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Exponential Attitude Stabilization and Synchronization of Rigid Bodies

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Abstract— **This paper considers the attitude stabilization and synchronization of rigid bodies. We use the logarithm map to exponentially stabilize the attitude of a rigid body with an almost global region of attraction. Interestingly, we are able to obtain a closed form solution for the kinematics. We extend this in a discontinuous manner and analyze global stability. Further, we present consensus protocols for exponential attitude and angular velocity synchronization under directed information flow. Simulation results are presented to validate the control laws.**

I. INTRODUCTION

Attitude stabilization of rigid bodies has seen a surge of interest due to advances in unmanned aerial vehicles and spacecrafts. The problem presents numerous challenges, and thus it has been extensively studied (see [1], [2], [3] and the references therein) with and without parametrizing the rotation matrices which describe the attitude. In particular, a parametrization in terms of the logarithm mapping on $SO(3)$ to its Lie algebra is used in [4]. The resulting model was stabilized using feedback linearization. In [5], it was shown that there are no continuous vector fields on compact manifolds which have a globally asymptotically stable equilibrium. Discontinuous control laws were proposed in [6] and [7]. In [8], the lack of robustness in such controllers was demonstrated, and a robust hybrid control was proposed.

The problem of synchronization or consensus of multiple autonomous systems has been of great interest [9], [10], [11] due to its utility in solving a large number of distributed control problems. Specifically, attitude synchronization of rigid bodies has applications to coordinated motion of satellite clusters and formation control of aerial and underwater vehicles. In [12], a synchronizing control for rigid bodies with undirected graphs using coupling potentials was proposed, in which the angular velocities were regulated to zero using a dissipation term. The results were extended to directed graphs in [13], which also dealt with the case where the final angular velocities are synchronized rather than regulated to zero. This result gives rise to an interesting motion where the group of synchronized rigid bodies acts as a single rigid body. The combined problem with both velocity synchronization and directed information flow remains unsolved in the formulation considered. In [14], a controller for consensus on $SO\,(3)$ was proposed using the Riemannian metric and the associated logarithm map. This was later extended to the case of manifolds with bounded curvature in [15]. Although our work also uses the logarithm map on $SO\,(3)$, it differs from these results as the synchronizing control is not obtained from a metric. In [16], attitude synchronization of multiple rigid bodies is solved without angular velocity measurements, while in [17], delay effects in communication links are taken into account. The modified Rodrigues parametrization is used in [18] for attitude synchronization using two approaches: in the first, a passivity based approach is used with undirected graphs, while in the second, a tracking control is extended to synchronization for directed graphs.

The exponential map from $\mathfrak{so}\,(3)$ to $SO\,(3)$ is a diffeomorphism on an open set containing the identity. In this paper, we use the logarithm map on this set to locally exponentially stabilize the attitude of a rigid body. We show that the closed-loop kinematics has a closed form solution. We extend this in a discontinuous manner and analyze global stability and robustness issues. The motivation for using the logarithm map stems from the fact that the image of the logarithm map is an open convex subset of $\mathbb{R}^3$. Accordingly, we apply existing Euclidean consensus protocols to solve the attitude and angular velocity synchronization problems under directed information flow. Finally, we comment on the connection between the consensus value reached under balance conditions, and a metric on a particular open geodesic ball.

The organization of the paper is as follows. Section II lists the required mathematical preliminaries. Section III presents the kinematic and dynamic attitude stabilization strategies with the logarithm map, while section IV describes attitude synchronization problems under two settings: velocity regulation and velocity synchronization. Simulation results are presented in Section V with concluding remarks in Section VI.

II. MATHEMATICAL PRELIMINARIES

For $x \in \mathbb{R}^n$, its Euclidean norm is denoted by $\|x\|_2$ while for a matrix $A \in \mathbb{R}^{n\times n}$, its Frobenius norm is defined as $\|A\|_F \triangleq \sqrt{\mathrm{Tr}(A^\top A)}$, where $\mathrm{Tr}(\cdot)$ is the Trace operation. The Kronecker product of two matrices M, N is denoted by $M \otimes N$. The inner-product of two matrices $A, B \in \mathbb{R}^{n\times n}$ is $\langle A, B\rangle_{Tr} \triangleq \mathrm{Tr}(A^\top B)$.

Let the orientation of a rigid body be denoted by $R(t) \in SO\,(3)$ relative to the reference inertial frame, where $SO\,(3)$ is the set of all rotation matrices in the special orthogonal group of rigid rotations in $\mathbb{R}^3$ and $\dot{R}(t) \in T_R SO\,(3)$, the tangent space to $SO\,(3)$ at R. Since $SO\,(3)$ is a Lie group, $T_I SO\,(3) \simeq \mathfrak{so}\,(3)$ acts as Lie algebra of the group, where I is the identity element of the group $SO\,(3)$, $\mathfrak{so}\,(3)$ is a vector space formed by skew-symmetric matrices. Since $\mathfrak{so}\,(3)$ is

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isomorphic to $\mathbb{R}^3$, the wedge operator $\wedge$, $x \in \mathbb{R}^3 \mapsto \widehat{x} \in \mathfrak{so}(3)$ is defined as

$$\widehat{x} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

with the bracket structure

$$[\widehat{\omega},\ \widehat{v}] = \widehat{\omega}\widehat{v} - \widehat{v}\widehat{\omega}, \quad \forall\, \widehat{v},\ \widehat{\omega} \in \mathfrak{so}(3)\,.$$

For $a, b \in \mathbb{R}^3$, $\mathrm{Tr}(\widehat{a}^\top \widehat{b}) = 2a^\top b$. Further, let $\vee$ be the inverse of the wedge operation and the Lie algebra isomorphism between $(\mathbb{R}^3, \times)$ and $(\mathfrak{so}(3), [\cdot,\ \cdot])$ is

$$[\widehat{\omega},\ \widehat{v}]^\vee = \omega \times v, \quad \forall\, v,\ \omega \in \mathbb{R}^3.$$

For every rotation matrix $R \in SO(3)$, one can associate a matrix $\log R = \theta\widehat{\eta} \in \mathfrak{so}(3)$, where $\theta \in S^1$ and $\eta \in S^2/\{-1, 1\} = \mathbb{P}^2$ are defined by Rodrigues' formula:

$$R = I + \widehat{\eta}\sin\theta + \widehat{\eta}^2(1 - \cos\theta).$$

Note that η and θ are the axis and angle associated with the rotation defined by R and $\mathbb{P}^2$ is the real projective plane. Rodrigues' formula follows from the general fact that the exponential of a matrix $A \in \mathbb{R}^{3\times 3}$, defined by

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!},$$

is an absolutely convergent series. The exponential map $\exp : \mathfrak{so}(3) \mapsto SO(3)$ is well-defined and surjective.

For $R \in \mathcal{D}$, where

$$\mathcal{D} = \left\{ R \in SO(3) \mid \frac{1}{\sqrt{2}}\|\log R\|_F < \pi \right\},$$

the exponential map is a diffeomorphism, with $\exp^{-1}(\cdot) = \log(\cdot)$, which is called the principal logarithm [19], and the injectivity radius of the exponential map is π. If $\theta = \pi$, $\log R \in \{\pi\widehat{\eta},\ -\pi\widehat{\eta}\}$.

The derivative of the exponential map on a Lie group is [20], [21]

$$\left.\frac{d}{dt}\right|_{t=0} e^{X+tY} = e^X \cdot \sum_{k=0}^{\infty} \frac{(-1)^k(\mathrm{ad}(X))^k}{(k+1)!}(Y)$$

where $X,\ Y$ are elements of the corresponding Lie algebra, and $\mathrm{ad}(X)(Y) = [X, Y]$. Since $R = \exp(\log R)$, we have

$$\begin{aligned} \dot{R} &= \exp(\log R) \sum_{k=0}^{\infty} \frac{(-1)^k(\mathrm{ad}(\log R))^k}{(k+1)!}\left(\frac{d}{dt}(\log R)\right) \\ \Rightarrow (R^\top \dot{R})^\vee &= \sum_{k=0}^{\infty} \frac{(-1)^k(\log R)^k}{(k+1)!}\left(\frac{d}{dt}(\log R)^\vee\right). \end{aligned}$$

On $SO(3)$, the infinite series can be simplified to

$$\begin{aligned} H(R) &\triangleq \sum_{k=0}^{\infty} \frac{(-1)^k(\log R)^k}{(k+1)!} \\ &= I + \frac{(\cos\theta - 1)}{\theta}\widehat{\eta} + \frac{(\theta - \sin\theta)}{\theta}\widehat{\eta}^2 \\ \Rightarrow H(R)\frac{d}{dt}(\log R)^\vee &= (R^\top \dot{R})^\vee. \end{aligned}$$

On $\mathcal{D}$, the inverse of $H(R)$ is given by

$$H^{-1}(R) = I + \frac{\theta}{2}\widehat{\eta} + \left(1 - \frac{\theta(1+\cos\theta)}{2\sin\theta}\right)\widehat{\eta}^2$$

We note the following:

$$\begin{aligned} H(R)(\log R)^\vee &= (\log R)^\vee, \\ H^{-1}(R)(\log R)^\vee &= (\log R)^\vee, \text{ and} \\ (H^{-1}(R))^\top(\log R)^\vee &= (\log R)^\vee. \end{aligned} \tag{1}$$

These follow from the fact that $(\log R)^\vee = \theta\eta$ and $\widehat{v}v = 0\ \forall\ v \in \mathbb{R}^3$.

In the synchronization problem, the individual rigid bodies are termed agents. The information flow between the agents is represented by a directed graph $\mathcal{G}$. An agent j is said to be a neighbour of agent i if $\mathcal{G}$ contains an edge from i to j. $\mathcal{G}$ may be represented by a matrix A, with $a_{ii} = 0$ and $a_{ij} > 0$ iff j is a neighbour of i. The digraph is undirected if A is symmetric, and balanced if

$$\sum_{j=1}^{N} a_{ij} = \sum_{j=1}^{N} a_{ji} \quad \forall\, i.$$

The graph Laplacian $L(\mathcal{G})$ is defined as $L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G})$, where $D(\mathcal{G})$ is the diagonal out-degree matrix defined as $d_{ii} = \sum_{j=1}^{N} a_{ij}$. $L(\mathcal{G})$ has atleast one eigenvalue at zero. Further, it has only one zero eigenvalue iff $\mathcal{G}$ contains a directed spanning tree.

III. Exponential attitude stabilization

We consider the attitude dynamics of a rigid body governed by the Euler–Poincaré equations

$$\dot{R} = R\widehat{\omega} \tag{2}$$

$$J\dot{\omega} = J\omega \times \omega + \tau, \tag{3}$$

where $R \in SO(3)$ represents the orientation of the body with respect to a fixed frame, $\omega \in \mathbb{R}^3$ is the body angular velocity, $J = J^\top \in \mathbb{R}^{3\times 3}$ is the positive definite inertia matrix and $\tau \in \mathbb{R}^3$ is the external torque.

Here, the control objective is to regulate the attitude $R(t)$ to a desired constant attitude R_d. Without loss of generality, we assume $R_d = I$. To this effect, we first propose a local smooth kinematic control law through the following proposition.

Proposition 1. *The control law*

$$\widehat{\omega} = -k_1 \log R,\ k_1 > 0. \tag{4}$$

locally exponentially stabilizes the kinematics (2) *to* $R = I$*. Moreover, the domain-of-attraction is given by* $\mathcal{D}$

Proof. In the domain $\mathcal{D}$, from equations (1) and (4),

$$\begin{aligned} \frac{d}{dt}(\log R)^\vee &= H^{-1}(R)\omega = -k_1 H^{-1}(R)(\log R)^\vee \\ &= -k_1(\log R)^\vee. \end{aligned}$$

Consider the candidate Lyapunov function $V_{\text{kin}} : \mathcal{D} \to \mathbb{R}$ defined by

$$V_{\text{kin}} = \frac{1}{2}\mathrm{Tr}\left((\log R)^\top(\log R)\right) = \frac{1}{2}\|\log R\|_F^2 = \theta^2. \tag{5}$$

Note that $V_{\text{kin}} = 0 \Rightarrow R = I$ and $V > 0$ for $R \neq I$.

Differentiating (5) along the trajectories of (2),

$$\begin{aligned}\dot{V}_{\text{kin}} &= \text{Tr}\left((\log R)^\top \frac{d}{dt}(\log R)\right)\\ &= \text{Tr}\left((\log R)^\top(-k_1\ \log R)\right)\\ &= -k_1\ \text{Tr}\left((\log R)^\top(\log R)\right)\\ &= -2\,k_1\ V_{\text{kin}} < 0\ \forall\ R \in \mathcal{D}\setminus\{I\}.\end{aligned}$$

The set $\mathcal{D}$ qualifies as the domain-of-attraction of the equilibrium $R = I$ since it is open, connected and positively invariant.

□

The closed-loop kinematic equation with control (4) is

$$\dot{R} = -k_1 R \log R. \tag{6}$$

We now show that this system has a closed form solution.

Proposition 2. *The solution of the system* (6) *for initial conditions such that* $R(0) \in \mathcal{D}$ *is given by*

$$\log R(t) = e^{-k_1 t} \log R(0), \tag{7}$$

or equivalently,

$$R(t) = \exp(e^{-k_1 t} \log R(0)).$$

Proof. The proof is by differentiation of (7) with respect to time,

$$\begin{aligned}\frac{d}{dt}\log R(t) &= -k_1 e^{-k_1 t}\log R(0)\\ &= -k_1 \log R(t)\\ &= R(t)^\top \dot{R}(t).\end{aligned}$$

Hence the claim holds. □

We now extend (4) to the boundary

$$\partial\mathcal{D} = \left\{R \in SO\,(3)\ \mid\ \frac{1}{\sqrt{2}}\|\log R\|_F = \pi\right\}.$$

For $\theta = \pi$, we define the control to be

$$\widehat{\omega} = -k_1\pi\widehat{\eta},\ k_1 > 0. \tag{8}$$

The closed-loop kinematic system (2) with the control laws (4) and (8) yield a discontinuous system on $SO\,(3)$

$$\dot{R} = f(R) = \begin{cases}-k_1\theta R\,\widehat{\eta}, & \theta \in (-\pi,\pi)\\ -k_1\pi R\,\widehat{\eta}, & \text{otherwise.}\end{cases}$$

We resort to the Filippov notion [22] of solution corresponding to the differential inclusion

$$\dot{R} \in F[f](R) = \begin{cases}\{-k_1 R\theta\widehat{\eta}\}, & \theta \in (-\pi,\pi)\\ \{-ak_1 R\pi\widehat{\eta}\mid a \in [-1,1]\}, & \theta = \pi.\end{cases} \tag{9}$$

Since $0 \in F[f](\exp(\pi\widehat{\eta}))$, every element $R = \exp(\pi\widehat{\eta})$ of $\partial\mathcal{D}$ is an equilibrium point of (9). However, each of these equilibria is unstable, and further, every other trajectory of (9) originating in $\partial\mathcal{D}$ enters $\mathcal{D}$.

This analysis points to the lack of robustness of the discontinuous control law, since it is possible that measurement errors may lead to the closed-loop trajectories being trapped in an open set around $\partial\mathcal{D}$. This phenomenon has been analyzed in a quaternion setting in [8, Theorem 2.1], which also presents a hybrid strategy to tackle the non-robustness.

We next extract the torque control law for (3) through the following proposition.

Proposition 3. *The control law*

$$\begin{aligned}\tau &= \omega\times J\omega - J(k_1 H^{-1}(R) + k_2 I)\omega\\ &\quad - (1 + k_1k_2)J(\log R)^\vee,\ k_1,k_2 > 0\end{aligned} \tag{10}$$

locally exponentially stabilizes (3) *to* $R = I$ *and* $\omega = 0$.

Proof. Consider the candidate Lyapunov function $V_{\text{dyn}} : \mathcal{D}\times I\!R^3 \to I\!R$ defined by

$$V_{\text{dyn}} = \frac{1}{2}V_{\text{kin}} + \frac{1}{2}\left(\omega + k_1(\log R)^\vee\right)^\top\left(\omega + k_1(\log R)^\vee\right) \tag{11}$$

with V_{kin} defined in (5).

$V_{\text{dyn}} = 0\ \Rightarrow\ R = I,\ \omega = 0$ and $V > 0$ for $R \neq I$ or $\omega \neq 0$.

We differentiate (11) with respect to time, using (1) to simplify the expression.

$$\begin{aligned}\dot{V}_{\text{dyn}} =& \frac{1}{2}\text{Tr}\left((\log R)^\top\frac{d}{dt}(\log R)\right)\\ &+ \left(\omega + k_1(\log R)^\vee\right)^\top\left(\dot{\omega} + k_1 H^{-1}(R)(R^\top\dot{R})^\vee\right)\\ =& \left(((\log R)^\vee)^\top H^{-1}(R)\omega\right) + \left(\omega + k_1(\log R)^\vee\right)^\top\\ &\left(J^{-1}(J\omega\times\omega + \tau) + k_1 H^{-1}(R)\omega\right)\\ =& \left((\log R)^\vee\right)^\top H^{-1}(R)\omega + \left(\omega + k_1(\log R)^\vee\right)^\top\\ &\left(-(k_1 H^{-1}(R) + k_2 I)\omega - (1 + k_1k_2)(\log R)^\vee\right.\\ &\left.+\ k_1 H^{-1}(R)\omega\right)\\ =& -k_2\left(\omega + k_1(\log R)^\vee\right)^\top\left(\omega + k_1(\log R)^\vee\right)\\ &- k_1\left((\log R)^\vee\right)^\top\left((\log R)^\vee\right)\\ \le& -2\,k_m\ V_{\text{dyn}} < 0\ \forall\ (R,\omega) \in (\mathcal{D}\times I\!R^3)\setminus\{(I,0)\}\end{aligned}$$

where $k_m = \min(k_1, k_2)$.

□

With the torque control (10), the closed-loop dynamics are described by

$$\begin{aligned}\dot{R} &= R\widehat{\omega}\\ \dot{\omega} &= -(k_1 H^{-1}(R) + k_2 I)\omega - (1 + k_1k_2)(\log R)^\vee.\end{aligned} \tag{12}$$

Since $SO\,(3) = \mathcal{D}\cup\partial\mathcal{D}$, and $\partial\mathcal{D}$ is an unstable submanifold of dimension two, the region of attraction of $R = I$ is almost global.

Remark 1. In [4], the logarithm map is used for feedback linearization of the rigid body dynamics. This allows for use of exponentially stabilizing linear controls. In contrast, we have an exponentially stabilizing control without feedback linearization.

IV. Attitude Synchronization

Consider a group of N identical rigid bodies (hereafter termed as agents) with orientations $R_i \in SO(3)$ and angular velocities $\omega_i \in I\!R^3$ whose dynamics are governed by

$$\dot{R}_i = R_i \widehat{\omega}_i \tag{13}$$
$$J\dot{\omega}_i = J\omega_i \times \omega_i + \tau_i. \tag{14}$$

The group of agents is said to synchronize or reach a consensus if the states of each agent agree on a common value. We consider two problems. In the first problem, termed as *attitude synchronization*, we propose a consensus protocol for (13), which achieves

$$\lim_{t\to\infty} R_i(t) - R_j(t) = 0, \ \forall\, i, j \tag{15}$$

In the second problem, termed as *angular velocity synchronization*, we consider both (13) and (14), wherein we present a consensus protocol which achieves (15) and

$$\lim_{t\to\infty} \omega_i(t) - \omega_j(t) = 0 \ \forall\, i, j.$$

We consider N agents in $SO(3)$, each governed by (13), and present an attitude synchronization protocol that locally exponentially stabilizes to the consensus value.

Proposition 4. *Let the information flow between the individual controllers be represented by a digraph $\mathcal{G}$ with weighted adjacency matrix $A(\mathcal{G}) = [a_{ij}]$. For every $R_i(0) \in \mathcal{D}$ and $\mathcal{G}$ containing a directed spanning tree, the control law*

$$\omega_i = \sum_{j=1}^{N} a_{ij} \left(H(R_i)(\log R_j)^\vee - (\log R_i)^\vee \right) \tag{16}$$

locally exponentially synchronizes the attitudes of the N agents.

Proof. For $R_i \in \mathcal{D}$,

$$H(R_i)\frac{d}{dt}(\log R_i)^\vee = (R_i^\top \dot{R}_i)^\vee = \omega_i.$$

With ω_i defined by (16), using (1),

$$H(R_i)\frac{d}{dt}(\log R_i)^\vee = \sum_{j=1}^{N} a_{ij} \left(H(R_i)\log R_j - \log R_i \right)^\vee$$
$$= H(R_i)\sum_{j=1}^{N} a_{ij} \left(\log R_j - \log R_i \right)^\vee$$
$$\Rightarrow \frac{d}{dt}(\log R_i)^\vee = \sum_{j=1}^{N} a_{ij} \left(\log R_j - \log R_i \right)^\vee$$

since $H(R_i)$ is invertible in $\mathcal{D}$.

Let $p_i \triangleq (\log R_i)^\vee$, and

$$p \triangleq \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{pmatrix}.$$

Then, we have

$$\dot{p}_i = \sum_{j=1}^{N} a_{ij} \left(p_j - p_i \right) \tag{17}$$

which may be expressed as

$$\dot{p} = -(L \otimes I_3)p,$$

where L is the graph Laplacian. Since the image of $\mathcal{D}$ under the logarithm map is a convex subset of $I\!R^3$, we can apply the results from [10] and [11] to show that (17) does converge to a configuration such that $\log R_i = \log R_j \ \forall\, i, j$, which implies $R_i = R_j \ \forall\, i, j$. □

The control torques may be obtained through backstepping by following a procedure similar to that in Proposition 3. However, this will force the angular velocities to zero. Instead, we use the following approach to synchronize both orientations and angular velocities.

Proposition 5. *Let the information transfer between the individual controllers be represented by a digraph $\mathcal{G}$ with weighted adjacency matrix $A(\mathcal{G}) = [a_{ij}]$. If $\mathcal{G}$ contains a directed spanning tree and*

$$k > \max_{\mu_i \neq 0} \sqrt{\frac{2}{|\mu_i| \cos\left(\frac{\pi}{2} - \tan^{-1} \frac{-Re(\mu_i)}{Im(\mu_i)} \right)}},$$

where $\mu_i, i = 1, \ldots, N$ are the eigenvalues of $-L$. Then, for every $(R_i(0), \omega(0)) \in \mathcal{D} \times I\!R^3$, the control

$$\begin{aligned} \tau_i &= \omega_i \times J\omega_i + J\frac{d}{dt}\left(H(R_i)\right) H^{-1}(R_i)\omega_i \\ &+ JH(R_i)\sum_{i=1}^{N} a_{ij} \left[k\left(H^{-1}(R_j)\omega_j - H^{-1}(R_i)\omega_i \right) \right. \\ &+ \left. (\log R_j - \log R_i)^\vee \right] \end{aligned} \tag{18}$$

achieves local angular velocity synchronization on $SO(3) \times I\!R^3$, that is, $R_i = R_j$ and $\omega_i = \omega_j \,\forall\, i, j$.

Proof. Substituting τ_i in the second equation of (3) for the i^{th} body and simplifying using the properties (1), we get

$$\frac{d^2}{dt^2}(\log R_i)^\vee = \sum_{i=1}^{N} a_{ij} \left[k\left(\frac{d}{dt}(\log R_j)^\vee - \frac{d}{dt}(\log R_i)^\vee \right) + (\log R_j - \log R_i)^\vee \right].$$

Denoting

$$\frac{d}{dt}(\log R_i)^\vee = \dot{p}_i,$$

we obtain the double integrator model

$$\ddot{p}_i = \sum_{i=1}^{N} a_{ij} \left[k\left(\dot{p}_j - \dot{p}_i \right) + (p_j - p_i) \right].$$

Then, from [23, Theorem 4.2], the system achieves consensus under the assumptions on L. □

In [14], a similar control

$$\widehat{\omega}_i = \sum_{j=1}^{N} a_{ij} \log(R_i^\top R_j) \tag{19}$$

was used to achieve consensus to the Karcher mean on $SO(3)$. However, the two controls have different properties. The control (19) is only valid for undirected graphs, while (16) can be used for any directed graph with a spanning tree. Also, while (16) has a domain of convergence $\mathcal{D}$, (19) has, in general, a smaller region of convergence given by the geodesic ball

$$\mathcal{B}(I, \pi/2) = \left\{ R \in SO(3) \mid \frac{1}{\sqrt{2}} \|\log R\|_F < \pi/2 \right\}$$

due to the non-convexity of the Karcher mean. However, (16) is not derived from a metric. This can lead to large rotations even if the neighboring bodies have small relative orientations. For instance, if R_1 and R_2 are such that $\theta_1 = 3\pi/4$, $\theta_2 = -3\pi/4$ and $\eta_1 = \eta_2$, then (16) would lead to rotations of more than $\pi/2$, which is the Riemannian distance. The difference between the controllers stems from the fact that $[\log R_i, \log R_j] \neq 0$ for arbitrary R_i, R_j.

In the case of weakly connected balanced graphs, one can use the results in [10] to show that (17) converges to the value

$$\overline{R} = \exp\left(\frac{\sum_{i=1}^{N} \log R_i}{N} \right). \tag{20}$$

This value can be thought of as a minimizer of the function

$$\phi(\overline{R}) = \sum_{i=1}^{N} \|\log \overline{R} - \log R_i\|_F .$$

We observe that $d(R_i, R_j) = \|\log R_i - \log R_j\|_F$ is a metric on the open geodesic ball $\mathcal{B}(I, \pi/2)$. In [19], different notions of means on $SO(3)$ were defined in terms of metrics. Here, we can interpret $\overline{R}$ as a mean on $\mathcal{B}(I, \pi/2)$.

V. Simulations

A. Exponential stabilization

The system parameter used for simulation is $J = \mathrm{diag}(0.2, 0.5, 0.1)\ \mathrm{kgm}^2$. The control gains in (10) are chosen as $k_1 = k_2 = 1$. The time-response of the closed-loop system (3)-(10) with the initial condition $\eta(0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{6}}\right)$, $\omega(0) = (0.3, -0.2, 0.5)\mathrm{rad/s}$ and two different $\theta(0)$ are shown in Figures 1-2. The diagonal entries of R, (r_{11}, r_{22}, r_{33}) each converge to 1 in both cases, implying $R = I$.

B. Synchronization

The simulation for synchronization is performed with $N = 4$ agents. The components of ω_i, η_i are denoted as $\omega_i^j, \eta_i^j, j = 1, 2, 3$ for each $i \in \{1, \ldots, N\}$. The system parameter of the agents used for simulation is $J = \mathrm{diag}(0.2, 0.5, 0.1)\ \mathrm{kgm}^2$. The control gains in (18) are chosen as $k = 3$,

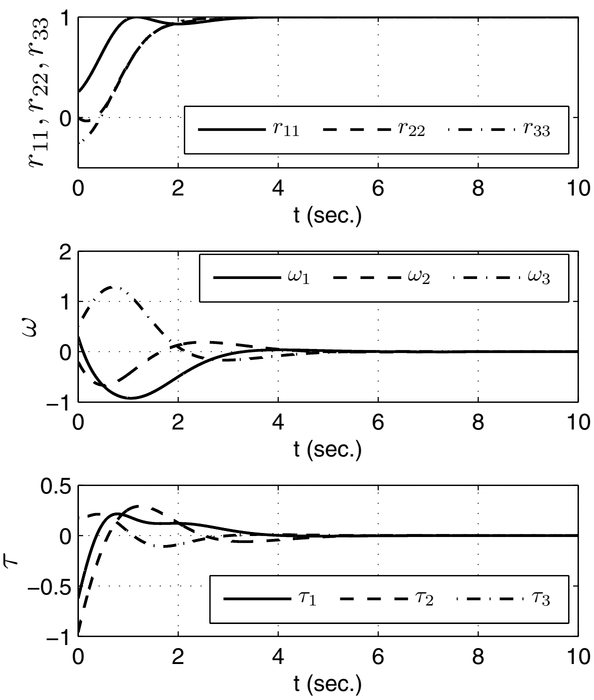

Fig. 1. Time-response of exponential stabilization for $\theta(0) = \frac{2\pi}{3}$

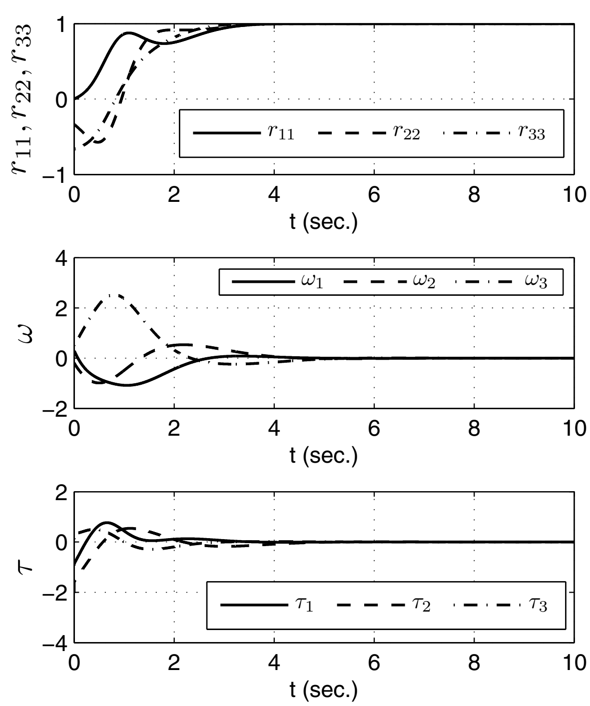

Fig. 2. Time-response of exponential stabilization for $\theta(0) = \pi$

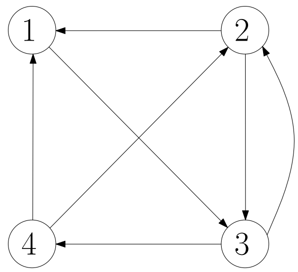

Fig. 3. Graph used in simulations

$A = \begin{bmatrix} 0 & 0 & 1.3 & 0 \\ 0.8 & 0 & 1.8 & 0 \\ 0 & 0.9 & 0 & 1.5 \\ 1.3 & 1.6 & 0 & 0 \end{bmatrix}$. The associated graph is shown in Figure 3. The initial conditions are $\theta_1(0) = \frac{\pi}{4}, \theta_2(0) = \frac{\pi}{6}, \theta_3(0) = \frac{\pi}{3}, \theta_4(0) = \frac{2\pi}{3}$,
$\eta_1(0) = (0, 0, 1), \eta_2(0) = \left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$,
$\eta_3(0) = \left(\frac{\sqrt{3}}{2}, \frac{-1}{2}, 0\right), \eta_4(0) = \left(\frac{-\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}, \frac{-1}{2}\right)$,
$\omega_1(0) = (-0.1, 0, 0)\mathrm{rad/s}, \omega_2(0) = (0, 0.2, 0.1)\mathrm{rad/s}$,
$\omega_3(0) = (-0.3, 0.2, 0)\mathrm{rad/s}, \omega_4(0) = (0.1, 0.1, -0.1)\mathrm{rad/s}$.
The time-response of the agents with control (18) is shown in Figure 4.

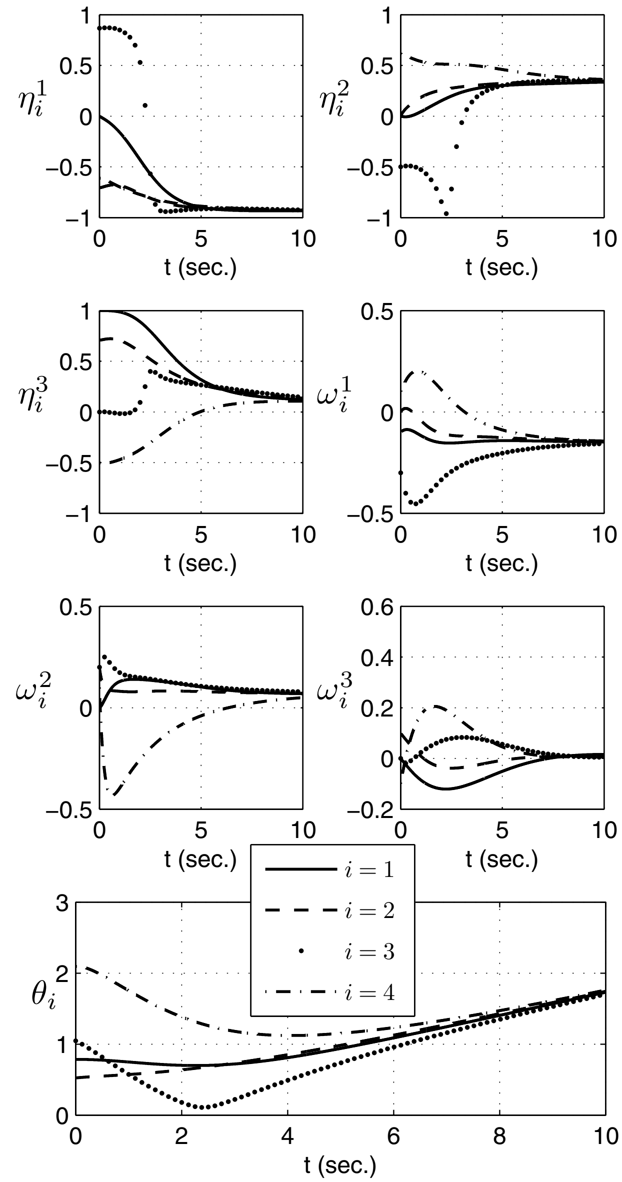

Fig. 4. Time-response of synchronization with 4 agents

VI. Conclusions

In this paper, we have addressed the rigid body attitude stabilization using the logarithm map. This has ensured local exponential stability with almost global domain of attraction for the desired orientation. The use of the logarithm map has also enabled the use of Euclidean consensus protocols for attitude and angular velocity synchronization.

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