitrary steady states in the admissible set

Notice that the first equation in system (4) implies that

$$\mathbb{1}^T \dot{x} = 0 \tag{42}$$

if the disturbance satisfies the matching condition (8).

In this section, instead of driving the state to consensus, we will make the state converge to any desirable point x^* in the admissible set, which is defined as

$$\mathcal{A} = \{x \mid \mathbb{1}^T x = \mathbb{1}^T x(0)\}. \tag{43}$$

We will achieve this by modifying our controller (6)

In [3], the authors construct a discontinuous controller with the information of desirable point to complete this task. Here we consider the following controller (with gain matrix $R = I$)

$$\begin{aligned} \dot{x}_c &= B^T \frac{\partial H}{\partial x}(x - x^*) \\ u &= -B^T \frac{\partial H}{\partial x}(x - x^*) - \frac{\partial H_c}{\partial x_c} \end{aligned} \tag{44}$$

which can measure the *difference* between the current states and the desirable ones. If the input is not saturated, then the closed-loop system is given as

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} -BB^T & -B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x}(x - x^*) \\ \frac{\partial H_c}{\partial x_c}(x_c) \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} d, \tag{45}$$

Theorem 13: Suppose $H(x)$ is strictly convex and has its minimum at the origin. Consider a desirable state x^* from admissible set $\mathcal{A}$. Assume the disturbance satisfies the matching condition (8). Then the trajectories of system (45) will converge to

$$\mathcal{E}_{tot} = \{(x^*, \bar{x}_c) \mid B \frac{\partial H_c}{\partial \bar{x}_c}(x_c) = E\bar{d}\} \tag{46}$$

if and only if the graph is weakly connected.

Proof:

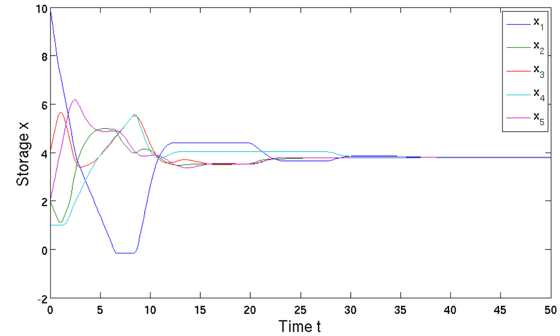


Fig. 4. The trajectories of state variables on vertices of the system in Example 4.1.

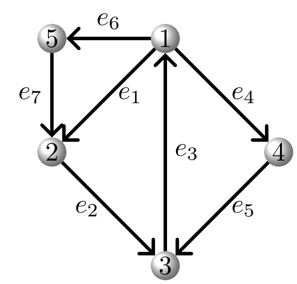


Fig. 5. Network of Example 4.1

Sufficiency: Consider the modified Hamiltonian

$$H^*(x,x_c) = H(x-x*)+H_c(x_c)-H_c(\bar{x}_c)-\frac{\partial^T H_c}{\partial \bar{x}_c}(x_c-\bar{x}_c) \tag{47}$$

as Lyapunov function. Then we have

$$\frac{dH^*}{dt} = -\frac{\partial^T H}{\partial x}(x-x^*)BB^T\frac{\partial H}{\partial x}(x-x^*) \leq 0 \tag{48}$$

By LaSalle's principle, the trajectories will converge into the largest invariant set, denoted as $\mathcal{I}$, in $\{(x,x_c) \mid B^T\frac{\partial H}{\partial x}(x-x^*)=0\}$. By the fact that the network is weakly connected, $\frac{\partial H}{\partial x}(x-x^*) = \alpha\mathbb{1}$ for some $\alpha \in \mathbb{R}$. Since H is strictly convex, we can write $x - x^* = \frac{\partial H^{-1}}{\partial x}(\alpha\mathbb{1})$. Furthermore, because H gets minimum at origin and $\mathbb{1}^T(x-x^*) = 0$, so $\alpha = 0$ and $x = x^*$. The rest of proof follows the standard way.

Necessary: If the network is not weakly connected, the controller (44) can only drive the states into the admissible set corresponding to each weakly connected components. Here end the proof ■

By denoting $x - x^*$ as $\tilde{x}$, we can write the system (45) as

$$\begin{bmatrix}\dot{\tilde{x}}\\ \dot{x}_c\end{bmatrix} = \begin{bmatrix}-BB^T & -B\\ B^T & 0\end{bmatrix}\begin{bmatrix}\frac{\partial H}{\partial \tilde{x}}(\tilde{x})\\ \frac{\partial H_c}{\partial x_c}(x_c)\end{bmatrix} + \begin{bmatrix}E\\ 0\end{bmatrix}d. \tag{49}$$

After absorbing the disturbance into constraint intervals as (14), the corresponding saturated case can be written as

$$\begin{aligned}\dot{\tilde{x}} &= B\,\mathrm{sat}\big(-B^T\frac{\partial H}{\partial \tilde{x}}(\tilde{x}) - x_c\,; u^-, u^+\big),\\ \dot{x}_c &= B^T\frac{\partial H}{\partial \tilde{x}}(\tilde{x}).\end{aligned} \tag{50}$$

By the fact that $\mathbb{1}^T\tilde{x} = 0$, and H is strictly convex and reach minimum at origin, we can show that $\frac{\partial H}{\partial \tilde{x}}(\tilde{x}) \to \alpha\mathbb{1} \Rightarrow \alpha = 0$ and $\tilde{x} \to 0$. This is a direct application of Theorem 9.

Corollary 14: Suppose the Hamiltonian H is strictly convex and reach minimum at origin. For any $x^* \in \mathcal{A}$, consider dynamical system (50) defined on a directed graph with compatible constraints $[u^-, u^+]$, then the trajectories will converge into

$$\mathcal{E}_{\mathrm{tot}} = \{(0,x_c) \mid B\,\mathrm{sat}(-x_c\,; u^-, u^+) = 0\}. \tag{51}$$

if and only if the network satisfies interior point condition.

VI. CONCLUSIONS

We have discussed a basic model of dynamical distribution networks where the flows through the edges are generated by distributed PI controllers. The main part of this paper focuses on the case where flow constraints are present. A key ingredient in this analysis is the construction of a C^1 Lyapunov function. After making the orientation and constraint interval compatible, we derived a sufficient and necessary condition, which depends on the graphic structure of the network and constraint intervals, for asymptotic load balancing with any given network and constraints. By modified PI controller, the states on nodes can be driven to any desirable state of admissible set.

An obvious open problem is how to put constraint on storage variables on vertices. This is currently under investigation. Many other questions can be addressed in this framework. For example, what is happening if the in/outflows are not assumed to be constant, but are e.g. periodic functions of time; see already [2].

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