

Control-Affine Systems Compatible with the Multi-Chained Form and Their x -Maximal Flatness

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Abstract—In this paper, we introduce the concept of x -maximal flatness. A control system is x -maximally flat if the number of new states gained by each successive derivation of the flat output is the largest possible. Firstly, we show that the only control-linear systems that are x -maximally flat are those that are static feedback equivalent to the m -chained form. Secondly, we generalize that result from control-linear systems to control-affine systems whose control-linear subsystem is static feedback equivalent to the m -chained form. We prove that they are x -maximally flat if and only if the drift exhibits a triangular form compatible with the m -chained form (and recently characterized in [1] and [2]). We also show that if we skip the assumption of the x -maximal flatness, the latter condition is not necessary for x -flatness of control-affine system whose associated control-linear subsystem is static feedback equivalent to the m -chained form.

I. INTRODUCTION

The notion of flatness introduced in control theory in the 1990's by Fliess, Lévine, Martin and Rouchon [3], [4], see also [5]–[7], has attracted a lot of attention because of its multiple applications in the problem of trajectory tracking and motion planning [8]–[14]. The fundamental property of flat systems is that all their solutions can be parametrized by m functions and their time-derivatives, m being the number of controls. More precisely, consider the nonlinear control system

$$\Xi : \dot{x} = F(x, u)$$

where x is the state defined on a open subset X of \mathbb{R}^n , u is the control taking values in an open subset U of \mathbb{R}^m (more generally, an n -dimensional manifold X and an m -dimensional manifold U , respectively) and the dynamics F are smooth (the word smooth will always mean C^∞ -smooth). The system Ξ is *flat* if we can find m functions, $\varphi_i(x, u, \dots, u^{(r)})$, for some $r \geq 0$, called *flat outputs*, such that

$$x = \gamma(\varphi, \dots, \varphi^{(s-1)}) \text{ and } u = \delta(\varphi, \dots, \varphi^{(s)}),$$

for a certain integer s , where $\varphi = (\varphi_1, \dots, \varphi_m)$. All state and control variables can thus be determined from the flat outputs without integration and all trajectories of the system can be completely parameterized.

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The differential weight of a flat output φ is, roughly speaking, the minimal number of derivatives of components of φ , needed to express x and u (see [12], [15], [16]). Here we propose another way of looking at that property. It is well known (see e.g. [5], [7], [9]) that for any $l \geq 0$, all time-derivatives $\varphi_i^{(j)}$, $1 \leq i \leq m$, $0 \leq j \leq l$, of flat outputs are independent. So the successive time-derivatives provide m new independent functions $\varphi_i^{(l+1)}$, $1 \leq i \leq m$. The problem that we are going to study is how many new functions of the state x do successive derivatives of the flat outputs provide? The system Ξ will be called *x -maximal flat* if each successive time-derivative of the flat output provides the largest possible number of independent functions of the state.

Observe, first, that x -maximally flat systems are simply static feedback linearizable systems (see Proposition 1). Secondly, we show that, within the class of control-linear systems, the only x -maximally flat systems are those that are static feedback equivalent to the m -chained form (see Proposition 2). Thirdly, we generalize that result from control-linear systems to control-affine systems whose control-linear subsystem is static feedback equivalent to the m -chained form. We prove that they are x -maximally flat if and only if the drift is triangular in the system of coordinates in which the controlled vector fields are in the m -chained form (see Theorem 1). In other words, they are x -maximally flat if and only if they are static feedback equivalent to a triangular form compatible with the m -chained form. That triangular form has been recently characterized by Silveira, Pereira da Silva and Rouchon [1] (for $m = 2$) and by the authors [2] for $m \geq 2$. We also show that if we skip the assumption of the x -maximal flatness, the compatibility condition is not necessary for x -flatness of control-affine system whose associated control-linear subsystem is static feedback equivalent to the m -chained form.

The paper is organized as follows. In Section II, we recall the definition of flatness, we introduce the notion of x -maximally flat system and we study the x -maximal flatness of general and then of control-linear systems. In Section III, we give our main result: we describe x -maximal flatness of control-affine systems whose control-linear subsystem is static feedback equivalent to the m -chained form. We illustrate our results by an example in Section IV and provide proofs in Section V.

II. PRELIMINARIES AND MOTIVATION

The fundamental property of flat systems is that all their solutions can be parametrized by a finite number of functions and their time-derivatives. Fix an integer $r \geq -1$ and denote $X^r = X \times U \times \mathbb{R}^{mr}$ and $\bar{u}^r = (u, \dot{u}, \dots, u^{(r)})$. For $r = -1$, we put $X^{-1} = X$ and \bar{u}^{-1} is empty.

Definition 1 *The system $\Xi : \dot{x} = F(x, u)$ is flat at $(x^*, \bar{u}^{r*}) \in X^r$, for $r \geq -1$, if there exists a neighborhood \mathcal{O}^r of (x^*, \bar{u}^{r*}) and m smooth functions $\varphi_i = \varphi_i(x, u, \dot{u}, \dots, u^{(r)})$, $1 \leq i \leq m$, defined in \mathcal{O}^r , having the following property: there exist an integer s and smooth functions γ_i , $1 \leq i \leq n$, and δ_j , $1 \leq j \leq m$, such that*

$$x_i = \gamma_i(\varphi, \dot{\varphi}, \dots, \varphi^{(s-1)}) \text{ and } u_j = \delta_j(\varphi, \dot{\varphi}, \dots, \varphi^{(s)})$$

along any trajectory $x(t)$ given by a control $u(t)$ that satisfy $(x(t), u(t), \dots, u^{(r)}(t)) \in \mathcal{O}^r$, where $\varphi = (\varphi_1, \dots, \varphi_m)$ is called a flat output.

In the particular case $\varphi_i = \varphi_i(x)$, for $1 \leq i \leq m$, we will say that the system is x -flat. In our study, the flat outputs will always depend on x only and r is 0 or -1.

The notion of differential weight of a flat system, introduced in [12], was discussed in [15], [16] in the context of system linearizable via one-fold prolongation. The differential weight of a flat output φ is, roughly speaking, the minimal number of derivatives of components of φ needed to express x and u and will be formalized as follows. By definition, for any flat output φ of Ξ there exist integers s_1, \dots, s_m such that

$$\begin{aligned} x &= \gamma(\varphi_1, \dot{\varphi}_1, \dots, \varphi_1^{(s_1)}, \dots, \varphi_m, \dot{\varphi}_m, \dots, \varphi_m^{(s_m)}) \\ u &= \delta(\varphi_1, \dot{\varphi}_1, \dots, \varphi_1^{(s_1)}, \dots, \varphi_m, \dot{\varphi}_m, \dots, \varphi_m^{(s_m)}), \end{aligned}$$

Moreover, we can choose (s_1, \dots, s_m) such that (see [12]) if for any other m -tuple $(\tilde{s}_1, \dots, \tilde{s}_m)$ we have

$$\begin{aligned} x &= \tilde{\gamma}(\varphi_1, \dot{\varphi}_1, \dots, \varphi_1^{(\tilde{s}_1)}, \dots, \varphi_m, \dot{\varphi}_m, \dots, \varphi_m^{(\tilde{s}_m)}) \\ u &= \tilde{\delta}(\varphi_1, \dot{\varphi}_1, \dots, \varphi_1^{(\tilde{s}_1)}, \dots, \varphi_m, \dot{\varphi}_m, \dots, \varphi_m^{(\tilde{s}_m)}), \end{aligned}$$

then $s_i \leq \tilde{s}_i$, for $1 \leq i \leq m$. We will call $\sum_{i=1}^m (s_i + 1) = \sum_{i=1}^m s_i + m$ the differential weight of φ . A flat output of Ξ is called *minimal* if its differential weight is the lowest among all flat outputs of Ξ . We define the *differential weight* of a flat system to be equal to the differential weight of a minimal flat output. Here we propose another way of looking at this property. Suppose that the control system $\Xi : \dot{x} = F(x, u)$ is flat at (x^*, \bar{u}^{r*}) and let $(\varphi_1, \dots, \varphi_m)$ be a flat output around (x^*, \bar{u}^{r*}) . It is well known (see e.g. [5], [7], [9]) that for any $l \geq 0$, all time-derivatives $\varphi_i^{(j)}$, $1 \leq i \leq m$, $0 \leq j \leq l$, of flat outputs are independent at (x^*, \bar{u}^{r*}) . So successive time-derivatives provide m new independent functions $\varphi_i^{(l+1)} = \varphi_i^{(l+1)}(x, u, \dot{u}, \dots, u^{(r+l+1)})$, $1 \leq i \leq m$. The problem that we are going to study is how many new functions of the state x do successive derivatives of the flat outputs provide?

To formalize that problem, for any $j \geq 0$, we denote

$$\begin{aligned} \Phi^j &= \text{span} \{d\varphi_i, \dots, d\varphi_i^{(j)}, 1 \leq i \leq m\}, \\ \mathcal{A}^j &= \Phi^j \cap T^*X \\ &= \text{span} \{d\varphi_i, \dots, d\varphi_i^{(j)}, 1 \leq i \leq m\} \cap T^*X, \end{aligned}$$

and define $a^j(x) = \dim \mathcal{A}^j(x)$. The vector $(a^0(x), a^1(x), \dots, a^\rho(x))$ will be called the x -growth vector of the nested sequence of codistributions $\Phi^0 \subset \Phi^1 \subset \dots \subset \Phi^\rho$ (equivalently, the growth vector of $\mathcal{A}^0 \subset \mathcal{A}^1 \subset \dots \subset \mathcal{A}^\rho$), where ρ is the smallest integer such that $\mathcal{A}^\rho = T^*X$. For two codistributions \mathcal{E} and \mathcal{F} , we define their pointwise intersection $\mathcal{E} \cap \mathcal{F}$ by $(\mathcal{E} \cap \mathcal{F})(x) = \mathcal{E}(x) \cap \mathcal{F}(x)$, for $x \in X$.

Definition 2 *A system Ξ flat at $(x^*, \bar{u}^{r*}) \in X^r$, for $r \geq -1$, is called x -maximally flat at (x^*, \bar{u}^{r*}) if there exists a flat output at (x^*, \bar{u}^{r*}) for which all codistributions \mathcal{A}^j do not depend on the control or the control derivatives and, in a neighborhood of x^* , the sequence $(a^0(x), a^1(x), \dots, a^\rho(x))$ is constant and the maximal possible among all flat systems for which $\dim U = m$ and $\dim X = n$.*

Flatness is closely related to the notion of feedback linearization. The control system $\Xi : \dot{x} = F(x, u)$ is linearizable by static feedback if it is equivalent via a diffeomorphism $z = \phi(x)$ and an invertible feedback transformation, $u = \psi(x, v)$, to a linear controllable system $\Lambda : \dot{z} = Az + Bv$. Jakubczyk and Respondek [17] and Hunt and Su [18] gave geometric necessary and sufficient conditions for a control system to be static feedback linearizable. It is well known that systems linearizable via invertible static feedback are flat. The expression of all states and controls uses the minimal possible, which is $n + m$, number of time-derivatives of the components of flat outputs φ_i . The following proposition gives an equivalent way to describe static feedback linearizable systems using the notion of x -maximal flatness.

Consider a control system $\Xi : \dot{x} = F(x, u)$, with m inputs and defined on a state space of dimension $n = km$. Let us first introduce some notations. To Ξ , we associate $\mathcal{F} = \{F_u : u \in U\}$, where $F_u = F(\cdot, u)$, i.e., \mathcal{F} stands for the family of all vector fields corresponding to constant controls u of Ξ . Define the following sequence of distributions on X : $\mathcal{D}^0(x, u) = \text{Im} \frac{\partial F}{\partial u}(x, u)$ and $\mathcal{D}^{i+1}(x, u) = \mathcal{D}^i(x, u) + \text{span} \{[F_u, g] : F_u \in \mathcal{F}, g \in \mathcal{D}^i\}$, for $i \geq 0$. If Ξ is a control-affine system, i.e., of the form $\dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x)$, we actually have $\mathcal{D}^0 = \text{span} \{g_1, \dots, g_m\}$ and $\mathcal{D}^{i+1} = \mathcal{D}^i + [f, \mathcal{D}^i]$.

Proposition 1 *The following conditions are equivalent:*

- (i) Ξ is x -maximally flat at (x^*, \bar{u}^{r*}) , for a certain $r \geq -1$;
- (ii) Ξ is x -maximally x -flat at x^* ;
- (iii) There exists a flat output of Ξ for which the x -growth vector is constant and equals $(m, 2m, \dots, km)$;

(iv) Ξ is static feedback equivalent to a linear system and, in particular, to the Brunovský canonical form

$$(Br) \begin{cases} \dot{z}_i^j &= z_i^{j+1} \\ \dot{z}_i^k &= v_i \end{cases}$$

where $1 \leq i \leq m$ and $1 \leq j \leq k-1$.

(v) The distribution \mathcal{D}^0 does not depend on u and for any $0 \leq i \leq k-1$, the distributions \mathcal{D}^i are involutive and of constant rank $(i+1)m$.

According to item (iii) of the above result, a control system is x -maximally flat if the number of new states (state functions) gained by successive derivations of the flat output is, at each step, the largest possible, which is m . For x -maximally flat systems, flatness and x -flatness coincide and moreover, both properties are equivalent to linearizability via an invertible static feedback transformation, and, in fact, one can bring the system into the Brunovský canonical form, see [19], with all controllability indices equal k . Item (v) recalls the geometric necessary and sufficient conditions for a general nonlinear control system to be static feedback linearizable, see [20]. If the considered control system is affine with respect to controls it is clear that \mathcal{D}^0 does not depend on u .

In general, a flat system is not linearizable by static feedback (with the exception of the single-input case, where flatness reduces to static feedback linearization, see [21]) and therefore it is not x -maximally flat. We can be interested, however, in x -maximal flatness within a particular class of systems \mathcal{C} . We will say that the system Ξ is x -maximally flat within the class \mathcal{C} if it satisfies the conditions of Definition 2 with the sequence (a^0, a^1, \dots, a^p) being the maximal possible among all flat systems belonging to the class \mathcal{C} for which $\dim U = m$ and $\dim X = n$. From now on, we will denote the number of controls by $m+1$ (and not by m) since, as we will see below, for all classes of systems that follow one control plays a particular role. Consider a control-linear system

$$\Sigma_{lin} : \dot{x} = \sum_{i=0}^m u_i g_i(x)$$

where the control u takes values in an open subset U of \mathbb{R}^{m+1} , the state space X is of dimension $n = km+1$ and g_0, \dots, g_m are smooth vector fields on X . To Σ_{lin} we associate the following distribution $\mathcal{G} = \text{span}\{g_0, \dots, g_m\}$. We define inductively the derived flag of \mathcal{G} by $\mathcal{G}^0 = \mathcal{G}$ and $\mathcal{G}^{i+1} = \mathcal{G}^i + [\mathcal{G}^i, \mathcal{G}^i]$, $i \geq 0$.

A flat control-linear system Σ_{lin} is never static feedback linearizable (unless the number of controls, $m+1$, equals the dimension of the state space) and therefore, according to Proposition 1, cannot admit a flat output with the x -growth vector $(m+1, 2(m+1), 3(m+1), \dots)$. In fact the x -growth vector may start with $m+1$ (if the system is x -flat) but, since the system is control-linear, the derivatives $\dot{\phi}_i$, for $0 \leq i \leq m$,

necessarily involve the control, hence the second component of the x -growth vector can be, at most, m . So the maximal possible x -growth vector is $(m+1, 2m+1, 3m+1, \dots, km+1)$ and it is, indeed, realized by control-linear systems static feedback equivalent to the m -chained form. An $(m+1)$ -input driftless control system Σ_{lin} , defined on a manifold X of dimension $km+1$, is said to be in the m -chained form if it is represented by

$$Ch_m^k \begin{cases} \dot{z}_0 = v_0 & \dot{z}_1^1 = z_1^2 v_0 & \dots & \dot{z}_m^1 = z_m^2 v_0 \\ & \dot{z}_1^2 = z_1^3 v_0 & & \dot{z}_m^2 = z_m^3 v_0 \\ & \vdots & & \vdots \\ & \dot{z}_1^{k-1} = z_1^k v_0 & \dots & \dot{z}_m^{k-1} = z_m^k v_0 \\ & \dot{z}_1^k = v_1 & \dots & \dot{z}_m^k = v_m \end{cases}$$

It is clear from this representation that one control, v_0 in this case, is indeed "special". To simplify the notations, from now on, z^i stands for $z^i = (z_1^i, \dots, z_m^i)$, for $1 \leq i \leq k$, and \bar{v} denotes the vector (v_1, \dots, v_m) . The problem of characterizing systems that are locally static feedback equivalent to the m -chained form has been studied and solved in [22] (see also [23]–[27]). It is immediate to see that systems locally feedback equivalent to the m -chained form are flat with $\varphi = (z_0, z_1^1, \dots, z_m^1)$ being a flat output, at any point $(z^*, v^*) \in X \times \mathbb{R}^{m+1}$ with $v_0^* \neq 0$, and in [12] all their minimal flat outputs have been described. Flat systems equivalent to Ch_m^k exhibit singularities in the control space defined by $U_{lin}^{sing}(x) = \{u(x) \in \mathbb{R}^{m+1} : \sum_{i=0}^m u_i(x) g_i(x) \in \mathcal{C}^1(x)\}$, where \mathcal{C}^1 is the characteristic distribution of \mathcal{G}^1 , see [12]. Clearly, $v_0^* = 0$ describes that singularity for Ch_m^k .

An invertible static feedback $u = \beta(x)\tilde{u}$ transforms the system Σ_{lin} into the form $\tilde{\Sigma}_{lin} : \dot{x} = \sum_{i=0}^m \tilde{u}_i \tilde{g}_i(x)$, where $\tilde{g} = g\beta$, with $g = (g_0, \dots, g_m)$ and $\tilde{g} = (\tilde{g}_0, \dots, \tilde{g}_m)$. To $\tilde{\Sigma}_{lin}$ we associate the $(k-1)$ -fold prolongation

$$\tilde{\Sigma}_{lin}^{(k-1,0,\dots,0)} : \begin{cases} \dot{x} &= y_1 \tilde{g}_0(x) + \sum_{i=1}^m u_i^p \tilde{g}_i(x) \\ \dot{y}_1 &= y_2 \\ &\vdots \\ \dot{y}_{k-2} &= y_{k-1} \\ \dot{y}_{k-1} &= u_{k-1}^p \end{cases}$$

with $y_1 = \tilde{u}_0$, $u_i^p = \tilde{u}_i$, for $1 \leq i \leq m$, obtained by prolonging $k-1$ times the control \tilde{u}_0 as $u_0^p = \tilde{u}_0^{(k-1)}$. Denote the drift and the controlled vector fields of the prolongation $\tilde{\Sigma}_{lin}^{(k-1,0,\dots,0)}$ by f_p and g_{pi} , $0 \leq i \leq m$, respectively. The distributions of the prolongation will be denoted using the subindex p , i.e., $\mathcal{D}_p^0 = \text{span}\{g_{p0}, \dots, g_{pm}\}$ and $\mathcal{D}_p^{i+1} = \mathcal{D}_p^i + [f_p, \mathcal{D}_p^i]$.

The following result characterizes control-linear systems that are locally static feedback equivalent to the m -chained form, from the point of view of x -maximal flatness.

Proposition 2 *The following conditions are equivalent:*

- (Lin 1) Σ_{lin} is x -maximally flat at (x^*, \bar{u}^{r*}) , for a certain $r \geq -1$, within the class of control-linear systems \mathcal{E} ;
- (Lin 2) Σ_{lin} is x -maximally x -flat at (x^*, u^*) within the class of control-linear systems \mathcal{E} ;
- (Lin 3) There exist a flat output of Σ_{lin} at (x^*, u^*) for which the x -growth is constant and equals $(m+1, 2m+1, 3m+1, \dots, km+1)$;
- (Lin 4) Σ_{lin} is locally, around x^* , static feedback equivalent to the m -chained form

$$Ch_m^k \left\{ \begin{array}{l} \dot{z}_0 = v_0 \quad \dot{z}^1 = z^2 v_0 \\ \dot{z}^2 = z^3 v_0 \\ \vdots \\ \dot{z}^{k-1} = z^k v_0 \\ \dot{z}^k = \bar{v} \end{array} \right.$$

and $u^* \notin U_{lin}^{sing}(x^*)$.

- (Lin 5) Σ_{lin} satisfies, around (x^*, u^*) , $u^* \notin U_{lin}^{sing}(x^*)$, the conditions:

(m-Ch1) $\mathcal{G}^{k-1} = TX$;

(m-Ch2) \mathcal{G}^{k-2} is of constant rank $(k-1)m+1$ and contains an involutive subdistribution \mathcal{L} that has constant corank one in \mathcal{G}^{k-2} ;

(m-Ch3) $\mathcal{G}^0(x^*)$ is not contained in $\mathcal{L}(x^*)$;

- (Lin 6) There exists, around x^* , an invertible static feedback transformation $u = \beta(x)\bar{u}$, bringing the system Σ_{lin} into the form $\tilde{\Sigma}_{lin} : \dot{x} = \sum_{i=0}^m \bar{u}_i \tilde{g}_i(x)$, such that for any $0 \leq i \leq k-2$, the intersections $\mathcal{D}_p^i \cap TX$ are involutive, of constant rank $m(i+1)$, and $\mathcal{D}_p^{k-1} \cap TX = TX$, where \mathcal{D}_p^i are the distributions of the $(k-1)$ -fold prolongation $\tilde{\Sigma}_{lin}^{(k-1,0,\dots,0)}$.

Proposition 2 states that the only control-linear systems that are x -maximally flat are those that are locally static feedback equivalent to the m -chained form and, as expected, x -maximal flatness and x -maximal x -flatness are equivalent. Conditions (m-Ch1)-(m-Ch3) are formally the same, independently of $m = 1$ or $m \geq 2$. Notice, however, they are checkable only if $m \geq 2$ because in that case \mathcal{L} , if it exists, is unique and can be calculated (see [23] and [2]). If $m = 1$, then an equivalent verifiable reformulation of the conditions (m-Ch1)-(m-Ch3) is, see [28]:

$$(m-Ch)' \dim \mathcal{G}^i(x) = \dim \mathcal{G}_i(x) = i+2, \text{ for } 0 \leq i \leq k-1, \text{ in a neighborhood of } x^*.$$

Conditions (m-Ch1)-(m-Ch3) characterize the m -chained form [23] (see also [22], [26]) and assure the existence of a change of coordinates $z = \phi(x)$ and of an invertible static feedback transformation of the form $u = \beta(x)\bar{u}$, after which the control vector fields are in the m -chained form. The set of singular controls U_{lin}^{sing} , i.e., the controls at which the system ceases to be flat, has been described in [26], where it was also shown that all singular controls u are mapped into $v = (v_0, \bar{v})$ such that $v_0 = 0$.

In item (Lin 6), the system $\tilde{\Sigma}_{lin}^{(k-1,0,\dots,0)}$ is obtained by prolonging $(k-1)$ -times the control \bar{u}_0 as $u_0^p = \bar{u}_0^{(k-1)}$ and it is clear that if we bring the original system Σ_{lin} into the m -chained form and we prolong the control v_0 , the associated prolongation verifies all conditions of (Lin 6). Moreover, in this case, it is easy to see that the associated $(k-1)$ -prolongation is, actually, static feedback linearizable. Since for any $i \geq 0$, $\mathcal{D}_p^i \cap TX$ are involutive, it can be shown that all distributions \mathcal{D}_p^i are, in fact, involutive and thus $\tilde{\Sigma}_{lin}^{(k-1,0,\dots,0)}$ is static feedback linearizable. Notice that item (Lin 6) is actually the dual of (Lin 3). Indeed, in the sequence of involutive distributions $\mathcal{D}_p^i \cap TX$ at each step we gain m new directions, which is the maximal possible and which is also the case for the x -growth vector (a^0, a^1, a^2, \dots) .

A natural question arises: under which conditions is x -maximal flatness of Σ_{lin} conserved if we perturb the system by adding a drift f , thus obtaining a control-affine system $\Sigma_{aff} : \dot{x} = f(x) + \sum_{i=0}^m u_i g_i(x)$? In other words, what are the conditions that the drift f should satisfy in order that the x -growth vector associated to Σ_{aff} (whose control-linear subsystem Σ_{lin} is static feedback equivalence to the m -chained form) is given by $(m+1, 2m+1, 3m+1, \dots, km+1)$? The next section of this paper answers that question and therefore generalizes Proposition 2 to the control-affine case.

III. MAIN RESULT

The purpose of this paper is to generalize Proposition 2 from control-linear systems Σ_{lin} to control-affine systems

$$\Sigma_{aff} : \dot{x} = f(x) + \sum_{i=0}^m u_i g_i(x)$$

defined on an open subset X of \mathbb{R}^{km+1} , with f and g_0, \dots, g_m smooth vector fields on X and such that the associated control-linear subsystem $\Sigma_{lin} : \dot{x} = \sum_{i=0}^m u_i g_i(x)$ satisfies Proposition 2.

In order to describe x -maximal flatness of control-affine systems whose control-linear subsystem is static feedback equivalent to the m -chained form, consider the following triangular form generalizing the m -chained form:

$$TCh_m^k \left\{ \begin{array}{l} \dot{z}_0 = v_0 \quad \dot{z}_i^1 = f_i^1(z_0, \bar{z}^2) + z_i^2 v_0 \\ \dot{z}_i^2 = f_i^2(z_0, \bar{z}^3) + z_i^3 v_0 \\ \vdots \\ \dot{z}_i^{k-1} = f_i^{k-1}(z_0, \bar{z}^k) + z_i^k v_0 \\ \dot{z}_i^k = v_i \end{array} \right.$$

where $1 \leq i \leq m$ and \bar{z}^j denotes $\bar{z}^j = (z_1^1, \dots, z_m^1, z_1^2, \dots, z_m^2, \dots, z_1^j, \dots, z_m^j)$, for $2 \leq j \leq k$. This form has been recently introduced and characterized by Silveira, Pereira da Silva and Rouchon [1] (for $m = 1$) and by the authors [2] for $m \geq 1$. It not only exhibits a formal compatibility of the triangular structure of the drift with the structure of the controlled chains but also a striking compatibility of its x -maximal flatness with

that of the m -chained form. This is seen in Theorem 1 below, which is the main result of the paper, where counterparts of conditions (Lin 1)-(Lin 6) are given as (Aff 1)-(Aff 6) for the control-affine case.

It is clear, see [2], that TCh_m^k is x -flat, with $\varphi = (z_0, z_1^1, \dots, z_m^1)$ being a flat output, at any point $(z^*, v^*) \in X \times \mathbb{R}^{m+1}$ satisfying $\text{rk } F^j(z^*) = m$, for $1 \leq j \leq k-1$, where F^j , for $1 \leq j \leq k-1$, is the $m \times m$ matrix given by $F_{iq}^j = \frac{\partial(f_q^j + z_q^{j+1}v_0^*)}{\partial z_i^{j+1}}$, for $1 \leq i, q \leq m$. Therefore, flat systems equivalent to TCh_m^k exhibit singularities in the control space (depending on the state) defined by (see [2])

$$U_{aff}^{sing}(x) = \cup_{i=0}^{k-2} U_{sing}^i(x),$$

with $U_{sing}^i(x) = \{u(x) \in \mathbb{R}^2 : \text{rk}(\mathcal{G}^i + [f + gu, \mathcal{L}^i])(x) < (i+2)m + 1\}$, for $0 \leq i \leq k-2$, where $gu = \sum_{i=0}^m u_i g_i$, the distribution $\mathcal{L}^i = \mathcal{C}^{i+1}$, for $0 \leq i \leq k-3$, is the characteristic distribution of \mathcal{G}^{i+1} and $\mathcal{L}^{k-2} = \mathcal{L}$ is the involutive subdistribution of corank one in \mathcal{G}^{k-2} , if $m \geq 2$ ($U_{sing}^{k-2}(x)$ is empty, if $m = 1$).

An invertible static feedback $u = \alpha(x) + \beta(x)\tilde{u}$, transforms the system Σ_{aff} into the form $\tilde{\Sigma}_{aff} : \dot{x} = \tilde{f}(x) + \sum_{i=0}^m \tilde{u}_i \tilde{g}_i(x)$, where $\tilde{f} = f + \alpha g$ and $\tilde{g} = g\beta$, with $g = (g_0, \dots, g_m)$ and $\tilde{g} = (\tilde{g}_0, \dots, \tilde{g}_m)$. To $\tilde{\Sigma}_{aff}$, we associate the $(k-1)$ -fold prolongation

$$\tilde{\Sigma}_{aff}^{(k-1,0,\dots,0)} \left\{ \begin{array}{l} \dot{x} = \tilde{f}(x) + y_1 \tilde{g}_0(x) + \sum_{i=1}^m u_i^p \tilde{g}_i(x) \\ \dot{y}_1 = y_2 \\ \vdots \\ \dot{y}_{k-2} = y_{k-1} \\ \dot{y}_{k-1} = u_0^p \end{array} \right.$$

with $y_1 = \tilde{u}_0$, $u_i^p = \tilde{u}_i$, for $1 \leq i \leq m$, obtained by prolonging $(k-1)$ -times the control \tilde{u}_0 as $u_0^p = \tilde{u}_0^{(k-1)}$. The linearizability distributions of the prolonged system $\tilde{\Sigma}_{aff}^{(k-1,0,\dots,0)}$ will be denoted using the subindex p , i.e., $\mathcal{D}_p^0 = \text{span}\{g_{p0}, \dots, g_{pm}\}$ and $\mathcal{D}_p^{i+1} = \mathcal{D}_p^i + [f_p, \mathcal{D}_p^i]$. Recall that z^j stands for $z^j = (z_1^j, \dots, z_m^j)$, for $1 \leq j \leq k$, and \bar{v} denotes the vector (v_1, \dots, v_m) .

Theorem 1 Consider the class \mathcal{C} of control-affine system $\Sigma_{aff} : \dot{x} = f(x) + \sum_{i=0}^m u_i g_i(x)$ whose control-linear subsystem $\Sigma_{lin} : \dot{x} = \sum_{i=0}^m u_i g_i(x)$ is static feedback equivalent to the m -chained form, that is, satisfies the conditions (m-Ch1)-(m-Ch3) of Proposition 2. For $\Sigma_{aff} \in \mathcal{C}$, the following conditions are equivalent:

- (Aff 1) Σ_{aff} is x -maximally flat at (x^*, \bar{u}^{r*}) , for a certain $r \geq -1$, within the class \mathcal{C} ;
- (Aff 2) Σ_{aff} is x -maximally x -flat at (x^*, u^*) within the class \mathcal{C} ;
- (Aff 3) There exists a flat output of Σ_{aff} at (x^*, u^*) for which the x -growth vector is constant and equals $(m+1, 2m+1, 3m+1, \dots, km+1)$ and all codistributions $\mathcal{A}^j(x)$, for $0 \leq j \leq k-1$, do not depend on the control or control derivatives;

(Aff 4) Σ_{aff} is locally, around x^* , static feedback equivalent to the triangular form TCh_m^k , compatible with the m -chained form, given by

$$TCh_m^k \left\{ \begin{array}{l} \dot{z}_0 = v_0 \quad \dot{z}^1 = f^1(z_0, z^1, z^2) + z^2 v_0 \\ \dot{z}^2 = f^2(z_0, z^1, z^2, z^3) + z^3 v_0 \\ \vdots \\ \dot{z}^{k-1} = f^{k-1}(z_0, z^1, \dots, z^{k-1}) + z^{k-1} v_0 \\ \dot{z}^k = \bar{v} \end{array} \right.$$

and $u^* \notin U_{aff}^{sing}(x^*)$;

(Aff 5) System Σ_{aff} satisfies, around (x^*, u^*) , with $u^*(x^*) \notin U_{aff}^{sing}(x^*)$, the following condition:

(m-Comp) $[f, \mathcal{C}^i] \subset \mathcal{G}^i$, for $1 \leq i \leq k-2$, where \mathcal{C}^i is the characteristic distribution of \mathcal{G}^i .

(Aff 6) There exists, around x^* , an invertible static feedback transformation $u = \alpha(x) + \beta(x)\tilde{u}$, bringing the system Σ_{aff} into the form $\tilde{\Sigma}_{aff} : \dot{x} = \tilde{f}(x) + \sum_{i=0}^m \tilde{u}_i \tilde{g}_i(x)$, such that for any $0 \leq i \leq k-2$, the intersections $\mathcal{D}_p^i \cap TX$ do not depend on y , are involutive, of constants rank $m(i+1)$ and $\mathcal{D}_p^{k-1} \cap TX = TX$, where \mathcal{D}_p^i are the distributions of the $(k-1)$ -fold prolongation $\tilde{\Sigma}_{aff}^{(k-1,0,\dots,0)}$.

Remarks: 1) We do not claim that Σ_{aff} satisfying one of the above conditions is x -maximally flat. Clearly, x -maximally flat control-affine systems are those that are static feedback linearizable, as assured by Proposition 1. The above theorem describes x -maximally flat systems within the class \mathcal{C} of control-affine ones whose control-linear subsystem is static feedback equivalent to the m -chained form.

2) Theorem 1 generalizes Proposition 2 and shows how x -maximal flatness of control-affine systems compatible with the m -chained form reminds, but also how it differs from, that of control-linear systems. As for control-linear systems, x -maximal flatness and x -maximal x -flatness are equivalent. Thus the x -growth vector starts with $m+1$, but since the control-linear subsystem is static feedback equivalent to the m -chained form, the second component can be at most m . Condition (Aff 6) of the above result is very similar to condition (Lin 6) of Proposition 2 but, in addition to (Lin 6), it requires that the involutive distributions $\mathcal{D}_p^i \cap TX$, associated to the prolongation $\tilde{\Sigma}_{aff}^{(k-1,0,\dots,0)}$, do not depend on $y = \tilde{u}_0$. For control-linear systems, adding that condition would be redundant, because it is a consequence of the involutivity and the proper growth vector of $\mathcal{D}_p^i \cap TX$, but for control-affine systems just involutivity and rank conditions do not give the desired triangular form. Actually, for any x -flat system whose prolongation $\tilde{\Sigma}_{aff}^{(k-1,0,\dots,0)}$ possesses involutive distributions $\mathcal{D}_p^i \cap TX$ of proper growth vector, the dependence (or not) on $y = \tilde{u}_0$ distinguishes between a general x -flat system and the class treated here.

3) According to item (Aff 4), the only x -maximally flat control-affine systems, compatible with the m -chain

form, are those that are static feedback equivalent to the triangular form TCh_m^k . Item (Aff 5), together with (m-Ch1)-(m-Ch3) assumed for the control-linear subsystem Σ_{lin} , provide an invariant geometric characterization of TCh_m^k . For two-inputs control systems, an equivalent description was given in [1]. In [2], the authors show that conditions (m-Ch1)-(m-Ch3) and (m-Comp) are necessary and sufficient for a control affine system to be static feedback equivalent to TCh_m^k , for any $m \geq 1$, and discuss flatness of that class of systems. While conditions (m-Ch1)-(m-Ch3) characterize the m -chained form, (m-Comp) takes into account the drift and gives the compatibility condition for the drift f to have the desired triangular form in the right coordinates, i.e., in those in which the controlled vector fields are in the m -chained form. The involutive subdistribution \mathcal{L} (which, for $m \geq 2$, is crucial for the m -chained form) is absent in the compatibility conditions, but plays a very important role in calculating minimal flat outputs and in describing singularities (see [2]). In order to verify the conditions (m-Ch1)-(m-Ch3), we have to verify whether the distribution \mathcal{G}^{k-2} contains an involutive subdistribution \mathcal{L} of corank one. Checkable necessary and sufficient conditions for the existence of \mathcal{L} (together with a construction), based on the work of Bryant [29], were given in [22] and is discussed in [2].

4) A natural question is whether the above theorem describes flat systems whose x -growth vector is $(m+1, 2m+1, 3m+1, \dots, km+1)$ (without assuming that their control-linear subsystem is static feedback equivalent to the m -chained form). The answer is negative and the problem of characterizing those systems will be discussed elsewhere.

5) Now assume that Σ_{aff} is x -flat with Σ_{lin} being static feedback equivalent to the m -chained form. Does Σ_{aff} satisfy the conditions of Theorem 1? In other words, are x -flat control-affine systems necessarily static feedback equivalent to TCh_m^k if the control-linear subsystem is static feedback equivalent to Ch_m^k ? The answer is negative as shown by the following example.

IV. EXAMPLE

Consider the following control-affine system whose associated distribution \mathcal{G}^0 is already in the chained form:

$$\Sigma : \begin{cases} \dot{z}_0 = v_0 & \dot{z}_1 = z_3 + z_2 v_0 \\ & \dot{z}_2 = -z_4 + z_3 v_0 \\ & \dot{z}_3 = b(z_0, z_1, z_2, z_3) + z_4 v_0 \\ & \dot{z}_4 = v_1 \end{cases}$$

where b is a smooth function non involving z_4 . Let us show that the pair $(\varphi_0, \varphi_1) = (z_0, z_1)$ is an x -flat output. Indeed, we have $\varphi_0 = z_0$, implying $\dot{\varphi}_0 = v_0$, and $\varphi_1 = z_1$, implying $\dot{\varphi}_1 = z_3 + z_2 \dot{\varphi}_0$ and $\ddot{\varphi}_1 = b(\varphi_0, \varphi_1, z_2, z_3) + z_3 \dot{\varphi}_0^2 + z_2 \ddot{\varphi}_0$. From these two relations, we express z_2 and z_3 , via the implicit function theorem, as: $z_2 = \gamma_2(\varphi_0^2, \varphi_1^2)$ and $z_3 = \gamma_3(\varphi_0^2, \varphi_1^2)$, where φ^j

denotes $(\varphi, \dot{\varphi}, \dots, \varphi^{(j)})$ and γ_2 and γ_3 are smooth functions. By differentiating z_3 , we deduce $z_4 = \gamma_4(\varphi_0^3, \varphi_1^3)$ which yields $v_1 = \delta_2(\varphi_0^4, \varphi_1^4)$. So we have determined all state and control variables with the help of φ_0 and φ_1 and their time-derivatives and it follows that $(\varphi_0, \varphi_1) = (z_0, z_1)$ is, indeed, an x -flat output. However, the first derivative of $\varphi = (\varphi_0, \varphi_1)$ gives no function depending only on the state z and the system is clearly not x -maximally flat. Moreover, the x -growth vector of the system is the maximal possible, i.e., equals $(m+1, 2m+1, \dots, km+1) = (2, 3, 4, 5)$, but the codistribution $\mathcal{A}^1 = \text{span}\{dz_0, dz_1, dz_3 + v_0 dz_2\}$ depends on the control. Equivalently, if we study the prolongation $\Sigma^{(3,0)}$ of the system, obtained by prolonging the control v_0 three times, we have $\mathcal{D}_p^1 \cap TX = \text{span}\{\frac{\partial}{\partial z_4}, y_1 \frac{\partial}{\partial z_3} - \frac{\partial}{\partial z_2}\}$, where $y_1 = v_0$, which clearly depends on the control. The above example shows that there are x -flat control-affine systems whose linear subsystem is static feedback equivalent to the m -chained form and whose drift is not compatible with the latter, i.e., the drift f does not admit the desired triangular form in the system of coordinates in which the controlled vector fields exhibit the m -chained structure.

V. PROOF OF THEOREM 1

(Aff 1) \Rightarrow (Aff 2). Assume that Σ_{aff} is x -maximally flat at (x^*, \bar{u}^{r*}) and let $(\varphi_0, \dots, \varphi_m)$ be a flat output such that the associated x -growth vector (a_0, a_1, a_2, \dots) is the maximal possible at any x in a neighborhood of x^* . We deduce immediately that $a_0 = m+1$ implying that all components φ_i of the flat output are functions of x only and thus the system is x -maximally x -flat.

(Aff 2) \Rightarrow (Aff 3). Assume that $\Sigma_{aff} : \dot{x} = f(x) + \sum_{i=0}^m u_i g_i(x)$ is x -maximally x -flat at (x^*, u^*) and let $\varphi = (\varphi_0, \dots, \varphi_m)$ be an x -flat output such that the associated x -growth vector (a_0, a_1, a_2, \dots) is the maximal possible at any x in a neighborhood of x^* . There exists open neighborhoods \mathcal{X} of x^* and \mathcal{U} of u^* such that φ is an x -flat output for any $(x, u) \in \mathcal{X} \times \mathcal{U}$.

Recall that the control-linear subsystem $\Sigma_{lin} : \dot{x} = \sum_{i=0}^m u_i g_i(x)$ is static feedback equivalent to the m -chained form. Thus $\bar{\mathcal{G}}^0$, the involutive closure of the distribution $\mathcal{G}^0 = \text{span}\{g_0, \dots, g_m\}$, satisfies $\bar{\mathcal{G}} = \mathcal{G}^{k-1} = TX$. Therefore, on an open and dense subset \mathcal{X}' of \mathcal{X} , for any flat output φ_i , $0 \leq i \leq m$, there exists at least one vector field g_j , $0 \leq j \leq m$, such that $L_{g_j} \varphi_i(x) \neq 0$. If not, then there exists i such that $L_{g_j} \varphi_i = 0$ on \mathcal{X}' , for $0 \leq j \leq m$, and by successive applications of Jacobi identity, it can be shown that $L_g \varphi_i = 0$ for any $g \in \mathcal{G}^{k-1} = TX$, implying that φ_i is identically zero, which contradicts flatness of $\varphi = (\varphi_0, \dots, \varphi_m)$. Consequently, a_1 can be at most $2m+1$ and the largest possible constant (see Definition 2) x -growth vector is $(m+1, 2m+1, 3m+1, \dots, km+1)$.

(Aff 3) \Rightarrow (Aff 4). Let $\varphi = (\varphi_0, \dots, \varphi_m)$ be a flat output at (x^*, u^*) such that condition (Aff 3) is satisfied. Since $a_0 = m+1$, it follows that $\varphi_i = \varphi_i(x)$,

$0 \leq i \leq m$ (in other words the system is actually x -flat in a neighborhood of x^*).

There exists an open neighborhood \mathcal{X} of x^* and an open neighborhood \mathcal{U} of u^* such that φ is an x -flat output at any $(x, u) \in \mathcal{X} \times \mathcal{U}$. Since the differentials of the components of flat outputs are independent at x^* , we can introduce new coordinates $z_0 = \varphi_0$, $z_i^1 = \varphi_i$, for $1 \leq i \leq m$, and complete them to a coordinate system $(z_0, z_1^1, \dots, z_m^1, z_1^2, \dots, z_1^k, \dots, z_m^k)$. We have just seen that for any flat output φ_i , $0 \leq i \leq m$, there exists at least one vector field g_j , $0 \leq j \leq m$, such that $L_{g_j} \varphi_i(x) \neq 0$, on an open and dense subset \mathcal{X}' of \mathcal{X} .

Let us now show that there exist integers i and j such that $L_{g_j} \varphi_i(x^*) \neq 0$. Suppose that for any flat output φ_i , we have $L_{g_j} \varphi_i(x^*) = 0$, for $0 \leq j \leq m$. We can always assume $u^* = 0$ (otherwise, apply the invertible feedback $\tilde{u} = u - u^*$ transforming u^* into $\tilde{u}^* = 0$). We get $\dot{\varphi}_i = f_i(z) + \sum_{j=0}^m g_i^j u_j$, for $0 \leq i \leq m$, where $g_i^j(z^*) = 0$, for $0 \leq i, j \leq m$. This yields $d\dot{\varphi}_i = df_i(z) + \sum_{j=0}^m (u_j dg_i^j + g_i^j du_j)$, which evaluated at $(z^*, u^*) = (z^*, 0)$ gives $d\dot{\varphi}_i(z^*, 0) = df_i(z^*)$, for $0 \leq i \leq m$. Thus $\Phi^1(z^*) = \text{span}\{d\varphi_i(z^*), d\dot{\varphi}_i(z^*), 0 \leq i \leq m\} = \text{span}\{dz_0, dz_i^1, df_i(z^*), 0 \leq i \leq m\}$ and is clearly of dimension $2m + 2$, because the differentials of flat outputs and their derivatives are independent everywhere. It follows that $\mathcal{A}^1(z^*) = \Phi^1(z^*) \cap T^*Z(z^*) = \Phi^1(z^*)$ and $a^1(z^*) = \dim \mathcal{A}^1(z^*) = 2m + 2$, contradicting the fact that a^1 is constant and equals $2m + 1$.

Without loss of generality, suppose $L_{g_0} \varphi_0(x^*) \neq 0$. After applying around x^* a suitable invertible feedback, $u = \alpha(x) + \beta(x)v$, transforming u^* into v^* , we get $\dot{\varphi}_0 = \dot{z}_0 = v_0$, $\dot{\varphi}_i = \dot{z}_i^1 = a_i^1(z) + b_i^1(z)v_0$, $1 \leq i \leq m$, where a_i^1 and b_i^1 are smooth functions. We continue to denote by f and by g_i , $0 \leq i \leq m$, the drift and, respectively, the controlled vector fields of the feedback modified system. Since the x -growth vector field grows always by m , it is immediate that $\frac{\partial \varphi_i^{(j)}}{\partial v_l} = 0$, for $1 \leq i, l \leq m$ and any $1 \leq j \leq k - 1$. Now, using the fact that the control-linear subsystem $\Sigma_{lin} : \dot{x} = \sum_{i=0}^m u_i g_i(x)$ is static feedback equivalent to the m -chained form, it can be shown that $\tilde{z}_0 = \varphi_0$, $\tilde{z}_i^1 = \varphi_i$, $\tilde{z}_i^2 = L_{g_0} \varphi_i$, \dots , $\tilde{z}_i^k = L_{g_0}^{k-1} \varphi_i$, for $1 \leq i \leq m$, is a valid local change of coordinates (to simplify notation, we continue to write z instead of \tilde{z}) in which Σ_{lin} is in the m -chained form, and after applying a suitable invertible feedback transformation, the system Σ_{aff} takes the form

$$\left\{ \begin{array}{l} \dot{z}_0 = v_0 \quad \dot{z}_i^1 = f_i^1(z) + z_i^2 v_0 \\ \quad \quad \quad \dot{z}_i^2 = f_i^2(z) + z_i^3 v_0 \\ \quad \quad \quad \vdots \\ \quad \quad \quad \dot{z}_i^{k-1} = f_i^{k-1}(z) + z_i^k v_0 \\ \quad \quad \quad \dot{z}_i^k = v_i \end{array} \right.$$

where $1 \leq i \leq m$. Since $dv_0 \in \Phi^1$, the codistribution $\mathcal{A}^1 = \Phi^1 \cap T^*X$ is given by $\mathcal{A}^1 =$

$\text{span}\{dz_0, dz_i^1, \omega_i, 1 \leq i \leq m\}$, where ω_i is the 1-form appearing as the i -th line of the vector $\Omega = (\frac{\partial f^1}{\partial z^2} + v_0 \text{Id}) dz^2 + \sum_{j=3}^k \frac{\partial f^1}{\partial z^j} dz^j$, where Id denotes the identity matrix, $\frac{\partial f^1}{\partial z^j}$ is the matrix $\frac{\partial f^1}{\partial z^j} = (\frac{\partial f_i^1}{\partial z_q^j})$, for $1 \leq i, q \leq k$, and $dz^j = (dz_1^j, dz_2^j, \dots, dz_m^j)^\top$, for $2 \leq j \leq m$. Notice that each ω_i can be written as $\omega_i = (\frac{\partial f_i^1}{\partial z_i^2} + v_0) dz_i^2 + \eta_i$, where η_i is a 1-form not involving v_0 . Since the codistribution $\mathcal{A}^1 = \text{span}\{dz_0, dz_i^1, (\frac{\partial f_i^1}{\partial z_i^2} + v_0) dz_i^2 + \eta_i, 1 \leq i \leq m\}$ does not depend on v , we have $\mathcal{A}^1(z, \bar{v}_0) = \mathcal{A}^1(z, \bar{v}_0)$, for any fixed $\bar{v}_0 \neq v_0$. It follows that $((\frac{\partial f_i^1}{\partial z_i^2} + \bar{v}_0) dz_i^2 + \eta_i) - ((\frac{\partial f_i^1}{\partial z_i^2} + v_0) dz_i^2 + \eta_i) = (\bar{v}_0 - v_0) dz_i^2 \in \mathcal{A}^1$, for $1 \leq i \leq m$, thus $dz_i^2 \in \mathcal{A}^1$ and $\eta_i \in \mathcal{A}^1$, for $1 \leq i \leq m$. From this and the fact that \mathcal{A}^1 is of rank $2m + 1$, we deduce $\mathcal{A}^1 = \text{span}\{dz_0, dz_i^1, dz_i^2, 1 \leq i \leq m\}$, and since $\eta_i \in \mathcal{A}^1, 1 \leq i \leq m$, it follows that η_i cannot involve dz_q^j , for $j \geq 3$. Hence $\frac{\partial f_i^1}{\partial z_q^j} = 0$, for $1 \leq i, q \leq m, 3 \leq j \leq k$, i.e., $f_i^1 = f_i^1(z_0, z^1, z^2)$, implying $\Omega = (\frac{\partial f^1}{\partial z^2} + v_0 \text{Id}) dz^2$, and the matrix $(\frac{\partial f^1}{\partial z^2} + v_0 \text{Id})$ is of full rank at (z^*, v_0^*) . By induction, we show that the drift f is triangular and that the regularity condition $u^* \notin U_{aff}^{sing}$ is satisfied.

(Aff 4) \Rightarrow (Aff 5). See [2].

(Aff 5) \Rightarrow (Aff 6). In [2], we have shown that conditions (m-Ch1)-(m-Ch3) and (m-Comp) of item (Aff 5) assure the existence of a change of coordinates $z = \phi(x)$ and of an invertible feedback transformation and $u = \alpha(x) + \beta(x)v$ that transform the system Σ_{aff} into TCh_m^k . Bring the system into TCh_m^k and prolong $(k - 1)$ -times the control v_0 . The obtained prolongation

$$\left\{ \begin{array}{ll} \dot{z}_0 = y_1 & \dot{z}_i^1 = f_i^1(z_0, \bar{z}^2) + z_i^2 y_1 \\ \dot{y}_1 = y_2 & \dot{z}_i^2 = f_i^2(z_0, \bar{z}^3) + z_i^3 y_1 \\ \vdots & \vdots \\ \dot{y}_{k-2} = y_{k-1} & \dot{z}_i^{k-1} = f_i^{k-1}(z_0, \bar{z}^k) + z_i^k y_1 \\ \dot{y}_{k-1} = v_0^p & \dot{z}_i^k = v_i^p \end{array} \right.$$

where $1 \leq i \leq m, y_1 = v_0$ and $v_i^p = v_i$, for $1 \leq i \leq m$, clearly satisfies (Aff 6).

(Aff 6) \Rightarrow (Aff 1). Assume that there exists, around x^* , an invertible static feedback transformation $u = \alpha(x) + \beta(x)\tilde{u}$, such that distributions \mathcal{D}_p^i associated to the $(k - 1)$ -fold prolongation $\tilde{\Sigma}_{aff}^{(k-1, 0, \dots, 0)}$, defined just before Theorem 1, do not depend on y , are involutive and of constants rank $m(i + 1)$, for any $0 \leq i \leq k - 2$, and $\mathcal{D}_p^{k-1} \cap TX = TX$. For simplicity of notation, we will drop the tildes.

Recall that the linear-control sub-system associated to Σ_{aff} is assumed to satisfy the conditions describing the m -chained form. An equivalent way to characterize the m -chained form is the following (see [23]): for

$0 \leq i \leq k-1$, each element \mathcal{G}^i of the derived flag has constant rank $(i+1)m+1$, contains an involutive subdistribution $\mathcal{L}^i \subset \mathcal{G}^i$ of corank one and each element \mathcal{G}_i of the Lie flag, where $\mathcal{G}_{i+1} = \mathcal{G}_i + [\mathcal{G}_0, \mathcal{G}_i]$ and $\mathcal{G}_0 = \text{span}\{g_0, \dots, g_m\}$, has constant rank $(i+1)m+1$. Moreover, the involutive subdistribution \mathcal{L}^i , for $0 \leq i \leq k-3$, is the characteristic distribution \mathcal{C}^{i+1} of \mathcal{G}^{i+1} , i.e., $\mathcal{C}^{i+1} = \mathcal{L}^i$. We will use that characterization to prove that control systems verifying item (Aff 6) are, in fact, feedback equivalent to TCh_m^k and hence, x -maximally flat. To this end, we will show that the involutive distributions $\mathcal{D}_p^i \cap TX$, for $0 \leq i \leq k-2$, are, in fact, of corank one in \mathcal{G}^i , so let us denote $\mathcal{L}^i = \mathcal{D}_p^i \cap TX$.

For $\Sigma_{aff}^{(k-1,0,\dots,0)}$, we have $\mathcal{D}_p^0 = \text{span}\{\frac{\partial}{\partial y_{k-1}}, g_i, 1 \leq i \leq m\}$, thus the distribution $\mathcal{L}^0 = \mathcal{D}_p^0 \cap TX = \text{span}\{g_i, 1 \leq i \leq m\}$ is involutive and of corank one in \mathcal{G}^0 . From this and since $\text{rk } \mathcal{G}^1 = \text{rk } \mathcal{G}_1 = 2m+1$, it follows that we necessarily have $\mathcal{G}^1 = \text{span}\{g_0, g_i, [g_0, g_i], 1 \leq i \leq m\}$, where all brackets $[g_0, g_i]$ are independent modulo \mathcal{G}^0 . We have $\mathcal{D}_p^1 = \text{span}\{\frac{\partial}{\partial y_{k-1}}, \frac{\partial}{\partial y_{k-2}}, g_i, ad_f g_i + y_1 [g_0, g_i], 1 \leq i \leq m\}$, thus the distribution $\mathcal{L}^1 = \mathcal{D}_p^1 \cap TX = \text{span}\{g_i, ad_f g_i + y_1 [g_0, g_i], 1 \leq i \leq m\}$, does not depend on y , is involutive and of rank $2m$. Since \mathcal{L}^1 does not depend on y , we have $\mathcal{L}^1(x, \tilde{y}_1) = \mathcal{L}^1(x, \bar{y}_1)$, for any fixed $\tilde{y}_1 \neq \bar{y}_1$. It follows that $(ad_f g_i + \tilde{y}_1 [g_0, g_i]) - (ad_f g_i + \bar{y}_1 [g_0, g_i]) = (\tilde{y}_1 - \bar{y}_1) [g_0, g_i] \in \mathcal{L}^1$, for $1 \leq i \leq m$, thus $\mathcal{L}^1 = \text{span}\{g_i, ad_f g_i, [g_0, g_i], 1 \leq i \leq m\}$. Since $\text{rk } \mathcal{G}^1 = \text{rk}(\text{span}\{g_0, g_i, [g_0, g_i], 1 \leq i \leq m\}) = 2m$, we obtain $ad_f g_i \in \mathcal{G}^1$ and we actually have $\mathcal{L}^1 = \text{span}\{g_i, [g_0, g_i], 1 \leq i \leq m\}$. Thus we have just shown that $[f, \mathcal{L}^0] \subset \mathcal{G}^1$. From the fact that \mathcal{L}^1 is involutive, we deduce that \mathcal{L}^1 is the corank one involutive subdistribution of \mathcal{G}^1 .

Repeating this argument, we prove that the involutive distributions $\mathcal{L}^i = \mathcal{D}_p^i \cap TX$, for $0 \leq i \leq k-2$, are of corank one in \mathcal{G}^i and $[f, \mathcal{L}^{i-1}] \subset \mathcal{G}^i$. According to the above remark, we deduce that $\mathcal{C}^i = \mathcal{L}^{i-1} = \mathcal{D}_p^{i-1} \cap TX$ and that we actually have $[f, \mathcal{L}^{i-1}] = [f, \mathcal{C}^i] \subset \mathcal{G}^i$. It follows that the system Σ_{aff} satisfies item (Aff 5) and thus, see [2], is static feedback equivalent the form TCh_m^k , which is clearly x -maximally flat.

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