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Hybrid State Observer for Time-Delay Systems under Intrinsic Impulsive Feedback

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***Abstract*— A hybrid static gain observer for systems described by a linear time-delay continuous part under impulsive feedback is suggested. The purpose of the observer is to asymptotically drive the state estimation error in the continuous states to zero and synchronize the sequence of modulated jump instants estimated by the observer with that of the plant. Conditions on the observer gain matrix to locally stabilize the observer error along an arbitrary periodic plant solution are obtained and the observer performance is illustrated by numerical simulations.**

***Index Terms*— hybrid systems, time-delay systems, state observer, impulsive systems**

***AMS Subj. Class.*—93C30, 93C57**

I. INTRODUCTION

Systems where continuous dynamics interact with discrete events in closed loop are ubiquitous in biology and medicine, [1]. A prominent example of such hybrid interaction comes from the field of neuroendocrinology [2]. The brain, especially the hypothalamus, controls the secretion of pituitary gland hormones and exhorts episodic (event-based) feedback action on the relatively slow dynamics of the hormone kinetics. The feedback mechanism implemented by the neurons can be modeled by frequency and amplitude modulation, as described e.g. in [3]. Then, general behavior of the neuroendocrine feedback system can be captured by linear time-invariant models with pulse modulation, [4], [5].

The presence of time delays in closed loop is an essential phenomenon in endocrinology. Time delays occur in endocrine systems mainly due to two circumstances. First, there is a delay due to the transport of the hormones in the blood stream from the secretion site to the site where the hormone molecules bind to the target receptors. Second, the time needed to synthesize the hormone before secretion when releasable pools of it are lacking also results in a a delay [3]. Time delays were introduced in numerous mathematical models of endocrine feedback [6], [7], [8], [9], [10], [11], [12] and pointed out as the main reason of sustained closed-loop oscillations in those systems, see e.g. [2]. However, the well-known pulsatile mechanism of non-basal regulation was not included in the models.

An impulsive model of testosterone regulation was proposed in [4] and analyzed in detail in [13]. Following the biological evidence, the model was based on the principles of pulse modulation [14]. It demonstrated a good agreement with clinical data [15], [16] and provided an explanation to the experimentally observed complex dynamical phenomena in endocrine systems, including deterministic chaos [17]. In fact, even without a time delay, the model lacks equilibria and appears to exhibit only periodical or chaotic behaviors. Yet, in order to improve the biological fidelity of the model, it was modified in [18], [19], [20] by introducing a time-delay into its continuous part. In further papers [21], [22], [23] these results were extended to the case of "a large delay", when the time delay can be greater than the time interval between two consecutive pulse modulation instants. Only "small delays" are considered in the present paper. Yet the theory developed for handling "large delays" comes in handy in the observer analysis below. It was also observed that the cascade structure of the continuous part, together with the impulsive feedback, allow for a significant simplification of the closed-loop hybrid dynamics by applying the notion of finite-dimensional (FD) reducibility.

Due to the intrinsic nature of the neuroendocrine feedback, the signal impacting the continuous part of the model (hormone kinetics) is not available for measurement and has to be estimated. A standard in endocrinology procedure for that is based on deconvolution cast in the form nonlinear least-squares optimization [24]. To incorporate the knowledge of the feedback law, a hybrid state observer for continuous oscillating systems under intrinsic pulse-modulated feedback was proposed in [5], without taking into account the time delay. The present paper extends the results of [5] to the case of a time delay in the continuous part of the plant and brings the observer closer to an application to biological data.

The paper is composed as follows. First the equations of the time-delay plant with a pulse modulated feedback portraying an endocrine system with non-basal regulation are summarized and a hybrid state observer for it is introduced. The existing theory is then generalized to the models whose continuous part is comprised by FD-reducible time-delay dynamics. A pointwise discrete mapping that governs the evolution of the observer state is derived and its properties are studied. Simulation results are provided to illustrate the performance of the proposed observer.

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II. SYSTEM EQUATIONS

Consider the impulsive time-delay model of non-basal endocrine regulation treated in [18], [19]:

$$\frac{dx(t)}{dt} = A_0x(t) + A_1x(t-\tau), \quad z(t) = Cx(t),$$
$$y(t) = Lx(t), \quad t_{n+1} = t_n + T_n, \quad x(t_n^+) = x(t_n^-) + \lambda_n B, \tag{1}$$
$$T_n = \Phi(z(t_n)), \quad \lambda_n = F(z(t_n)),$$

where $A_0 \in \mathbb{R}^{n_x \times n_x}$, $A_1 \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times 1}$, $C \in \mathbb{R}^{1 \times n_x}$, $L \in \mathbb{R}^{n_y \times n_x}$ are constant matrices, z is the scalar controlled output, y is the vector measurable output, x is the state vector (1), and τ is a constant time delay. Here $x(t_n^-)$, $x(t_n^+)$ are one-sided limits of $x(t)$ (left and right, correspondingly) at the time instant t_n.

System (1) is considered for $t \geqslant 0$ and subject to the initial condition $x(t) = \varphi(t)$, $-\tau \leqslant t < 0$, where $\varphi(t)$ is a continuous initial vector function. The matrix relationships

$$CB = 0, \quad LB = 0 \tag{2}$$

apply to (1) and are essential for further analysis. Suppose that the matrix A_0 is Hurwitz stable. Let the modulation functions of the feedback $\Phi(\cdot)$ and $F(\cdot)$ be continuous, strictly monotonic and bounded with strictly positive lower bounds. The latter condition implies that system (1) has no equilibria.

The states $x(t)$ of system (1) experience jumps at times $t = t_n$. However, because of the imposed conditions on the system matrices expressed by (2), the outputs $y(t)$, $z(t)$ are continuous.

Time delay values that are less than the minimal time interval between two consecutive pulse modulation instants are considered

$$\inf_z \Phi(z) > \tau,$$

so that $T_n > \tau$ for all n. Therefore, only "small delays" in the continuous part of (1), compared to the jump period of the pulse-modulated feedback, are treated below, cf. [21], [22], [23].

Henceforth, the continuous (linear) part of system (1) is assumed to be finite dimension (FD) reducible [18], [19], i.e.

$$A_1A_0^kA_1 = 0 \quad \text{for} \quad k = 0, 1, \ldots, n_x - 1. \tag{3}$$

Any solution $x(t)$ of an FD-reducible linear time-delay equation

$$\frac{dx(t)}{dt} = A_0x(t) + A_1x(t-\tau) \tag{4}$$

defined for $t \geqslant 0$ with some initial function $\varphi(t)$, $-\tau \leqslant t \leqslant 0$, satisfies a delay-free linear equation

$$\frac{dx(t)}{dt} = Dx(t)$$

for $t \geqslant \tau$, where $D = A_0 + A_1 e^{-A_0\tau}$ (see [18], [19], [20]). For an FD-reducible system, the eigenvalue spectrum of the matrix A_0 coincides with that of D and, thus, the spectrum of D is independent of τ, i.e.

$$\det(sI_{n_x} - A_0 - A_1 e^{-A_0\tau}) = \det(sI_{n_x} - A_0)$$

for all complex s and any τ. Additionally, time-delay linear system (4) has a finite (pole) spectrum, since

$$\det(sI_{n_x} - A_0 - A_1 e^{-\tau s}) = \det(sI_{n_x} - A_0),$$

for all complex s. However, FD-reducibility cannot be reduced to the finite spectrum property of system (4) but also involves the structure of the system matrices, as illustrated by the following example.

Example: Consider (4) with the system matrices

$$A_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{5}$$

Obviously, the characteristic polynomial $\det(sI_{n_x} - A_0 - A_1 e^{-\tau s})$ has a finite set of roots, namely $\{-1, -1, -1\}$. However,

$$A_1A_0^kA_1 = (-1)^k A_1^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (-1)^k & 0 & 0 \end{bmatrix},$$

for any integer k, so system (4) is not FD-reducible.

The property of FD-reducibility can be characterized as follows.

Lemma 1 ([20]): System (4) is FD-reducible iff there exists an invertible $n_x \times n_x$ matrix S such that

$$S^{-1}A_0S = \begin{bmatrix} U & 0 \\ W & V \end{bmatrix}, \quad S^{-1}A_1S = \begin{bmatrix} 0 & 0 \\ \bar{W} & 0 \end{bmatrix}, \tag{6}$$

where the blocks U, V are square, the blocks W, $\bar{W}$ are of the same dimension.

In the matrices of (5), a partition described by (6) cannot be achieved since the position of the nonzero elements of A_1 do not allow for two square blocks (U,V) on the main diagonal of A_0.

Introduce a new notion of observability that is more restrictive than conventional spectral observability (see, e.g., [25], [26]) and requires, in addition, FD-reducibility .

Definition 1: The linear part of system (1) will be called spectrally FD-observable if for any given complex self-conjugate set of numbers $\Lambda = \{\lambda_j, \quad j = 1, \ldots, n_x\}$ there exists a matrix K such that the eigenvalue spectrum of $A_0 - KL$ coincides with Λ, and, moreover,

$$A_1(A_0 - KL)^k A_1 = 0 \quad \text{for} \quad k = 0, 1, \ldots, n_x - 1. \tag{7}$$

In other words, (7) parallels condition (3) but for A_0 replaced with $A_0 - KL$. Relationships (7) imply that

$$\det(sI_{n_x} - D + KL) = \det(sI_{n_x} - A_0 + KL)$$

for all complex s. Further on, this property is supposed to hold with respect to the continuous part of (1).

Notice that since FD-reducibility is a more limiting property than finite spectrum, spectral FD-observability cannot be reduced to finite spectrum assignability (see, e.g. [27], [28], [29], [30]) that is guaranteed by spectral controllability or observability, depending on the considered design problem. Besides, the notion of FD-reducibility is applicable as defined to only autonomous systems (4) while spectral controllability (observability) covers input-to-state (state-to-output) properties. Sufficient conditions for spectral FD-observability are given by the following lemma.

Lemma 2: For system (4) represented in the block form of (6), write L in terms of the matrix blocks

$$L = [L_1 \quad L_2],$$

where the sizes of the blocks L_1, L_2 correspond to those of the blocks in (6). Let the matrix pair (L_2, V) be observable and assume that there exists a matrix L_0 such that $L_0 L_2 = 0$ and the matrix pair $(L_1 L_0, U)$ is also observable. Then system (4) is spectrally FD-observable.

Proof: Consider a matrix

$$K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}, \quad K_1 = \tilde{K}_1 L_0,$$

where matrices $\tilde{K}_1$, K_2 are to be determined. By the choice of L_0, one has $K_1 L_2 = 0$ and

$$A_0 - KL = \begin{bmatrix} U - \tilde{K}_1 L_0 L_1 & 0 \\ W - K_2 L_1 & V - K_2 L_2 \end{bmatrix}.$$

From Lemma 1, it follows that (7) is valid for any $\tilde{K}_1$, K_2. By the observability assumptions, arbitrary eigenvalues can be assigned to the matrices $U - \tilde{K}_1 L_0 L_1$, $V - K_2 L_2$ by the choice of $\tilde{K}_1$, K_2. ■

The purpose of observation in hybrid closed-loop system (1) is to produce estimates $(\hat{t}_n, \hat{\lambda}_n)$ of the pulse modulation parameters (t_n, λ_n). Obviously, given the sequence $(t_n, \lambda_n), n = 0, \ldots, \infty$, estimates of the state vector x of the continuous part can be obtained by any of the conventional state estimation techniques.

Notice that the results of [5] cannot be directly applied to the present case, even under the assumption of the linear part of (1) being FD-reducible. Although the time-delay linear plant can indeed be reduced to a finite dimensional one, for the time-delay hybrid system such a reduction is only valid on certain time intervals. Thus, an observer for the infinite dimensional system given by (1) requires special consideration.

In order to estimate the state vector of (1), a hybrid observer is formulated as:

$$\begin{gathered} \frac{d\hat{x}}{d\hat{t}} = A_0 \hat{x}(t) + A_1 \hat{x}(t-\tau) + K(y(t) - \hat{y}(t)), \\ \hat{y}(t) = L\hat{x}(t), \quad \hat{z}(t) = C\hat{x}(t), \quad \hat{x}(\hat{t}_n^+) = \hat{x}(\hat{t}_n^-) + \hat{\lambda}_n B, \\ \hat{t}_{n+1} = \hat{t}_n + \hat{T}_n, \quad \hat{T}_n = \Phi(\hat{z}(\hat{t}_n)), \quad \hat{\lambda}_n = F(\hat{z}(\hat{t}_n)). \end{gathered} \tag{8}$$

With $\tau = 0$, the observer above is equivalent to the one treated in [5] .

Introduce $D_K = A_0 - KL$, where K is a static feedback gain. If the linear part of (1) is spectrally FD-observable, the matrix K can be chosen in such a way that the matrix D_K is Hurwitz stable and fulfills (7).

III. POINTWISE MAPPING AND ITS PROPERTIES

Consider the pointwise mapping describing the evolution of the observer (hybrid) state:

$$(\hat{x}(\hat{t}_n^-),\, \hat{t}_n) \mapsto (\hat{x}(\hat{t}_{n+1}^-),\, \hat{t}_{n+1}). \tag{9}$$

For any integer numbers k and s, $0 \leqslant k \leqslant s$, define the set

$$\begin{aligned} S_{k,s} = \{(\zeta, \theta) \,:\, \theta \in \mathbb{R}, \quad \zeta \in \mathbb{R}^{n_x}, \quad t_k \leqslant \theta < t_{k+1}, \\ t_s \leqslant \theta + \Phi(C\zeta) < t_{s+1}\}. \end{aligned}$$

Denote

$$G(\theta) = \begin{cases} e^{A_0 \theta}, & 0 \leqslant \theta \leqslant \tau, \\ e^{D(\theta - \tau)} e^{A_0 \tau}, & \tau \leqslant \theta, \end{cases}$$

$$\tilde{G}(\theta) = \begin{cases} e^{D_K \theta}, & 0 \leqslant \theta \leqslant \tau, \\ e^{\tilde{D}_K(\theta - \tau)} e^{D_K \tau}, & \tau \leqslant \theta, \end{cases}$$

and

$$\tilde{R}(\theta_1, \theta_2) = \begin{cases} \mathrm{e}^{\tilde{D}_K \theta_2} \left[\mathrm{e}^{\tilde{D}_K(\theta_1 - \tau)} \mathrm{e}^{D_K \tau} - \mathrm{e}^{D_K \theta_1} \right], & \\ & 0 \leqslant \theta_1 \leqslant \tau, \\ 0, & \tau \leqslant \theta_1, \end{cases}$$

where $\tilde{D}_K = D_K + A_1 \mathrm{e}^{-D_K \tau}$.

Define $P(\zeta, \theta) = P_{k,s}(\zeta, \theta)$ at $(\zeta, \theta) \in S_{k,s}$, with

$$\begin{aligned} & P_{k,s}(\zeta, \theta) \\ & \quad = \mathrm{e}^{D(\theta + \Phi(C\zeta) - t_s)} x(t_s^-) - \mathrm{e}^{\tilde{D}_K \Phi(C\zeta)} \left(\mathrm{e}^{D(\theta - t_k)} x(t_k^-) - \zeta \right) \\ & \qquad - \lambda_k \left(\mathrm{e}^{\tilde{D}_K \Phi(C\zeta)} G(\theta - t_k) + \tilde{R}(\theta - t_k, \Phi(C\zeta)) \right) B \\ & \qquad\qquad + F(C\zeta)\, \tilde{G}(\Phi(C\zeta)) B \\ & - \sum_{j=k+1}^{s} \lambda_j \tilde{G}(\theta + \Phi(C\zeta) - t_j) B + \lambda_s G(\theta + \Phi(C\zeta) - t_s) B. \end{aligned}$$

Apply the for brevity the shorthand notation $x_k = x(t_k^-)$, $\hat{x}_n = \hat{x}(\hat{t}_n^-)$. The mapping defined in the proposition below completely describes the evolution of the observer (hybrid) state (i.e (9)) from one jump time in the intrinsic feedback to another, thus reducing the hybrid dynamics of (1) to a discrete (asynchronous) system.

Theorem 1: Pointwise mapping (9) is given by the equations

$$\hat{x}_{n+1} = P(\hat{x}_n, \hat{t}_n), \quad \hat{t}_{n+1} = \hat{t}_n + \Phi(C\hat{x}_n). \tag{10}$$

Proof: Omitted.

Theorem 2: The mapping $P(\zeta, \theta)$ is continuous.

Proof: See Appendix A.

Theorem 3: If the scalar functions $F(\cdot)$, $\Phi(\cdot)$ have continuous derivatives, then the partial derivatives

$$P'_\zeta = \frac{\partial P}{\partial \zeta}, \quad P'_\theta = \frac{\partial P}{\partial \theta}$$

are continuous everywhere.

Proof: See Appendix B.

Notice that the property proved by Thorem 3 is crucial for the observer design and does not generally apply. For instance, as demonstrated in [31], [32], for the time-delay-free and minimal-dimension case (i.e. $\tau = 0$ and $n_x = 1$), (2) is not satisfied and, thus, continuity of the mapping is not guaranteed. The latter phenomenon results in conditional stability of the observer, i.e. the state estimation error converges for some observer initial conditions but not for other.

Introduce additional notation referring to mapping (9). Define a function

$$Q_{k,s}(q) = \begin{bmatrix} P_{k,s}(\zeta,\theta) \\ \theta + \Phi(C\zeta) \end{bmatrix}, \quad \text{where} \quad q = \begin{bmatrix} \zeta \\ \theta \end{bmatrix}.$$

Set $Q(q) = Q_{k,s}(q)$ for $t_k \leqslant \theta < t_{k+1}$, $t_s \leqslant \theta + \Phi(C\zeta) < t_{s+1}$. Then $\hat{q}_{n+1} = Q(\hat{q}_n)$, where

$$\hat{q}_n = \begin{bmatrix} \hat{x}_n \\ \hat{t}_n \end{bmatrix}, \quad Q(q) = \begin{bmatrix} P(\zeta,\theta) \\ \theta + \Phi(C\zeta) \end{bmatrix}.$$

Iterations of the operator Q will be also considered and defined as

$$Q^{(m)}(q) = \underbrace{Q(Q(\ldots(Q(}_{m}q))\ldots)).$$

According to the definition, P'_ζ is a $n_x \times n_x$-matrix, and P'_θ is a n_x-dimensional column. Then the Jacobian matrix of $Q(q)$ is calculated as

$$Q'(q) = \begin{bmatrix} P'_\zeta(\zeta,\theta) & P'_\theta(\zeta,\theta) \\ \Phi'(C\zeta)C & 1 \end{bmatrix}.$$

By the chain rule, the Jacobian matrix of the m-th iteration of the mapping is given by the expression

$$\begin{aligned} \left(Q^{(m)}\right)'(q) = Q'\left(Q^{(m-1)}(q)\right) Q'\left(Q^{(m-2)}(q)\right) \times \cdots \\ \times Q'\left(Q(q)\right) Q'(q). \end{aligned} \quad (11)$$

IV. SYNCHRONOUS MODE

Let $(x(t), t_n)$ be a solution of plant equations (1) with the parameters λ_k, T_k, and $x_k = x(t_k^-)$. Suppose that the plant is already running at the moment when the observer is initiated, i.e. $t_a \leqslant \hat{t}_0 < t_{a+1}$, for some $a \geqslant 1$.

Considering the solution $(\hat{x}(t), \hat{t}_n)$ of observer equations (8) subject to the initial conditions

$$\hat{t}_0 = t_a, \qquad \hat{x}(\hat{t}_0^-) = x(t_a^-),$$

yields

$$\hat{x}_n = x_{n+a}, \quad \hat{t}_n = t_{n+a}, \quad \hat{\lambda}_n = \lambda_{n+a}, \quad n = 0, 1, 2, \ldots,$$

and $\hat{x}(t) = x(t)$ for $t \geqslant t_a$. Such a solution $(\hat{x}(t), \hat{t}_n)$ will be called *a synchronous mode* of the observer with respect to $(x(t), t_n)$ (see [5] for a more detailed discussion).

A synchronous mode will be called *locally asymptotically stable* if for any solution $(\hat{x}(t), \hat{t}_n)$ of (8) such that the initial estimation errors $|\hat{t}_0 - t_a|$ and $\|\hat{x}(\hat{t}_0^-) - x(t_a^-)\|$ are sufficiently small, it follows that $\hat{t}_n - t_{n+a} \to 0$ and $\|\hat{x}(\hat{t}_n^-) - x(t_{n+a}^-)\| \to 0$ as $n \to \infty$. The latter implies $\hat{\lambda}_n - \lambda_{n+a} \to 0$ as $n \to \infty$.

For brevity, denote $n_a = n + a$, $\Phi'_k = \Phi'(Cx_k)$, $F'_k = F'(Cx_k)$. The synchronous mode with respect to $x(t)$ is completely characterized by the vector sequence

$$\hat{q}_n^0 = \begin{bmatrix} x_{n_a} \\ t_{n_a} \end{bmatrix}. \quad (12)$$

For for all $k \geqslant 0$, define matrices J_k comprised of the following matrix blocks

$$\begin{aligned} (J_k)_{11} &= \Phi'_k D x_{k+1} C + \mathrm{e}^{\tilde{D}_K T_k}\left(I_{n_x} + F'_k \mathrm{e}^{-\tilde{D}_K \tau} \mathrm{e}^{D_K \tau} BC\right), \\ (J_k)_{12} &= D x_{k+1} - \mathrm{e}^{\tilde{D}_K T_k}\left(D x_k + \lambda_k \tilde{D}_K \mathrm{e}^{-\tilde{D}_K \tau} \mathrm{e}^{D_K \tau} B\right), \\ (J_k)_{21} &= \Phi'_k C, \qquad (J_k)_{22} = 1. \end{aligned}$$

Recall the mapping $Q(q)$ introduced in the previous section.

Theorem 4: For any $n \geqslant 0$, the Jacobian of $Q(\cdot)$ at $\hat{q}_n^0$ is calculated as

$$Q'(\hat{q}_n^0) = J_{n+a}. \quad (13)$$

Proof: Omitted.

From (11) it follows that for any $m \geqslant 1$

$$\left(Q^{(m)}\right)'(\hat{q}_n^0) = J_{n_a+m-1} J_{n_a+m-2} \ldots J_{n_a+1} J_{n_a}. \quad (14)$$

V. LOCAL STABILITY OF A SYNCHRONOUS MODE WITH RESPECT TO AN *M*-CYCLE

A solution of (1) is called m-cycle if it is periodic with exactly m pulse modulation instants in the least period. The existence conditions of an m-cycle in pulse-modulated time-delay system (1) were studied in [18], [19].

Let $(x(t), t_n)$ be an m-cycle of plant (1), where m is some integer, $m \geqslant 1$. Then $x_{n+m} \equiv x_n$, $\lambda_{n+m} \equiv \lambda_n$, $T_{n+m} \equiv T_n$. Consider a synchronous mode of observer (8) with respect to $(x(t), t_n)$ and let $\hat{q}_n^0$ be the corresponding vector sequence as in (12), such that $\hat{q}_{n+1}^0 = Q(\hat{q}_n^0)$ is satisfied.

In order to ensure feasibility of the observer, stability properties of the synchronous mode have to be investigated and guaranteed by design. Consider previously defined matrices J_n. Then $J_{n+m} \equiv J_n$, so that the sequence $\{J_n\}_{n=0}^\infty$ contains no more than m distinct matrices, namely $J_0, \ldots, J_{m-1}$.

Theorem 5: Let the matrix product $J_0 \ldots J_{m-1}$ be Schur stable, i.e. all the eigenvalues of this matrix lie strictly inside the unit circle. Then the synchronous mode with respect to $(x(t), t_n)$ is locally asymptotically stable.

Proof: Along the lines of Theorem 3 in [5].

Theorem 6: Suppose that the linear part of system (1) is spectrally FD-observable. Let $(x(t), t_n)$ be an m-cycle and the notation of Section IV apply. Suppose

$$-1 < \prod_{k=0}^{m-1} (\Phi'_k C D x_{k+2} + 1) < 1. \quad (15)$$

Then there is always a matrix K such that D_K is Hurwitz and the matrix product $J_0 \ldots J_{m-1}$ is Schur stable.

Proof: Along the lines of Theorem 4 in [5].

Clearly, the stability properties of a synchronous mode depend on the parameters of the observed m-cycle in the plant.

Notice that in (15) $x_m = x_0$ and $x_{m+1} = x_1$. In the case of a 1-cycle, (15) turns into

$$-2 < \Phi'_0 C D x_0 < 0$$

and in the case of a 2-cycle (15) becomes

$$-1 < (\Phi'_0 C D x_0 + 1)(\Phi'_1 C D x_1 + 1) < 1. \quad (16)$$

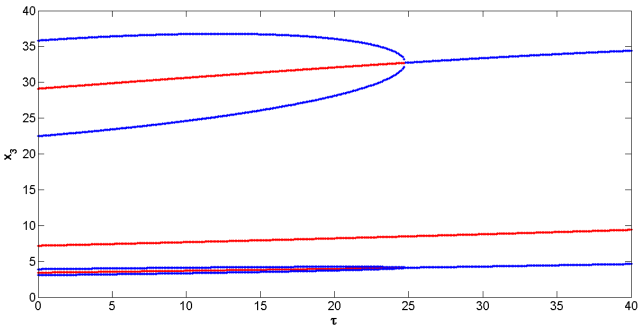

Fig. 1. Bifurcation diagram of hybrid plant (1) with respect to τ. The values of x_3 at the feedback jump times t_n are depicted as function of τ. Blue dots — stable solutions, red dots — unstable solutions.

VI. NUMERICAL EXAMPLES

Assume the following values in model (1)

$$A_0 = \begin{bmatrix} -b_1 & 0 & 0 \\ g_1 & -b_2 & 0 \\ 0 & 0 & -b_3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & g_2 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

To demonstrate that in this case the linear part of the system is spectrally FD-observable, Lemma 2 is applied. One has

$$U = \begin{bmatrix} -b_1 & 0 \\ g_1 & -b_2 \end{bmatrix}, \quad V = [-b_3], \quad W = [0 \quad 0],$$

$$\bar{W} = [0 \quad g_2], \quad L_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Choose $L_0 = [1 \quad 0]$. Then $L_0 L_1 = 0$ and the matrix pair $(L_0 L_1, U)$ is observable, provided that $g_1 \neq 0$. The matrix pair (L_2, V) is evidently observable. Hence the assumptions of Lemma 2 are fulfilled.

Let $h = 2.7$, $b_1 = 0.02$, $b_2 = 0.15$, $b_3 = 0.1$, $g_1 = 0.6$, $g_2 = 1.5$, and

$$\Phi(z) = 40 + 80\frac{(z/h)^2}{1+(z/h)^2}, \quad F(z) = 0.05 + \frac{5}{1+(z/h)^2}.$$

Since $\inf_z \Phi(z) = 40$, then $0 \leqslant \tau < 40$. As shown in Fig. 1, the plant has a stable 4-cycle for $0 \leqslant \tau \leqslant 24.7$ and a stable 2-cycle for $24.8 \leqslant \tau < 40$. Interestingly, the dynamics of the closed-loop system described by (1) are simplified in this case with a higher time-delay value. This is the opposite to what usually happens in a closed-loop dynamical system with an increase of the time delay.

Let $L_m = \prod_{k=0}^{m-1} (\Phi'_k CDx_{k+2} + 1)$. Note that for 2-cycle $L_4 = (L_2)^2$and, hence, $|L_4| < 1$ implies $|L_2| < 1$. Numerical calculations show, see Fig. 2, that condition (15) is fulfilled. Hence, the existence of a stabilizing observer gain K is guaranteed for the interval $0 \leqslant \tau < 40$.

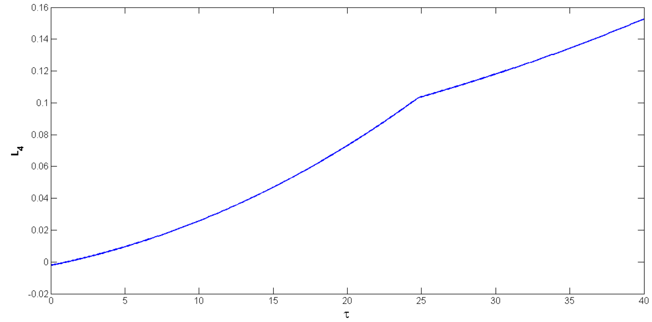

Fig. 2. The dependence of L_4 on τ.

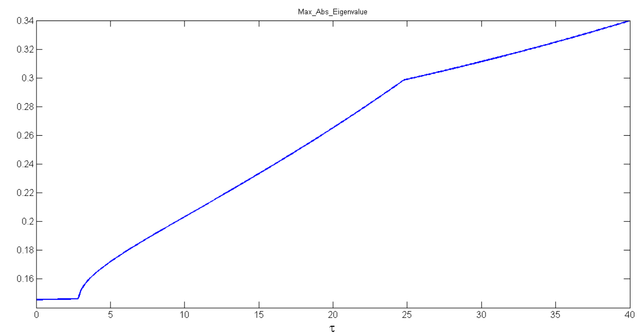

Fig. 3. The dependence of the spectral radius of the product $J_0 J_1 J_2 J_3$ on τ. Notice that for $24.8 \leqslant \tau < 40$, a 4-cycle is reduced to a 2-cycle ($m = 2$) and $J_2 = J_0$, $J_3 = J_1$.

Choose the observer feedback gain as

$$K = \begin{bmatrix} 0 & 0 \\ 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

Then D_K is Hurwitz stable, $(A_0 - KL, A_1)$ is FD-reducible and the synchronous mode is locally asymptotically stable as can be easily checked in Fig. 3.

A. Observation of a 4-cycle

Let $\tau = 20$. Then, the plant has a stable 4-cycle with

$$x_0^\top = \begin{bmatrix} 0.0290 & 0.1337 & 3.7387 \end{bmatrix}^\top,$$

$$x_1^\top = \begin{bmatrix} 0.2815 & 1.2994 & 36.0704 \end{bmatrix}^\top,$$

$$x_3^\top = \begin{bmatrix} 0.0329 & 0.1518 & 4.2459 \end{bmatrix}^\top,$$

$$x_4^\top = \begin{bmatrix} 0.2190 & 1.0106 & 28.1169 \end{bmatrix}^\top,$$

where $(\cdot)\top$ denotes transpose.

Fig. 4 illustrates the transients in the sequence $\hat{\lambda}_n$ produced by the observer relative to the sequence λ_n of the plant, caused by a mismatch in the initial conditions of the plant and those of the observer.

The red vertical lines of height $-\hat{\lambda}_n$ positioned at $\hat{t}_n$ correspond to the observer jump sequence. The pulse modulation of the plant is shown by blue lines of height λ_n positioned at t_n. It can be seen that the jump instants of

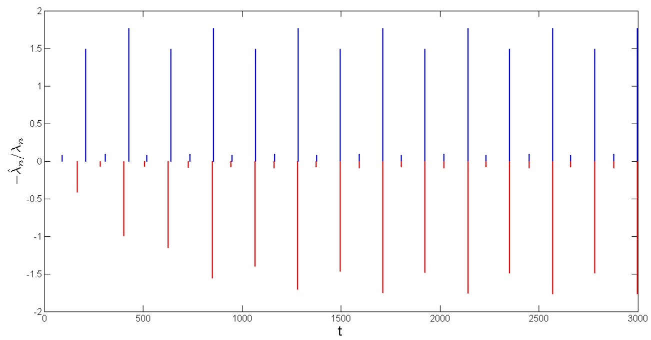

Fig. 4. Transients due to non-zero initial conditions in the times and magnitudes of jumps of the plant and the observer ($\tau = 20$): Blue lines (upper part of the figure) mark the jump times of the observer $\hat{t}_n$ with the height equal to $\hat{\lambda}_n$. Red lines (lower part of the figure) correspond to the pulse modulation of the plant in 4-cycle with the jump times t_n and the jumps $-\lambda_n$.

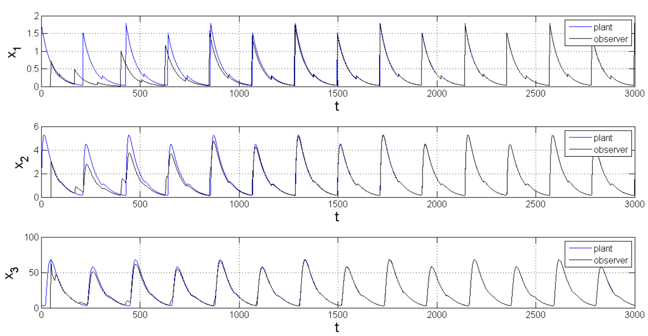

Fig. 5. Transients in the continuous components of the plant (blue lines) and the observer (black lines) for $\tau = 20$.

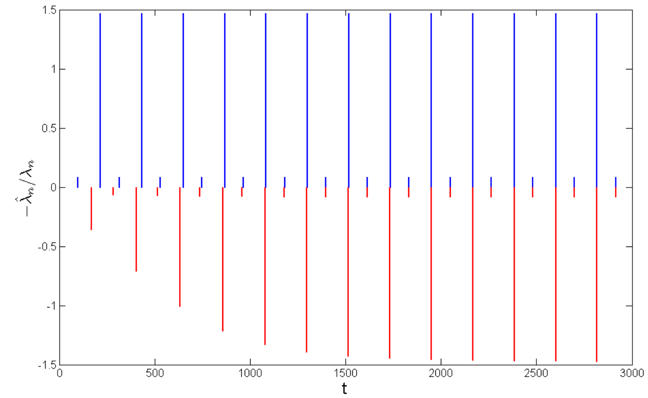

Fig. 6. Transients due to non-zero initial conditions in the times and magnitudes of jumps of the plant and the observer ($\tau = 30$).

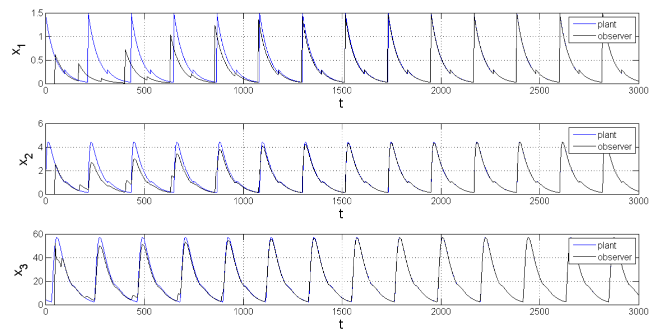

Fig. 7. Transients in the continuous components of the plant (blue lines) and the observer (black lines) for $\tau = 30$.

the observer become synchronized with those of the plant and the magnitudes of jumps $\hat{\lambda}_n$ asymptotically converge to λ_n.

The transients in the continuous states of the plant and the observer due to initial conditions mismatch are depicted in Fig. 5.

B. *Observation of a 2-cycle*

Let $\tau = 30$ yielding a stable 2-cycle with

$$x_0^{\mathsf{T}} = \begin{bmatrix}0.0272 & 0.1255 & 4.2853\end{bmatrix}^{\mathsf{T}},$$

$$x_1^{\mathsf{T}} = \begin{bmatrix}0.2141 & 0.9883 & 33.3766\end{bmatrix}^{\mathsf{T}}.$$

Fig. 6 illustrates the transients in the sequence $\hat{\lambda}_n$ relative to λ_n, caused by a mismatch between the initial conditions of the plant and those of the observer.

The transients in the continuous states of the plant and the observer due to initial conditions mismatch are depicted in Fig. 7.

VII. CONCLUSIONS

An earlier suggested observer structure is adopted for the state observation in hybrid impulsive systems with a time delay in the continuous part of the plant. By local stability analysis of the observer, it is shown that with a proper choice of the observer static gain and knowledge of the plant solution parameters, one can obtain an asymptotically converging estimate of the continuous system states and synchronization between the periodic pulse modulation sequence of the plant and that of the observer.

APPENDIX

A. *Proof of Theorem 2*

To prove the result of Theorem, the following property of the functions $P_{k,s}(\zeta,\theta)$ is used.

Lemma 3: The function $P_{k,s}(\zeta,\theta)$ can be represented as

$$P_{k,s}(\zeta,\theta) = u_k(\zeta,\theta) + v_s(\zeta,\theta) + w(\zeta), \qquad (17)$$

where

$$\begin{aligned} u_k(\zeta,\theta) &= \mathrm{e}^{\tilde{D}_K \Phi(C\zeta)} \left[-\mathrm{e}^{D(\theta - t_k)} x(t_k^-) - \lambda_k G(\theta - t_k) B \right] \\ &\quad - \lambda_k \tilde{R}(\theta - t_k, \Phi(C\zeta)) B + \sum_{j=1}^{k} \lambda_j \tilde{G}(\theta + \Phi(C\zeta) - t_j) B, \\ v_s(\zeta,\theta) &= \mathrm{e}^{D(\theta + \Phi(C\zeta) - t_s)} x(t_s^-) - \sum_{j=1}^{s} \lambda_j \tilde{G}(\theta + \Phi(C\zeta) - t_j) B \\ &\quad + \lambda_s G(\theta + \Phi(C\zeta) - t_s) B, \\ w(\zeta) &= \mathrm{e}^{\tilde{D}_K \Phi(C\zeta)} \zeta + F(C\zeta) \tilde{G}(\Phi(C\zeta)) B. \end{aligned}$$

Moreover, for $(\zeta,\theta)\in S_{k,s}$ the following recursions hold

$$\begin{aligned} u_k(\zeta,\theta)-u_{k-1}(\zeta,\theta)=\lambda_k\Big[&-\mathrm{e}^{\tilde D_K\Phi(C\zeta)}G(\theta-t_k)\\ &-\tilde R(\theta-t_k,\Phi(C\zeta))+\tilde G(\theta+\Phi(C\zeta)-t_k)\Big]B, \end{aligned} \tag{18}$$

$$\begin{aligned} v_s(\zeta,\theta)-v_{s-1}(\zeta,\theta)=\lambda_s\big[&G(\theta+\Phi(C\zeta)-t_s)\\ &-\tilde G(\theta+\Phi(C\zeta)-t_s)\big]B. \end{aligned} \tag{19}$$

Proof: Omitted.

Now Theorem 2 can be proved. Since the functions $G(\theta)$, $\tilde G(\theta)$, $\tilde R(\theta,\cdot)$ are continuous for $\theta\geqslant 0$, then the function $P(\zeta,\theta)$ can have gaps only on the surfaces

$$M_k=\{(\zeta,\theta):\theta=t_k\},\quad N_s=\{(\zeta,\theta):\theta+\Phi(C\zeta)=t_s\}.$$

Let $(\zeta,\theta)\in M_k$ for some k. From (18) it follows that $u_k(\zeta,t_k)=u_{k-1}(\zeta,t_k)$, because

$$G(0)=I,\quad \tilde R(0,\Phi(C\zeta))=\mathrm{e}^{\tilde D_K(\Phi(C\zeta)-\tau)}\mathrm{e}^{D_K\tau}-\mathrm{e}^{\tilde D_K\Phi(C\zeta)}$$

and $\tilde G(\Phi(C\zeta))=\mathrm{e}^{\tilde D_K(\Phi(C\zeta)-\tau)}\mathrm{e}^{D_K\tau}$. Hence, $P(\zeta,\theta)$ has no gaps on M_k.

Let $(\zeta,\theta)\in N_s$ for some s. From (19) it follows that

$$v_s(\zeta,t_s-\Phi(C\zeta))=v_{s-1}(\zeta,t_s-\Phi(C\zeta)),$$

because $G(0)-\tilde G(0)=0$. Hence, $P(\zeta,\theta)$ has no gaps on N_s.

If $(\zeta,\theta)\in M_k\cap N_s$ for some k s, then $t_k=t_s-\Phi(C\zeta)$, hence, $P_{k,s}(\zeta,t_k)=P_{k-1,s-1}(\zeta,t_k)$.

B. *Proof of Theorem 3*

The following fact is needed for the proof.

Lemma 4: The partial derivatives

$$\frac{\partial P_{k,s}(\zeta,\theta)}{\partial\zeta},\quad \frac{\partial P_{k,s}(\zeta,\theta)}{\partial\theta}$$

are continuous for $(\zeta,\theta)\in S_{k,s}$ whenever the scalar functions $F(\cdot)$, $\Phi(\cdot)$ have continuous derivatives.

Proof: Omitted.

It follows from Lemma 4 that the functions P'_ζ, P'_θ can have gaps either on the surfaces $M_k=\{(\zeta,\theta)\,:\,\theta=t_k\}$ or on the surfaces $N_s=\{(\zeta,\theta)\,:\,\theta+\Phi(C\zeta)=t_s\}$.

Let $(\zeta,\theta)\in M_k$ for some k. Then

$$\begin{aligned} \frac{\partial(u_k(\zeta,\theta)-u_{k-1}(\zeta,\theta))}{\partial\zeta}\bigg|_{\theta=t_k}&=\lambda_k\Phi'(C\zeta)\Big[-\tilde D_K\mathrm{e}^{\tilde D\Phi(C\zeta)}\\ -\tilde D_K\mathrm{e}^{\tilde D_K\Phi(C\zeta)}(\mathrm{e}^{-D\tau}\mathrm{e}^{D_K\tau}-I)&+\tilde D_K\mathrm{e}^{\tilde D_K(\Phi(C\zeta)-\tau)\mathrm{e}^{D_K\tau}}\Big]BC=0, \end{aligned}$$

$$\begin{aligned} \frac{\partial(u_k(\zeta,\theta)-u_{k-1}(\zeta,\theta))}{\partial\theta}\bigg|_{\theta=t_k}&=\lambda_k\Big[-\mathrm{e}^{\tilde D_K\Phi(C\zeta)}A_0\\ -\mathrm{e}^{\tilde D_K\Phi(C\zeta)}(\tilde D_K\mathrm{e}^{-D\tau}\mathrm{e}^{D_K\tau}-D_K)&+\tilde D_K\mathrm{e}^{\tilde D_K(\Phi(C\zeta)-\tau)\mathrm{e}^{D_K\tau}}\Big]B=0, \end{aligned}$$

because $\mathrm{e}^{\tilde D_K\Phi(C\zeta)}\tilde D_K=\tilde D_K\mathrm{e}^{\tilde D_K\Phi(C\zeta)}$ and $(A_0-D_K)B=KLB=0$.

Thus

$$\frac{\partial u_k}{\partial\zeta}=\frac{\partial u_{k-1}}{\partial\zeta},\quad \frac{\partial u_k}{\partial\theta}=\frac{\partial u_{k-1}}{\partial\theta}$$

at a point (ζ,t_k). Consequently, P'_ζ and P'_θ have no gaps on this surface.

Let $(\zeta,\theta)\in N_s$ for some s.

$$\begin{aligned} \frac{\partial(v_s(\zeta,\theta)-v_{s-1}(\zeta,\theta))}{\partial\zeta}\bigg|_{\theta+\Phi(C\zeta)=t_s}&\\ =\lambda_s\Phi'(C\zeta)(A_0-D_K)BC=0,& \end{aligned}$$

$$\frac{\partial(v_s(\zeta,\theta)-v_{s-1}(\zeta,\theta))}{\partial\theta}\bigg|_{\theta+\Phi(C\zeta)=t_s}=\lambda_s(A_0-D_K)B=0.$$

Thus,

$$\frac{\partial v_s}{\partial\zeta}=\frac{\partial v_{s-1}}{\partial\zeta},\quad \frac{\partial v_s}{\partial\theta}=\frac{\partial v_{s-1}}{\partial\theta}$$

at a point $(\zeta,t_s-\Phi(C\zeta))$. Hence, P'_ζ and P'_θ have no gaps on this surface.

Clearly, if $(\zeta,\theta)\in M_k\cap N_s$, for some k and s, then $t_k=t_s-\Phi(C\zeta)$, and hence

$$\frac{\partial P_{k,s}}{\partial\zeta}=\frac{\partial P_{k-1,s-1}}{\partial\zeta},\quad \frac{\partial P_{k,s}}{\partial\theta}=\frac{\partial P_{k-1,s-1}}{\partial\theta}$$

at a point (ζ,t_k). This means that partial derivatives of $P(\zeta,\theta)$ have no gaps on $(\zeta,\theta)\in M_k\cap N_s$. This completes the proof of Theorem 3.

REFERENCES

[1] L. Mailleret and V. Lemesle, "A note on semi-discrete modeling in the life sciences," *Phil. Trans. R. Soc. A*, vol. 367, no. 1908, pp. 4779–4799, 2009.
[2] L. S. Farhy, "Modeling of oscillations in endocrine networks with feedback," *Methods in Enzymology*, vol. 384, pp. 54–81, 2004.
[3] J. J. Walker, J. R. Terry, K. Tsaneva-Atanasova, S. P. Armstrong, C. A. McArdle, and S. L. Lightman, "Encoding and decoding mechanisms of pulsatile hormone secretion," *J. Neuroendocrinoly*, vol. 22, no. 12, pp. 1226–1238, 2009.
[4] A. Churilov, A. Medvedev, and A. Shepeljavyi, "Mathematical models of testosterone regulation," *Stochastic optimization in informatics*, vol. 2, pp. 147–158, 2006.
[5] ——, "A state observer for continuous oscillating systems under intrinsic pulse-modulated feedback," *Automatica*, vol. 45, no. 6, pp. 1117–1122, 2012.
[6] W. R. Smith, "Qualitative mathematical models of endocrine systems," *Am. J. Physiol.*, vol. 245, no. 4, pp. R473–R477, 1983.
[7] M. Cartwright and M. Husain, "A model for the control of testosterone secretion," *J. Theor. Biol.*, vol. 123, pp. 239–250, 1986.
[8] P. Das, A. B. Roy, and A. Das, "Stability and oscillations of a negative feedback delay model for the control of testosterone secretion," *BioSystems*, vol. 32, no. 1, pp. 61–69, 1994.
[9] D. M. Keenan and J. D. Veldhuis, "A biomathematical model of time-delayed feedback in the human male hypothalamic-pituitary-Leydig cell axis," *Amer. J. Physiology. Endocrinology and Metabolism*, vol. 275, no. 1, pp. E157–E176, 1998.
[10] S. Ruan and J. Wei, "On the zeros of a third degree exponential polynomial with applications to a delayed model for the control of testosterone secretion," *IMA J. Mathematics Applied in Medicine and Biology*, vol. 18, no. 1, pp. 41–52, 2001.
[11] B. Mukhopadhyay and R. Bhattacharyya, "A delayed mathematical model for testosterone secretion with feedback control mechanism," *Int. J. Mathematics and Math. Sciences*, vol. 2004, no. 3, pp. 105–115, 2004.
[12] J. D. Murray, *Mathematical Biology, I: An Introduction (3rd ed.)*. Springer, New York, 2002.
[13] A. Churilov, A. Medvedev, and A. Shepeljavyi, "Mathematical model of non-basal testosterone regulation in the male by pulse modulated feedback," *Automatica*, vol. 48, no. 1, pp. 78–85, 2009.
[14] A. K. Gelig and A. N. Churilov, *Stability and Oscillations of Nonlinear Pulse-modulated Systems*. Boston: Birkhäuser, 1998.

[15] E. Hidayat and A. Medvedev, "Laguerre domain identification of continuous linear time delay systems from impulse response data," *Automatica*, vol. 48, no. 11, pp. 2902–2907, 2012.

[16] P. Mattsson and A. Medvedev, "Modeling of testosterone regulation by pulse-modulated feedback: an experimental data study," in *2013 Int. Symp. Comput. Models Life Sci., AIP Conf. Proc.*, vol. 1559. Melville, New York: AIP Publishing, 2013, pp. 333–342.

[17] Z. Zhusubaliyev, A. Churilov, and A. Medvedev, "Bifurcation phenomena in an impulsive model of non-basal testosterone regulation," *Chaos*, vol. 22, no. 1, pp. 013 121–1—013 121–11, 2012.

[18] A. Churilov, A. Medvedev, and P. Mattsson, "Periodical solutions in a time-delay model of endocrine regulation by pulse-modulated feedback," in *Proceedings of the 51st IEEE Conference on Decision and Control*, Maui, Hawaii, December 10–13 2012, pp. 362–367.

[19] ——, "Finite-dimensional reducibility of time-delay systems under pulse-modulated feedback," in *Proceedings of the 52nd IEEE Conference on Decision and Control*, Florence, Italy, December 10–13 2013, pp. 2078–2083.

[20] ——, "Periodical solutions in a pulse-modulated model of endocrine regulation with time-delay," *IEEE Transactions on Automatic Control*, vol. 59, no. 3, pp. 728–733, 2014.

[21] Z. Zhusubaliyev, A. Churilov, and A. Medvedev, "Time delay induced multistability and complex dynamics in an impulsive model of endocrine regulation," in *Proceedings of the 13th European Control Conference (ECC)*, Strasbourg, France, June 24 – 27 2014, (accepted).

[22] A. Churilov, A. Medvedev, and Z. Zhusubaliyev, "Periodic modes and bistability in an impulsive Goodwin oscillator with large delay," in *Proceedings of the 19th IFAC World Congress*, Cape Town, South Africa, August 24–29 2014, (accepted).

[23] A. Churilov and A. Medvedev, "An impulse-to-impulse discrete-time mapping for a time-delay impulsive system," *Automatica (IFAC journal)*, 2014, (accepted).

[24] M. L. Johnson, L. Pipes, P. P. Veldhuis, L. S. Farhy, R. Nass, M. O. Thorner, and W. S. Evans, "AutoDecon: A robust numerical method for the quantification of pulsatile events," in *Methods in Enzymology: Computer methods, Volume A*, M. L. Johnson and L. Brand, Eds., 2009, vol. 454, pp. 367–404.

[25] D. Salamon, "On controllability and observability of time delay systems," *IEEE Trans. Autom. Contr.*, vol. 29, no. 5, pp. 432–439, 1984.

[26] O. Sename, "New trends in design of observers for time-delay systems," *Kybernetika*, vol. 74, no. 4, pp. 427–458, 2001.

[27] A. Manitius and A. Olbrot, "Finite spectrum assignment problem for systems with delays," *IEEE Transactions on Automatic Control*, vol. 24, no. 4, pp. 541–553, 1979.

[28] K. Watanabe, "Finite spectrum assignment and observer for multivariable systems with commensurate delays," *IEEE Transactions on Automatic Control*, vol. 31, no. 6, pp. 543–550, 1986.

[29] A. Medvedev, "Disturbance attenuation in finite-spectrum-assignment controllers," *Automatica (IFAC Journal)*, vol. 33, no. 6, pp. 1163–1168, 1997.

[30] M. Jankovic, "Forward, backstepping, and finite spectrum assignment for time delay systems," *Automatica (IFAC journal)*, vol. 45, no. 1, pp. 2–9, 2009.

[31] A. Churilov, A. Medvedev, and Z. Zhusubaliev, "Conditional stability of a state observer for a low-order hybrid plant," in *Proceedings of the 5th IFAC International Workshop on Periodic Control Systems (PSYCO'2013)*, University of Caen Basse-Normandie, Caen, France, July 3 – 5 2013, pp. 113–118.

[32] A. Churilov, A. Medvedev, and Z. Zhusubaliyev, "State observer for a low-order plant under intrinsic pulse-modulated feedback: a case study," in *Proceedings of the 12th European Control Conference (ECC)*, Zürich, Switzerland, July 17 – 19 2013, pp. 2122–2127.