21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

An Orthogonal Projection Method for Computing Active, Reactive, and Scattered Power and its Application to Compensator Design

Jacob van der Woude and Dimitri Jeltsema

Abstract— **In this paper, we present an alternative method to determine the active, reactive, and scattered power in an electrical network driven by an ideal non-sinusoidal AC voltage source. The method is based on orthogonally projecting the current demanded by the load onto the space of all (anti)derivatives of the source voltage of odd order. It is applicable in both the time-domain and frequency-domain. The obtained projection is the reactive current that can be compensated by a finite-dimensional lossless linear time-invariant compensator placed in parallel to the load. The parameters of the compensator can also be obtained from the projection. With the compensator, the apparent power delivered by the source can be reduced, while the active power demanded by the load remains unchanged. In this way, the power factor of the energy transport from source to load can be improved.**

I. Introduction

The usage of alternative sources of power has caused that the problem of energy transfer optimization is increasingly involved with nonsinusoidal signals and nonlinear loads. The power factor (PF) is used as a measure of the effectiveness of the transfer of power between an electrical source and a load. It is defined as the ratio between the power consumed by a load (real or active power), denoted as P, and the power delivered by a source (apparent power), denoted as S, i.e.,

$$\mathrm{PF} := \frac{|P|}{S}. \tag{1}$$

The active power is defined as the average of the instantaneous power and apparent power as the product of the root-mean-square norms of the source current and voltage. The standard approach to improve the power factor is to place a passive compensator, such as a capacitor or an inductor, parallel to the load. Conceptually, the design of the compensator typically assumes that the source is ideal, i.e., the internal (Thevenin) impedance is negligible, producing a fixed sinusoidal voltage. If the load is linear and time-invariant (LTI) and the source voltage is sinusoidal, the resulting stationary current generally is a shifted sinusoid, and the power factor is the cosine of the phase-shift angle between the source voltage and current. Classically, the remaining part of the power is called reactive power, and is denoted as Q. The relationship between the three types of power is given by $S^2 = P^2 + Q^2$. Any improvement of the PF is accomplished by the reduction of the absolute value of the reactive power, hence reducing the phase shift between the current and the voltage.

For nonsinusoidal voltages and currents, the problem of decomposing the apparent power into active and reactive components is much more involved. Starting from the work of Budeanu [1], many authors have aimed to improve the concept of reactive power in the most general case; see e.g., [6], [7], and the references therein. Most of these contributions aim at decompositions of the load current into physical meaningful orthogonal quantities. One of the most detailed works to date appears to be that of Czarnecki [6], and is commonly known as the Currents' Physical Components (CPC) method. In essence, for LTI loads, the CPC method decomposes the current into three components associated to three distinctive physical phenomena in the load: permanent energy conversion (active current), change of load conductance with harmonic order (scattered current), and phase-shift between the voltage and current harmonics (reactive current). The CPC method uses techniques from both the time-domain and the frequency domain, and can therefore be considered as a hybrid approach.

In this paper, we present an alternative method that can be applied in either the time-domain or the frequency-domain. The method is based on orthogonally projecting the current demanded by the load onto the space of all (anti)derivatives of the source voltage of odd order. The obtained projection coincides with the reactive current that can be compensated by a finite-dimensional lossless LTI compensator placed in parallel to the load. An advantage of the projection method is that it automatically generates the structure and the parameters of the compensator.

II. Framework

We consider an electrical network with an AC voltage source without internal resistance to which a load is coupled. Throughout the paper we assume that the source voltage $v(t)$ is a linear combination of a finite number of harmonics, i.e.,

$$v(t) = \sum_{l=1}^{L} \alpha_l \cos(lt) + \beta_l \sin(lt), \tag{2}$$

for some finite positive integer L and real-valued coefficients α_l, β_l, for $l = 1, \ldots, L$.

Clearly, the above representation can be seen as a real-valued (finite) Fourier series of the voltage $v(t)$. Later on we will also use the complex-valued version of the Fourier series. Furthermore, we assume that there is no constant (DC) term in the source voltage. In fact, we assume this to hold for all signals in this paper. We also assume that the signals in the network are described by means of a finite number of harmonics. This is for instance true when the load can be described as an asymptotically stable finite-dimensional linear time-invariant system.

J. van der Woude and D. Jeltsema are with the Delft Institute of Applied Mathematics, Delft University of Technology, The Netherlands.

ISBN: 978-90-367-6321-9

We denote by $v^{(k)}(t)$ the k-th order derivative of $v(t)$, with k some positive integer. With $v(t)$ as defined above, it is clear that $v^{(k)}(t)$ is also a linear combination of $\cos(lt)$ and $\sin(lt)$, for $l = 1, \ldots, L$. In case k is a negative integer, we denote by $v^{(k)}(t)$ the $(-k)$-th order antiderivative of $v(t)$, where the usual integration constants have been omitted. Hence, the polynomial expressions in t resulting from these constants are neglected, and $v^{(k)}(t)$ consists of only a linear combination of $\cos(lt)$ and $\sin(lt)$, for $l = 1, \ldots, L$.

III. Powers and inner product

Before we introduce the various types of powers, being specific products of voltages and currents, we introduce the following notion of inner product. Assuming that functions/signals f and g have suitable properties, for instance being represented by a finite Fourier series, we take as their inner product

$$\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(t)g(t)\mathrm{d}t,$$

and define the norm of f by $||f|| := \sqrt{\langle f, f\rangle}$, and likewise for $||g||$.

With the source voltage $v(t)$ as defined above, we claim that

$$\langle v^{(k)}, v^{(m)}\rangle = 0,$$

for all integer k, m, with $k-m$ odd. Indeed, assume without loss of generality that $k > m$. Then, by integration by parts, it follows that

$$\int_0^{2\pi} v^{(k)}(t)v^{(m)}(t)\mathrm{d}t =$$

$$v^{(k-1)}(t)v^{(m)}(t)\Big|_0^{2\pi} - \int_0^{2\pi} v^{(k-1)}(t)v^{(m+1)}(t)\mathrm{d}t$$

$$= -\int_0^{2\pi} v^{(k-1)}(t)v^{(m+1)}(t)\mathrm{d}t,$$

where the term $v^{(k-1)}(t)v^{(m)}(t)\Big|_0^{2\pi} = 0$ by the periodicity of $v^{(k-1)}(t)$ and $v^{(m)}(t)$. Repeated application of the previous yields that

$$\int_0^{2\pi} v^{(k)}(t)v^{(m)}(t)\mathrm{d}t = (-1)^{k-m}\int_0^{2\pi} v^{(m)}(t)v^{(k)}(t)\mathrm{d}t.$$

Hence, if $k-m$ is odd, it follows that

$$\int_0^{2\pi} v^{(k)}(t)v^{(m)}(t)\mathrm{d}t = 0 \text{ and } \langle v^{(k)}, v^{(m)}\rangle = 0.$$

IV. Orthogonal projection

The above property has the following application. Let $v(t)$ be the source voltage as defined in (2), let $i(t)$ denote the current demanded by the load and let the load be described by means of a finite-dimensional LTI system such that its spectrum does not contain any of the frequencies in $v(t)$. Assume that $i(t)$ is written as a linear combination of $v^{(k)}(t)$, for $k = -N, \ldots, N$ with N sufficiently large.

The idea behind the latter is that the voltage $v(t)$ can be seen as the input and the current $i(t)$ as the output, related to each other by a (stable) single-input/single-output (SISO) transfer function. Writing this transfer function in its Maclaurin series, it can be seen that the output $i(t)$ is a linear combination of the input $v(t)$ and all its (anti)derivatives. Further, computing $i(t)$ from $v(t)$ using a method like the method of undetermined coefficients, it can be seen that only a finite number of (anti)derivatives of $v(t)$ are actually needed to compute $i(t)$. Then, $i(t)$ can be decomposed as

$$i(t) = i_{\mathrm{e}}(t) + i_{\mathrm{o}}(t),$$

with

$$i_{\mathrm{e}}(t) := \sum_{\substack{k=-N \\ k \text{ even}}}^{N} \gamma_k v^{(k)}(t),\ i_{\mathrm{o}}(t) := \sum_{\substack{k=-N \\ k \text{ odd}}}^{N} \gamma_k v^{(k)}(t),$$

where γ_k are real-valued coefficients, for $k = -N, \ldots, N$.

Clearly, because $\langle v^{(k)}, v^{(m)}\rangle = 0$ whenever $k-m$ is odd, it follows that $\langle i_{\mathrm{e}}, i_{\mathrm{o}}\rangle = 0$. Hence, the decomposition $i(t) = i_{\mathrm{e}}(t) + i_{\mathrm{o}}(t)$ is orthogonal. Note that also $\langle v, i_{\mathrm{o}}\rangle = 0$. The latter means that there is no loss of energy involved in the interaction between v and $i_{\mathrm{o}}(t)$.

The current $i_{\mathrm{o}}(t)$ is also known in literature as the *reactive current*, and will be denoted in this paper as $i_r(t)$, see e.g., [6]. It follows that $\langle i_r, i_{\mathrm{e}}\rangle = 0$ and $\langle v, i_r\rangle = 0$. The latter implies that any controller that is driven by $v(t)$ and that can compensate $i_r(t)$, will be able to do so without loss of energy. Therefore, the reactive current can be seen as the current that can be compensated completely and losslessly by means of a finite-dimensional LTI compensator that is placed parallel to the load.

In $i_{\mathrm{e}}(t)$ two subcomponents can be identified, namely the so-called *active current* $i_a(t)$ and the so-called *scattered current* $i_s(t)$. These two currents are defined as follows:

$$i_a(t) := \frac{\langle i_{\mathrm{e}}, v\rangle}{\langle v, v\rangle}v(t),$$
$$i_s(t) := i_{\mathrm{e}}(t) - i_a(t),$$

see e.g., [6], for more details.

Since $\langle v, i_{\mathrm{o}}\rangle = 0$, the active current can be seen as the orthogonal projection of the current $i(t)$ on the voltage $v(t)$. It is immediate that $\langle i_a, v\rangle = \langle i_{\mathrm{e}}, v\rangle$ and, consequently, that $\langle i_s, v\rangle = 0$.

Next distinguish the following two cases.

- When $\langle i_a, v\rangle \neq 0$, then $i_a(t)$ is an actual multiple of $v(t)$, implying that also $\langle i_s, i_a\rangle = 0$. Further, since $\langle i_r, v\rangle = 0$, it follows that also $\langle i_r, i_a\rangle = 0$. Finally, because of $\langle i_o, i_e\rangle = \langle i_r, i_a + i_s\rangle = 0$, it follows that $\langle i_r, i_s\rangle = 0$.

- When $\langle i_a, v\rangle = 0$, then $i_a(t) = 0$ and $i_s(t) = i_e(t)$ for all t. Consequently, $\langle i_a, i_r\rangle = 0$, $\langle i_a, i_s\rangle = 0$ and $\langle i_r, i_s\rangle = \langle i_o, i_e\rangle = 0$.

Hence, it follows that $i = i_a + i_r + i_s$, with $\langle i_a, i_r\rangle = 0$, $\langle i_a, i_s\rangle = 0$ and $\langle i_r, i_s\rangle = 0$, implying that

$$||i||^2 = ||i_a||^2 + ||i_r||^2 + ||i_s||^2.$$

So, the *apparent power* S can be decomposed as

$$S^2 = ||v||^2||i||^2 = S_a^2 + S_r^2 + S_s^2, \quad (3)$$

with $S_a^2 = ||v||^2||i_a||^2, S_r^2 = ||v||^2||i_r||^2, S_s^2 = ||v||^2||i_s||^2$.

Since $i = i_a + i_r + i_s$ and $(P_r =)\langle v, i_r\rangle = 0$ and $(P_s =) \langle v, i_s\rangle = 0$, the *active power* P_a, defined as $P_a = \langle v, i_a\rangle$, equals the *power required by the load* P, defined as $P = \langle v, i\rangle$. Indeed

$$P = \langle v, i\rangle = \langle v, i_a + i_r + i_s\rangle = \langle v, i_a\rangle = P_a.$$

Finally, it can be shown in general, for the above inner product and suitable functions $v(t), i(t)$, see also [9], that

$$||v||^2||i||^2 - \langle v, i\rangle^2 = \frac{1}{2}\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\big(v(t)i(s) - i(t)v(s)\big)^2 \mathrm{d}s\mathrm{d}t.$$

Hence, $S^2 - P^2 = Q^2$, with

$$Q^2 = \frac{1}{2}\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\big(v(t)i(s) - i(t)v(s)\big)^2 \mathrm{d}s\mathrm{d}t.$$

Since $i_a(t)$ is a multiple of $v(t)$ and due to the fact that $\langle i_r, i_s\rangle = 0$, $\langle i_r, v\rangle = 0$ and $\langle i_s, v\rangle = 0$, it follows that

$$\int_0^{2\pi}\int_0^{2\pi}\big(v(t)i(s) - i(t)v(s)\big)^2 \mathrm{d}s\mathrm{d}t = \int_0^{2\pi}\int_0^{2\pi}\big(v(t)i_r(s) - i_r(t)v(s)\big)^2 \mathrm{d}s\mathrm{d}t + \int_0^{2\pi}\int_0^{2\pi}\big(v(t)i_s(s) - i_s(t)v(s)\big)^2 \mathrm{d}s\mathrm{d}t.$$

Thus, $Q^2 = Q_r^2 + Q_s^2$, where the *reactive power* Q_r is defined as

$$Q_r^2 := \frac{1}{2}\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\big(v(t)i_r(s) - i_r(t)v(s)\big)^2 \mathrm{d}s\mathrm{d}t$$

and the *scattered power* Q_s is defined as

$$Q_s^2 := \frac{1}{2}\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\big(v(t)i_s(s) - i_s(t)v(s)\big)^2 \mathrm{d}s\mathrm{d}t,$$

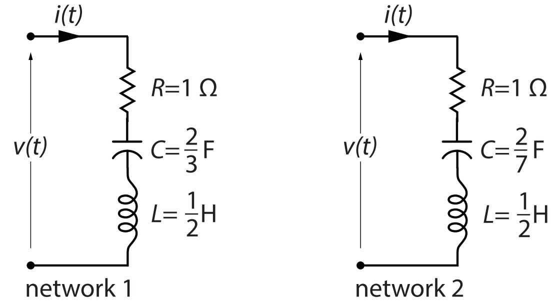

Fig. 1. Example RLC networks.

or, equivalently,

$$Q_r^2 = ||v||^2||i_r|| - \langle v, i_r\rangle^2 = ||v||^2||i_r||^2,$$
$$Q_s^2 = ||v||^2||i_s|| - \langle v, i_s\rangle^2 = ||v||^2||i_s||^2.$$

Hence, it follows that

$$S^2 = P^2 + Q_r^2 + Q_s^2,$$

which matches the identity for S^2 in (3), in which $P^2 = ||v||^2||i_a||^2$, because $P = P_a = \langle v, i_a\rangle = ||v||||i_a||$, and $Q_r^2 = ||v||^2||i_r||^2, Q_s^2 = ||v||^2||i_s||^2$.

Under the condition that the voltage source is ideal, for the computation of the active, reactive and scattered power, it suffices to know (the norm of) the active, reactive and scattered current, respectively.

V. Examples

In this section, we illustrate our main idea by means of the two RLC networks depicted in Figure 1. These networks have been studied in the literature to show the need for a proper current and power decomposition; see e.g., [4]. Using the notation of this paper, the equation for the RLC circuit is

$$Ri(t) + Li^{(1)}(t) + \frac{1}{C}i^{(-1)}(t) = v(t).$$

Recall that $i^{(1)}(t)$ denotes the first order derivative of $i(t)$ and $i^{(-1)}(t)$ the first order antiderivative of $i(t)$. As source voltage we take

$$v(t) = \sqrt{2}\big(100\sin(t) + 100\cos(3t)\big).$$

It is immediate that $||v|| = 100\sqrt{2}$. Note that for $m \geq 0$

$$v^{(2m+1)}(t) = (-1)^m\sqrt{2}\Big(100\cos(t) - 3^{2m+1}100\sin(3t)\Big)$$

and

$$v^{(-2m-1)}(t) = (-1)^m\sqrt{2}\Big(- 100\cos(t) + \frac{100}{3^{2m+1}}\sin(3t)\Big).$$

It follows easily that the span of odd ordered (anti)derivatives of $v(t)$ is given by

$$\begin{aligned}\mathbb{P}_o v(t) &= \text{span}\{v^{(k)}(t)\big|k \text{ integer \& odd}\}\\ &= \text{span}\{v^{(1)}(t), v^{(-1)}(t)\}\\ &= \text{span}\{\cos(t), \sin(3t)\}.\end{aligned}$$

A. Network 1

According to the network equations, it follows easily for network 1 that (using the method of undetermined coefficients)

$$i(t) = \sqrt{2}\big(50\cos(t) + 50\sin(t) + 50\cos(3t) + 50\sin(3t)\big).$$

Clearly $||i|| = 100$. Projecting $i(t)$ orthogonally onto the span $\mathbb{P}_{\mathrm{o}}v(t)$, it is immediate that

$$i_{\mathrm{o}}(t) = \sqrt{2}\big(50\cos(t) + 50\sin(3t)\big),$$

and, consequently,

$$i_{\mathrm{e}}(t) = \sqrt{2}\big(50\sin(t) + 50\cos(3t)\big).$$

Observe that here $i_{\mathrm{e}}(t) = \frac{1}{2}v(t)$. Further, it follows that

$$\begin{aligned} i_r(t) &= \sqrt{2}\big(50\cos(t) + 50\sin(3t)\big), && ||i_r|| = 50\sqrt{2},\\ i_a(t) &= \sqrt{2}\big(50\sin(t) + 50\cos(3t)\big), && ||i_a|| = 50\sqrt{2},\\ i_s(t) &= 0, && ||i_s|| = 0. \end{aligned}$$

Computing the powers, it is straightforward to see that $P = ||v||||i_a|| = 10\,\mathrm{kW}$, $S = ||v||||i|| = 10\sqrt{2}\,\mathrm{kVA}$, $Q_r = ||v||||i_r|| = 10$ kVAr and $Q_s = ||v||||i_s|| = 0$ kVA. Hence, the power factor, defined in (1), is $\frac{1}{2}\sqrt{2}$.

Recall that

$$v^{(1)}(t) = \sqrt{2}\big(100\cos(t) - 300\sin(3t)\big)$$

and

$$v^{(-1)}(t) = \sqrt{2}\big(-100\cos(t) + \frac{100}{3}\sin(3t)\big).$$

A simple calculation reveals that

$$i_r(t) = -\frac{1}{4}v^{(1)}(t) - \frac{3}{4}v^{(-1)}(t).$$

Hence, a compensator consisting of an inductor with inductance $\frac{4}{3}$ and a capacitor with capacity $\frac{1}{4}$, both placed parallel to the load, is able to compensate the current $i_r(t)$, i.e., $i_r(t)$ can be compensated by the controller

$$i_c(t) = C_c v^{(1)}(t) + \frac{1}{L_c}v^{(-1)}(t),$$

with $C_c = \frac{1}{4}$ and $L_c = \frac{4}{3}$.

With the compensator, the reactive power is eliminated, implying that the power factor, being the ratio of P and $\sqrt{S^2 - Q_c^2}$, becomes 1. Hence, there is complete power transfer.

B. Network 2

For network 2 it follows easily that

$$i(t) = \sqrt{2}\big(30\cos(t) + 10\sin(t) + 90\cos(3t) + 30\sin(3t)\big)$$

Note that again $||i|| = 100$. Performing the orthogonal projection of $i(t)$ on $\mathbb{P}_{\mathrm{o}}v(t)$, as above, it follows in a similar way that

$$\begin{aligned} i_r(t) &= \sqrt{2}\big(30\cos(t) + 30\sin(3t)\big), && ||i_r|| = 30\sqrt{2},\\ i_a(t) &= \sqrt{2}\big(50\sin(t) + 50\cos(3t)\big), && ||i_a|| = 50\sqrt{2},\\ i_s(t) &= \sqrt{2}\big(-40\sin(t) + 40\cos(3t)\big), && ||i_s|| = 40\sqrt{2}. \end{aligned}$$

Computing the powers, it is straightforward to see that $P = ||v||||i_a|| = 10\,\mathrm{kW}$, $S = ||v||||i|| = 10\sqrt{2}\,\mathrm{kVA}$, $Q_r = ||v||||i_r|| = 6$ kVAr and $Q_s = ||v||||i_s|| = 8$ kVA. Also for this network the power factor without compensation is $\frac{1}{2}\sqrt{2}$.

A closer inspection reveals that

$$i_r(t) = -\frac{3}{20}v^{(1)}(t) - \frac{9}{20}v^{(-1)}(t).$$

Hence, a compensator consisting of an inductor with inductance $\frac{20}{9}$ and a capacitor with capacity $\frac{3}{20}$, both placed parallel to the load, is able to compensate the current $i_r(t)$, i.e., $i_r(t)$ can be compensated by the controller

$$i_c(t) = C_c v^{(1)}(t) + \frac{1}{L_c}v^{(-1)}(t),$$

with $C_c = \frac{3}{20}$ and $L_c = \frac{20}{9}$.

With the compensator the power factor becomes 0.78087, the ratio of P and $\sqrt{S^2 - Q_c^2} = (\sqrt{P^2 + Q_s^2})$. Hence, not all supplied power is consumed by the load.

VI. Computational issues

In the previous sections, we described a method to determine the reactive power and current. The method boils down to an orthogonal projection in some appropriate function space. However, since all functions/signals are supposed to have a finite Fourier series, the projection can also be formulated in terms of the coefficients present in the Fourier series. To introduce this alternative, computationally more practical method, consider a real-valued 2π-periodic function f, i.e., $f(t) \in \mathbb{R}$ and $f(t) = f(t+2\pi)$, for all $t \in \mathbb{R}$. Suppose that f consists of a linear combination of a cosine, sine and a finite number of their higher harmonics. Then using complex arithmetic with $j = \sqrt{-1}$, f can be written as follows

$$f(t) = \sum_{l=-L}^{L} e^{lkt} f_l,$$

with $f(t)$ real-valued, implying that $f_l = \overline{f_{-l}}$, for all $l = -L, \ldots, L$. Also (replace l by $-l$)

$$f(t) = \sum_{l=-L}^{L} \overline{f_l} e^{-jlt}.$$

Because we assume that the functions/signals do not have a DC-term, it follows that in the above representation $f_0 = 0$. Now define

$$e(t) = \Big(e^{-jLt}, \ldots, e^{-jt}, e^{jt}, \ldots, e^{jLt}\Big),$$

and

$$F = \begin{pmatrix} f_{-L} \\ \vdots \\ f_{-1} \\ f_1 \\ \vdots \\ f_L \end{pmatrix}.$$

Then, $f(t) = e(t)F$, but also $f(t) = F^*e^*(t)$, where

$$F^* = \Big(\overline{f_{-L}}, \ldots, \overline{f_{-1}}, \overline{f_1}, \ldots, \overline{f_L}\Big) = \Big(f_L, \ldots, f_1, f_{-1}, \ldots, f_{-L}\Big)$$

and

$$e^*(t) = \begin{pmatrix} \overline{e^{-jLt}} \\ \vdots \\ \overline{e^{-jt}} \\ \overline{e^{jt}} \\ \vdots \\ \overline{e^{jLt}} \end{pmatrix} = \begin{pmatrix} e^{jLt} \\ \vdots \\ e^{jt} \\ e^{-jt} \\ \vdots \\ e^{-jLt} \end{pmatrix}.$$

Observe that $F^*F = \text{trace } FF^*$. Also note that

$$\frac{1}{2\pi}\int_0^{2\pi} e^*(t)e(t)\mathrm{d}t = I_{2L},$$

where I_{2L} denotes that $2L \times 2L$ identity matrix.

In general, for two real-valued 2π-periodic functions f and g, written as $f(t) = e(t)F$ and $g(t) = e(t)G$, there holds $F^*G = \text{trace } GF^*$. Also note that

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} F^*e^*(t)e(t)G\mathrm{d}t = F^*G.$$

Further, for such f and g the chain of equalities (4) (on the next page) holds true, where $||\cdot||_{\text{Frob}}$ denotes the Frobenius norm. Hence, the quantity Q can be determined once the Fourier series of f and g is known.

A. Further orthogonality and decomposition properties

To stress the dependency on f and g, we write $Q = Q(f, g)$. If $f(t) = e(t)F$ and $g(t) = e(t)G$, we then have $Q^2(f, g) = \frac{1}{2}||GF^* - GF^*||^2_{\text{Frob}}$. Recall that, in general, $||f||^2||g||^2 = \langle f, g\rangle^2 + Q^2(f, g)$. If $\langle f, g\rangle = 0$, then $||f||^2||g||^2 = Q^2(f, g)$.

Assume that f, g and h are periodic functions, such that $\langle f, g\rangle = \langle f, h\rangle = \langle g, h\rangle = 0$. Then, using orthogonality, it is easily seen that

$$Q^2(f, h + g) = Q^2(f, g) + Q^2(f, h).$$

Now, in the context of out networks, assume that $f(t) = e(t)F, g(t) = e(t)G$ and $h(t) = e(t)H$. Then $\langle f, g\rangle = \langle f, h\rangle = \langle g, h\rangle = 0$ implies that $F^*G = F^*H = G^*H = 0$. Furthermore, from (4), it follows that

$$||F(G + H)^* - (G + H)F^*||^2_{\text{Frob}} = ||FG^* - GF^*||^2_{\text{Frob}} + ||FH^* - HF^*||^2_{\text{Frob}}.$$

So, if $v(t) = e(t)V$ is the source voltage and $i(t) = i_a(t) + i_r(t) + i_s(t)$ is the current demanded by the load, see before for the meaning of $i_a(t), i_r(t)$ and $i_s(t)$, with $i_a(t) = e(t)I_a, i_r(t) = e(t)I_r$ and $i_s(t) = e(t)I_s$, then

$$Q^2(v, i_r + i_s) = Q^2(v, i_r) + Q^2(v, i_s),$$

since $\langle v, i_r\rangle = 0$, $\langle v, i_s\rangle = 0$ and $\langle i_r, i_s\rangle = 0$. In terms of Fourier coefficients, it follows that

$$||V(I_r + I_s)^* - (I_r + I_s)V^*||^2_{\text{Frob}} = ||VI_r^* - I_rV^*||^2_{\text{Frob}} + ||VI_s^* - I_sV^*||^2_{\text{Frob}}.$$

Recall that $i_a(t)$ is a multiple of $v(t)$, and I_a is a multiple of V, implying that

$$Q^2(v, i_a + i_r + i_s) = Q^2(v, i_r + i_s)$$

and

$$||V(I_a + I_r + I_s)^* - (I_a + I_r + I_s)V^*||^2_{\text{Frob}} = ||V(I_r + I_s)^* - (I_r + I_s)V^*||^2_{\text{Frob}}.$$

Hence, it follows with the orthogonality assumptions that

$$Q^2(v, i) = Q^2(v, i_r) + Q^2(v, i_s)$$

and

$$||VI^* - IV^*||^2_{\text{Frob}} = ||VI_r^* - I_rV^*||^2_{\text{Frob}} + ||VI_s^* - I_sV^*||^2_{\text{Frob}}.$$

B. Orthogonal projection

Next consider a real-valued 2π-periodic function f with a finite number of harmonics, without constant term, written as

$$f(t) = \sum_{\substack{l=-L \\ l\neq 0}}^{L} e^{jlt}f_l = e(t)F.$$

Next consider the derivative of $f(t)$ given by

$$f'(t) = \sum_{\substack{l=-L \\ l\neq 0}}^{L} e^{jlt}(jl)f_l = e(t)WF,$$

where

$$W = \text{ diag }\Big(-jL, \ldots, -j, j, \ldots, jL\Big).$$

Note that $W^* + W = 0$ and that W is invertible.

Repeated differentiation yields

$$f^{(k)}(t) = e(t)W^kF,$$

For $k < 0$ we denote $f^{(k)}(t)$ for the $(-k)$-th anti derivative of $f(t)$. Since matrix W is invertible, the above expression for $f^{(k)}(t)$ clearly also holds for negative values of k.

Now we return to our network. Let $v(t)$ denote the source voltage and $i(t)$ the current demanded by the load. Assume that both are made up of a finite number of harmonics, say L, and that both do not contain a constant term. Then write

$$v(t) = e(t)V \text{ and } i(t) = e(t)I.$$

To obtain $i_r(t)$ we orthogonally project $i(t)$ onto the space spanned by the odd ordered (anti)derivatives of $v(t)$ to obtain

$$i_r(t) = \sum_{\substack{k=-N \\ k \text{ odd}}}^{N} \gamma_k v^{(k)}(t).$$

$$
\begin{aligned}
Q^2 &= \frac{1}{2}\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\big(f(t)g(s)-g(t)f(s)\big)^2\mathrm{d}s\mathrm{d}t \\
&= \frac{1}{2}\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\big(f(t)g(s)-g(t)f(s)\big)\big(g(s)f(t)-f(s)g(t)\big)\mathrm{d}s\mathrm{d}t \\
&= \frac{1}{2}\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\big(f(t)G^*-g(t)F^*\big)e^*(s)e(s)\big(Gf(t)-Fg(t)\big)\mathrm{d}s\mathrm{d}t \\
&= \frac{1}{2}\frac{1}{2\pi}\int_0^{2\pi}\big(f(t)G^*-g(t)F^*\big)\Big(\frac{1}{2\pi}\int_0^{2\pi}e^*(s)e(s)\mathrm{d}s\Big)\big(Gf(t)-Fg(t)\big)\mathrm{d}t \\
&= \frac{1}{2}\frac{1}{2\pi}\int_0^{2\pi}\big(f(t)G^*-g(t)F^*\big)\big(Gf(t)-Fg(t)\big)\mathrm{d}t \\
&= \frac{1}{2}\frac{1}{2\pi}\int_0^{2\pi}\text{trace }\big(Gf(t)-Fg(t)\big)\big(f(t)G^*-g(t)F^*\big)\mathrm{d}t \\
&= \frac{1}{2}\text{trace }\frac{1}{2\pi}\int_0^{2\pi}\big(GF^*-FG^*\big)e^*(t)e(t)\big(FG^*-GF^*\big)\mathrm{d}t \\
&= \frac{1}{2}\text{trace }\big(GF^*-FG^*\big)\Big(\frac{1}{2\pi}\int_0^{2\pi}e^*(t)e(t)\mathrm{d}t\Big)\big(FG^*-GF^*\big) \\
&= \frac{1}{2}\text{trace }\big(GF^*-FG^*\big)\big(FG^*-GF^*\big) \\
&= \frac{1}{2}||GF^*-FG^*||^2_{\text{Frob}}.
\end{aligned}
\tag{4}
$$

with N large enough. For the coefficients in the Fourier series this means

$$I_r = \sum_{\substack{k=-N \\ k \text{ odd}}}^{N} \gamma_k W^k V,$$

where we wrote $i_r(t) = e(t)I_r$. Likewise, we can write for the scattered current $i_s(t) = e(t)I_s$.

Hence, I_r is the orthogonal projection of I onto

$$\text{span}\{W^k V | k = -N, \ldots, N, k \text{ is odd}\}.$$

If the source voltage contains a finite number of harmonics, the above span is finite dimensional, and the reactive current contains also a finite number of harmonics.

If the load is described by means of a finite dimensional linear system, the scattered current also contains a finite number of harmonics. However, if the load is nonlinear, for instance a switch, the scattered current may contain an infinite amount of harmonics.

VII. Examples (cont'd)

Consider the two load examples of Figure 1. Recall that the source voltage in both examples contains $\sin(t)$ and $\cos(3t)$. Taking $L = 3$, the voltage in both examples can be written as $v(t) = e(t)V$, with

$$e(t) = \Big(e^{-3jt}, e^{-2jt}, e^{-jt}, e^{jt}, e^{2jt}, e^{3jt}\Big)$$

and

$$V = \sqrt{2}\begin{pmatrix} 50 \\ 0 \\ 50j \\ -50j \\ 0 \\ 50 \end{pmatrix}.$$

The current $i(t)$ in the two networks can be written as $i(t) = e(t)I$, where I is obtained by solving the equation

$$(RI_6 + LW + \frac{1}{C}W^{-1})I = V,$$

where I_6 denotes the 6×6 identity matrix, and

$$W = \text{ diag }\Big(-3j, -2j, -j, j, 2j, 3j\Big).$$

For the two networks, we obtain

$$I = \sqrt{2}\begin{pmatrix} 25+25j \\ 0 \\ 25+25j \\ 25-25j \\ 0 \\ 25-25j \end{pmatrix}, \quad I = \sqrt{2}\begin{pmatrix} 45+15j \\ 0 \\ 15+5j \\ 15-5j \\ 0 \\ 45-15j \end{pmatrix},$$

respectively. The Fourier coefficients vector I of the current in each networks has to be projected orthogonally onto

$$\text{span}\{W^k V | k = -L, \ldots, L, k \text{ is odd}\}.$$

Here it turns out that this space is spanned by WV and $W^{-1}V$, given by

$$WV = \sqrt{2}\begin{pmatrix} -150j \\ 0 \\ 50 \\ 50 \\ 0 \\ 150j \end{pmatrix}, \quad W^{-1}V = \sqrt{2}\begin{pmatrix} \frac{50}{3}j \\ 0 \\ -50 \\ -50 \\ 0 \\ -\frac{50}{3}j \end{pmatrix}.$$

The orthogonal projection of the vector I onto the above space gives I_r, defining $i_r(t) = e(t)I_r$, where

$$I_r = \sqrt{2}\begin{pmatrix} 25j \\ 0 \\ 25 \\ 25 \\ 0 \\ -25j \end{pmatrix}, \quad I_r = \sqrt{2}\begin{pmatrix} 15j \\ 0 \\ 15 \\ 15 \\ 0 \\ -15j \end{pmatrix},$$

respectively. The active current $i_a(t) = e(t)I_a$ is obtained by projecting $i(t)$ on $v(t)$. For both the two networks, we obtain

$$I_a = \sqrt{2}\begin{pmatrix} 25 \\ 0 \\ 25j \\ -25j \\ 0 \\ 25 \end{pmatrix}.$$

Finally, for the scattered current $i_s(t) = e(t)I_s$, we get

$$I_s = \sqrt{2}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad I_s = \sqrt{2}\begin{pmatrix} 20 \\ 0 \\ -20j \\ 20j \\ 0 \\ 20 \end{pmatrix},$$

respectively.

With the above Fourier coefficients vectors, the various norms and powers can now be computed easily. For network 1, it follows

$$\begin{aligned} ||v|| &= \sqrt{V^*V} = 100\sqrt{2}, \\ ||i|| &= \sqrt{I^*I} = 100, \\ ||i_r|| &= \sqrt{I_r^*I_r} = 50\sqrt{2}, \\ ||i_a|| &= \sqrt{I_a^*I_a} = 50\sqrt{2}, \\ ||i_s|| &= \sqrt{I_s^*I_s} = 0. \end{aligned}$$

Observe that $I_r = -\frac{1}{4}WV - \frac{3}{4}W^{-1}V$. Similarly, for network 2, we obtain

$$\begin{aligned} ||v|| &= \sqrt{V^*V} = 100\sqrt{2}, \\ ||i|| &= \sqrt{I^*I} = 100, \\ ||i_r|| &= \sqrt{I_r^*I_r} = 30\sqrt{2}, \\ ||i_a|| &= \sqrt{I_a^*I_a} = 50\sqrt{2}, \\ ||i_s|| &= \sqrt{I_s^*I_s} = 40\sqrt{2}. \end{aligned}$$

Now, note that $I_r = -\frac{3}{20}WV - \frac{9}{20}W^{-1}V$.

Clearly, the obtained results correspond with the results obtained earlier. This also holds for the (coeffcients of the) compensator. However, the method with complex vectors and matrices is easier to use/extend in larger or more complicated situations than the method with the function spaces, described before.

VIII. Concluding remarks

In this paper, we presented a method to obtain the active, reactive and scattered power by orthogonal projection methods. The obtained powers can be used to improve the power factor of an electrical network. The method is based on orthogonally projecting the current demanded by the load onto a function space generated by the voltage supplied by the source. The method can be applied in both the time-domain and the frequency-domain. The frequency-domain alternative is computationally more practical and also able to deal with more complicated examples. It has been transformed from a method based on periodic functions into a method based on their complex Fourier coefficients. Basic to this transformation is Parseval's identity, relating the inner product used in this paper to the appropriate inner product in the space of Fourier coefficients. Furthermore, we have assumed that the load can be modeled as a finite-dimensional linear time-invariant system. However, the method also works in case the load is nonlinear or not time-invariant. In general, then an infinite number of terms in the Fourier series has to be taken into account. Also the notion of reactive power has then to be defined differently and the projection has to be modified.

In the examples, the compensator could be implemented by means of a inductor and capacitor. Although not so simple anymore, the latter holds in general, e.g., [13]. Finally, the assumption of an ideal voltage source, i.e., a voltage source without internal resistance, do not in essence seem to complicate the approach in this paper. Even with an internal resistance, the projection of the current required by the load onto the space of odd ordered (anti)derivatives of the source voltage can still be performed. However, now the voltage is not fixed, but is dependent on the current required by the load. The latter makes the method more involved and a closed form for the solution is the topic of future research.

In this paper, the non-sinusoidal situation is studied in which the source voltage contains a fundamental frequency and one or more of its harmonics. Hence, all frequencies present do have a common ground frequency. In many practical situations there will be frequencies present in the input voltage that do not have such a common ground frequency. In the context of this paper, one possible approach to deal with such a situation is to approximate all frequencies present in the signals by rational ones and to find a frequency that may serve as good as possible as common ground frequency for these rational approximates. This can be done in many ways, for instance by means of continued fraction expansions and the greatest common divisor techniques. Also this will be a topic of future research.

The present paper is inspired by the CPC approach in [6], which contains both time-domain and frequency domain aspects. For a version of the CPC approach that only requires time-domain considerations, we refer to [10] and [11].

References

[1] C.I. Budeanu. Puissances reactives et fictives. *Inst. Romain de l'Energie, Bucharest, Romania*, 1927.

[2] L.S. Czarnecki. Considerations on the reactive power in nonsinusoidal situations. *IEEE Trans. Instr. Meas.*, IM-34(3):399–404, 1984.

[3] L.S. Czarnecki. What is wrong with the Budeanu concept of reactive and distortion power and why it should be abandoned. *IEEE Trans. Instr. Meas.*, IM-36(3):834–837, 1987.

[4] L.S. Czarnecki. Budeanu and Fryze: Two frameworks for interpreting power properties of circuits with nonsinusoidal voltages and currents. *Arch. Elektrotechn.*, 80(6):359–367, 1997.

[5] L.S. Czarnecki. Energy flow and power phenomena in electrical circuits: illusions and reality. *Electrical Engineering (Archiv fur Elektrotechnik)*, 82:119–126, 2000.

[6] L.S. Czarnecki. Currents' physical components (CPC) concept: a fundamental power theory. *Electrical Review*, R84(6):28–37, 2008.

[7] A.E. Emmanuel. Power in nonsinusoidal situations: A review of definitions and physical meaning. *IEEE Trans. Power Delivery*, 5(3):1377–1383, 1990.

[8] E. García-Canseco, R. Griño, R. Ortega, M. Salichis, and A.M. Stanković. Power-factor compensation of electrical circuits. *IEEE Control Systems Magazine*, 99:46–59, 2007.

[9] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, 1934.

[10] D. Jeltsema and J. van der Woude. Currents' Physical Components (CPC) in the Time-Domain: Single-Phase Systems. in Proc. European Control Conference, Strasbourg, June 24-27, 2014.

[11] D. Jeltsema, J. van der Woude and M.T. Hartman. A Novel Time-Domain Perspective of the CPC Power Theory: Single-Phase Systems. arXiv:1403.7842

[12] A. Menti, T. Zacharias, and J. Milias-Argitis. Geometric algebra: A powerful tool for representing power under nonsinusoidal conditions. *IEEE Transactions on Circuits and Systems*, 54:601–609, 2007.

[13] A.J. van der Schaft, Time-reversible Hamiltonian systems. *IEEE System & Control letters*, 1:295–300, 1982.