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Property-preserving convergent sequences of invariant sets for linear discrete-time systems

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***Abstract*— New sequences of monotonically increasing sets are introduced, for linear discrete-time systems subject to input and state constraints. The elements of the set sequences are controlled invariant and admissible regions of stabilizability. They are generated from the iterative application of the inverse reachability mapping, its geometric generalization, called the inverse directional reachability mapping, and mappings constructed by parts of the one-step inverse reachability and the one-step inverse directional reachability set. The four proposed set sequences converge to the maximal region of stabilizability.**

I. INTRODUCTION

Several problems in control theory and its applications, especially when systems subject to constraints are involved, can be solved by assigning properties to subsets of the state and input space, related to the system under study [1]–[3]. For linear systems and when the input and constraint sets contain the origin in their interior, existence of polytopic invariant / controlled invariant sets is guaranteed for asymptotically stable / stabilizable systems.

A plethora of works deals with the computation of polytopic invariant sets in this setting. The proposed solutions can be divided in two groups. The first group exploits the algebraic necessary and sufficient existence conditions [4]–[8] of invariant sets and contractive sets. The second group of methods, in which the approach presented in this article also belongs, consists in applying iteratively set mappings [9]–[19].

In detail, the latter group of methods provides set sequences which converge to the maximal invariant set or the maximal region of stabilizability. For example, under a controllability assumption, application of the inverse reachability mapping starting from the zero vector converges to the maximal region of stabilizability of the system [10], [11]. Moreover, the set sequence with initial element the state constraint set, generated by application of the inverse reachability mapping intersected with the state constraint set, converges to the maximal controlled invariant set [14]. An appropriately modified set-sequence converges to the maximal controlled λ-contractive set. The aforementioned works, e.g. [11], [14], are the only, to the best of the authors' knowledge, systematic approaches for computing the N-step safe sets or the N-step controllability sets. However, they might not constitute the most suitable choice in the different setting of computing, or approximating if this is not possible, the admissible region of stabilizability.

In this article, we propose a new set mapping which is not based on N-step reachability computations, but rather exploits the geometric properties of the sets involved in addition to the dynamics of the system. Intuitively, given a controlled invariant set which is also a region of stabilizability, application of the proposed mapping recovers all vectors that can be transferred to the convex hull of the set and their previous position. The set sequence induced by the inverse directional reachability set mapping is proven to converge to the region of stabilizability in the general setting, i.e., for n-th order dimensional systems. Additionally, to tackle the induced computational burden of computing the newly defined set mapping, we also provide alternative set sequences, which are constructed from parts of the inverse reachability and inverse directional reachability sets. These alternative set sequences are strictly monotonic with respect to the set inclusion relation and converge to the maximal admissible region of stabilizability as well.

Section II provides the notation and basic definitions. In Section III a motivating example regarding a family of first order systems is presented. In Section IV, the set mappings that are utilized in the article are formally introduced. The main results are presented in Section V, while conclusions are drawn in Section VI.

II. PRELIMINARIES

Let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ denote the field of real numbers, the set of non-negative reals and the set of nonnegative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$ we define the set $\Pi_{\geq c} := \{k \in \Pi : k \geq c\}$. The set $\Pi_{\leq c}$ is defined similarly. Moreover, we define $\mathbb{R}_\Pi := \Pi$ and $\mathbb{N}_\Pi := \mathbb{N} \cap \Pi$. Given two real matrices $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{n\times m}$, the inequality $A \leq B$ ($A < B$) holds componentwise. The i-th row and the j-th column of a matrix $H \in \mathbb{R}^{n\times m}$ are denoted by $h_i^\top$, $i \in \mathbb{N}_{[1,n]}$ and h^j, $j \in \mathbb{N}_{[1,m]}$ respectively. The identity matrix is denoted by $I_n \in \mathbb{R}^{n\times n}$, the zero matrix is denoted by $0_{n\times m} \in \mathbb{R}^{n\times m}$ and the vector with all elements equal to one is denoted by $1_n \in \mathbb{R}^n$. A proper $\mathcal{C}$-set $\mathcal{S} \subset \mathbb{R}^n$ is a compact and convex set which contains the origin in its interior. Given a set $\mathcal{S} \subset \mathbb{R}^n$ and a real scalar $\alpha \in \mathbb{R}$, the set $\alpha\mathcal{S}$ is defined by $\alpha\mathcal{S} := \{x \in \mathbb{R}^n : (\exists y \in \mathcal{S} : x = \alpha y)\}$. Given two sets $\mathcal{A} \subset \mathbb{R}^n, \mathcal{B} \subset \mathbb{R}^n$, the strict inclusion $\mathcal{A} \subset \mathcal{B}$ holds if and only if $a \in \mathcal{A}$ implies $a \in \mathcal{B}$ and there exists at least one $b^\star \in \mathcal{B}$ such that $b^\star \notin \mathcal{A}$. The boundary, the interior and

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the closure of a set $\mathcal{S} \subset \mathbb{R}^n$ are denoted by $\partial\mathcal{S}$, $\mathrm{interior}(\mathcal{S})$ and $\mathrm{closure}(\mathcal{S})$ respectively. The set of all subsets of $\mathbb{R}^n$ is denoted by $\mathcal{R}^n$. For a set $\mathcal{S} \subset \mathbb{R}^n$ with a finite number of elements, the number of its elements is denoted by $\mathrm{card}(\mathcal{S})$. For the empty set $\emptyset$, it holds that $\mathrm{card}(\emptyset) := 0$. The convex hull of a set $\mathcal{S} \subset \mathbb{R}^n$ will be denoted by $\mathrm{conv}(\mathcal{S})$.

A polytope is the bounded intersection of a finite number of closed half-spaces. Proper $\mathcal{C}$-polytopic sets are described in half-space or vertex representation. Adopting the notation from [20], the half-space representation of a proper $\mathcal{C}$-polytopic set $\mathcal{S} \subset \mathbb{R}^n$ is

$$P(G) := \{x \in \mathbb{R}^n : Gx \leq 1_p^\top\}, \tag{1}$$

for a suitable full column-rank matrix $G \in \mathbb{R}^{p\times n}$. The vertex representation of a proper $\mathcal{C}$-polytopic set $\mathcal{S} \subset \mathbb{R}^n$ is

$$Q(V) := \mathrm{conv}(\{v^i\}_{i\in\mathbb{N}_{[1,q]}}), \tag{2}$$

where v^i, $i \in \mathbb{N}_{[1,q]}$ are the columns of a suitable full row-rank matrix $V \in \mathbb{R}^{n\times q}$. If $q \in \mathbb{N}_{\geq n+1}$ is the smallest integer needed to describe the set $\mathcal{S}$ in form (2), the vectors v^i, $i \in \mathbb{N}_{[1,q]}$, are called the vertices of the polytope $\mathcal{S}$. Consider a proper $\mathcal{C}$-polytopic set described in half-space representation (1). Given an index set $\mathcal{I} \subset \mathbb{N}_{[1,p]}$, a face of the set $\mathcal{S}$ is any nonempty set of the form $\{x \in \mathcal{S} : g_i^\top x = 1, i \in \mathcal{I}\}$. The faces of dimensions $0, 1, n-2$ and $n-1$ are called vertices, edges, ridges and facets, respectively. The convex hull of a proper $\mathcal{C}$-polytopic set $\mathcal{S} \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$ is denoted by $\mathcal{S}^x$, i.e.,

$$\mathcal{S}^x := \mathrm{conv}(\{\{v^i\}_{i\in\mathbb{N}_{[1,q]}}, x\}). \tag{3}$$

We consider linear discrete-time systems with inputs

$$x_{t+1} = Ax_t + Bu_t, \tag{4}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$ and $t \in \mathbb{N}$ is the time variable.

The state and input vectors are confined in proper C-polytopic sets $\mathcal{X} \subset \mathbb{R}^n, \mathcal{U} \subset \mathbb{R}^m$, of the form

$$\mathcal{X} := P(G_x) := Q(V_x), \tag{5}$$
$$\mathcal{U} := P(G_u) := Q(V_u), \tag{6}$$

where $G_x \in \mathbb{R}^{p_x\times n}$, $V_x \in \mathbb{R}^{n\times q_x}$, $G_u \in \mathbb{R}^{p_u\times m}$, $V_u \in \mathbb{R}^{m\times q_u}$.

Assumption 1 *The matrix pair (A, B) is stabilizable.*

Assumption 2 *The state constraint set $\mathcal{X} \subset \mathbb{R}^n$ and the input constraint set $\mathcal{U} \subset \mathbb{R}^m$ are proper $\mathcal{C}$-polytopic sets, described by* (5) *and* (6) *respectively.*

The definition of the admissible controlled invariant set [21] and the admissible region of stabilizability, which is similar to the definition of the Ω-invariant set in [11], follows.

Definition 1 *Consider the system* (4) *and the state and input constraints $\mathcal{X}$* (5) *and $\mathcal{U}$* (6) *respectively. The set $\mathcal{S} \subseteq \mathcal{X}$ is called an admissible controlled invariant set, or simply, a controlled invariant set, if and only if there exists an admissible state-feedback control $u = f(x)$, $f(\cdot) : \mathcal{S} \to \mathcal{U}$, such that for any initial state $x_0 \in \mathcal{S}$ the corresponding trajectory x_t of the resulting closed-loop system satisfies $x_t \in \mathcal{S}$, for all $t \in \mathbb{N}$.*

Definition 2 *Consider the system* (4) *and the state and input constraints $\mathcal{X}$* (5) *and $\mathcal{U}$* (6) *respectively. The set $\mathcal{S} \subseteq \mathcal{X}$ is called an admissible region of stabilizability, or simply, a region of stabilizability, if and only if there exists an admissible state-feedback control $u = f(x)$, $f(\cdot) : \mathcal{X} \to \mathcal{U}$, such that for any initial state $x_0 \in \mathcal{S}$ the corresponding trajectory x_t of the resulting closed-loop system satisfies $x_t \in \mathcal{X}$, for all $t \in \mathbb{N}$, and moreover, $\lim_{t\to\infty} \|x_t\| = 0$.*

The definitions of the maximal admissible controlled invariant set and the maximal admissible region of stabilizability follow.

Definition 3 *Consider the system* (4) *and the state and input constraints $\mathcal{X}$* (5) *and $\mathcal{U}$* (6) *respectively. The set $\mathcal{S}_\mathrm{m} \subseteq \mathcal{X}$ is called the maximal admissible controlled invariant set, or simply, the maximal controlled invariant set, if and only if $\mathcal{S}_\mathrm{m}$ is controlled invariant, and moreover, for any controlled invariant set $\mathcal{S} \subset \mathcal{X}$, $\mathcal{S} \neq \mathcal{S}_\mathrm{m}$, it holds that $\mathcal{S} \subset \mathcal{S}_\mathrm{m}$.*

Definition 4 *Consider the system* (4) *and the state and input constraints $\mathcal{X}$* (5) *and $\mathcal{U}$* (6) *respectively. The set $\mathcal{M} \subseteq \mathcal{X}$ is called the maximal admissible region of stabilizability, or simply, the maximal region of stabilizability, if and only if $\mathcal{M}$ is a region of stabilizability, and moreover, for any region of stabilizability $\mathcal{S} \subset \mathcal{X}$, $\mathcal{S} \neq \mathcal{M}$, it holds that $\mathcal{S} \subset \mathcal{M}$.*

The following Proposition clarifies the relationship between the maximal controlled invariant set $\mathcal{S}_\mathrm{m}$ and the maximal region of stabilizability $\mathcal{M}$, under Assumptions 1 and 2.

Proposition 1 *Suppose that Assumptions 1 and 2 hold. Consider the system* (4) *and the state and input constraints $\mathcal{X}$*(5) *and $\mathcal{U}$* (6) *respectively. The following statements hold.*

i. *The set $\mathcal{S}_\mathrm{m}$ is a proper $\mathcal{C}$-set.*
ii. $\mathcal{M} \subseteq \mathcal{S}_\mathrm{m}$.
iii. $\mathrm{interior}(\mathcal{S}_\mathrm{m}) \subseteq \mathcal{M}$.
iv. $\mathrm{closure}(\mathcal{M}) = \mathcal{S}_\mathrm{m}$.

Proof sketch: (i) This statement can be proven by [14, Propositions 3.1, 3.2, Theorem 3.1], setting $\lambda = 1$. (ii) By definition of the region of stabilizability, for any $x_0 \in \mathcal{M}$, there exists a sequence $\{u_i\}_{i\in\mathbb{N}}$ such that $u_i \in \mathcal{U}$, $x_i \in \mathcal{X}$, for all $i \in \mathbb{N}$, and $\lim_{i\to\infty} \|x_i\| = 0_n$. Thus, since $\mathcal{M}$ is the maximal region of stabilizability, it follows that $x_i \in \mathcal{M}$, for all $i \in \mathbb{N}$, for all $x_0 \in \mathcal{M}$. Thus, $\mathcal{M}$ is controlled invariant. Since $\mathcal{S}_\mathrm{m}$ is the maximal controlled invariant set, then relation $\mathcal{M} \subseteq \mathcal{S}_\mathrm{m}$ necessarily holds. (iii) The statement can be induced from [22, Lemma 12] (moreover, it can be proven without modifications that for any proper $\mathcal{C}$-set $\mathcal{S}$ which is also controlled invariant, $\mathrm{interior}(\mathcal{S})$ is a region of stabilizability). (iv) From Proposition 1(ii) and 1(iii), it follows that $\text{interior}(\mathcal{S}_\text{m}) \subseteq \mathcal{M} \subseteq \mathcal{S}_\text{m}$. Since both $\mathcal{S}_\text{m}$ and $\mathcal{M}$ are convex, bounded, contain the origin in their interior, and, moreover, by Proposition 1(i), $\mathcal{S}_\text{m}$ is closed, it follows that $\text{closure}(\mathcal{M}) = \mathcal{S}_\text{m}$. □

III. A MOTIVATING EXAMPLE

Consider the following family of scalar linear discrete-time systems

$$x_{t+1} = ax_t + bu_t,$$

where $a := 1 + \delta$, $b := \delta$, $\delta > 0$. The system is subject to state constraints $x \in \mathcal{X} := \mathbb{R}_{[-\beta,\beta]}$ and input constraints $u \in \mathcal{U} := \mathbb{R}_{[-\gamma,\gamma]}$, with $0 < \gamma < \beta$.

We investigate the problem of computing the maximal controlled invariant set $\mathcal{S}_\text{m} \subset \mathcal{X}$ and the maximal region of stabilizability $\mathcal{M} \subset \mathcal{X}$. For the simple system under consideration, it is straightforward to verify that $\mathcal{S}_\text{m} = \mathbb{R}_{[-\gamma,\gamma]}$ and $\mathcal{M} = \mathbb{R}_{(-\gamma,\gamma)}$.

First, consider the method in [11], which consists in computing monotonically expanding inner invariant approximations $\mathcal{Y}_i$, $i \in \mathbb{N}$, of the admissible region of stabilizability $\mathcal{M}$. For this family of systems, the sets $\{\mathcal{Y}_i\}_{i\in\mathbb{N}}$ can be computed analytically, i.e., $\mathcal{Y}_0 := \{0\}$ and

$$\mathcal{Y}_i = \left[-b\gamma \sum_{j=1}^{i} a^{-j}, \quad b\gamma \sum_{j=1}^{i} a^{-j} \right],$$

for all $i \in \mathbb{N}_{\geq 1}$. The sets $\mathcal{Y}_i$ are also the i-step controllability sets. They contain the states that can be transferred to the origin in at most i steps with admissible control actions[1].

Next, consider the algorithm which computes monotonically shrinking outer, non-invariant, approximations $\mathcal{Z}_i$, $i \in \mathbb{N}$, of the maximal controlled invariant set [14]. The sets $\{\mathcal{Z}_i\}_{i\in\mathbb{N}}$ can be analytically computed after some manipulations as well, i.e., $\mathcal{Z}_0 := \mathcal{X}$ and

$$\mathcal{Z}_i := \left[-\beta a^{-i} - b\gamma \sum_{j=1}^{i} a^{-j}, \quad \beta a^{-i} + b\gamma \sum_{j=1}^{i} a^{-j} \right],$$

for all $i \in \mathbb{N}_{\geq 1}$. The sets $\mathcal{Z}_i$ are also the i-step safe sets and they include the states for which there exists an admissible control sequence such that they remain in $\mathcal{X}$ for at least i instants.

For this example, *there do not exist finite integers* $i^\star \in \mathbb{N}, j^\star \in \mathbb{N}$ *such that* $\mathcal{Y}_{i^\star} = \mathcal{M}$, $\mathcal{Z}_{j^\star} = \mathcal{S}_m$. In fact, $\mathcal{Y}_i \subset \mathcal{M}$, for all $i \in \mathbb{N}$, while $\mathcal{Z}_j \supset \mathcal{S}_\text{m}$, for all $j \in \mathbb{N}$. Thus, the standard methods converge only asymptotically. Nevertheless, convergence to the limit of the aforementioned set sequences is uniform, thus an ϵ-close approximation of the limit is possible. However, this might be achieved after arbitrarily many iterations, which will be the case in the studied setting when δ is arbitrarily small.

Let us explore an alternative way of approximating the maximal admissible region of stabilizability. First, consider a symmetric controlled invariant set which is also a region of stabilizability $\mathcal{S}_0 = \{x \in \mathbb{R} : -\varepsilon \leq x \leq \varepsilon\}$, for a suitable scalar $\varepsilon \in \mathbb{R}_{>0}$. Such a set can be computed e.g. by taking $\mathcal{S}_0 := \mathcal{Y}_1$ [10], or by exploiting the methods presented in [5]–[8].

The idea is to retrieve all points $x^\star$ in $\mathcal{X}$ for which there exist admissible inputs $u^\star \in \mathcal{U}$ that drive $x^\star$ to the convex hull of $x^\star$ and $\mathcal{S}_0$, i.e., $ax^\star + bu^\star \in \mathcal{S}_0^{x^\star}$, and, moreover, $ax^\star + bu^\star \neq x^\star$. Due to the convexity of the input and state constraint sets $\mathcal{U}$ and $\mathcal{X}$ and the linearity of the dynamics, the resulting set $\mathcal{S}_1$, calculated as the convex union of these points and the set $\mathcal{S}_0$, will also be controlled invariant and a region of stabilizability. The procedure can be applied iteratively until no further addition of points is possible.

In order to apply the proposed construction to the scalar example under study, it suffices to find the "farthest" points from the set $\mathcal{S}_0$ that have the aforementioned properties, for all directions in $\mathbb{R}$, i.e., $\{-1, +1\}$. First, the direction -1 is considered. Thus, the smallest scalar $x^\star \in \mathcal{X}$ for which there exists a control input $u^\star$ such that $ax^\star + bu^\star \in \mathbb{R}_{(x^\star,\varepsilon]}$ is calculated. To this end, we compute a triplet $(\lambda^\star, u^\star, x^\star) \in \mathbb{R}_{[0,1)} \times \mathcal{U} \times \mathcal{X}$ such that $ax^\star + bu^\star = \lambda^\star x^\star + (1 - \lambda^\star)\varepsilon$, or, $x^\star = \frac{bu^\star - (1-\lambda^\star)\varepsilon}{\lambda^\star - a}$. Since the scalars b, ε, a are positive and $\lambda^\star$ is nonnegative, the minimum value is obtained at $x^\star = -\gamma$ with $u^\star = \gamma$, by solving the related linear programming problem. Consequently, the set $\mathcal{S}_{\{-1\}}$, where $\mathcal{S}_{\{-1\}} = \{x \in \mathbb{R} : x^\star < x \leq \varepsilon\} = \{x \in \mathbb{R} : -\gamma < x \leq \varepsilon\}$, is a controlled invariant set and a region of stabilizability.

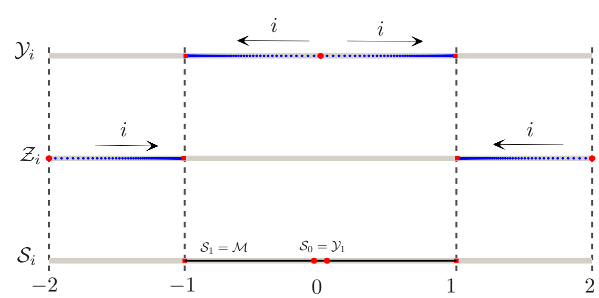

Fig. 1. The extreme points of the set sequences $\{\mathcal{Y}_i\}_{i\in\mathbb{N}_{[1,100]}}$, $\{\mathcal{Z}_i\}_{i\in\mathbb{N}_{[1,100]}}$ and $\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[0,1]}}$. The extreme points of the sets $\mathcal{Y}_0$, $\mathcal{Z}_0$ and $\mathcal{S}_0$ are shown in red circles, the extreme points of the sets $\mathcal{Y}_i, \mathcal{Z}_i$, $i \in \mathbb{N}_{[1,99]}$ are shown in blue dots and the extreme points of the sets $\mathcal{Y}_{100}$, $\mathcal{Z}_{100}$ and $\text{closure}(\mathcal{S}_1)$ are shown in red squares. The maximal region of stabilizability $\mathcal{M}$ is shown in black line, while the state constraint set $\mathcal{X}$ is shown in grey lines.

Following the same steps for the direction $+1$, we obtain similarly the set $\mathcal{S}_{\{+1\}} = \{x \in \mathbb{R} : -\varepsilon \leq x < \gamma\}$ which is also controlled invariant and a region of stabilizability. Computing the convex union of the two sets, we have

$$\mathcal{S}_1 := \mathcal{S}_{\{-1\}} \cup \mathcal{S}_{\{+1\}} = \{x \in \mathbb{R} : -\gamma < x < \gamma\} = \mathcal{M}.$$

Consequently, the maximal region of stabilizability $\mathcal{M}$ is calculated in one sweep through all directions in $\mathbb{R}$, inde-

[1]The system is controllable in the studied example. If the system is stabilizable but not controllable, $\mathcal{Y}_0$ can be a region of stabilizability. For this case, $\mathcal{Y}_i$ will contain the states that can be transferred to $\mathcal{Y}_0$ in at most i steps.

pendently of the value of δ. This comes in contrast to the existing approaches [11], [14] that converge asymptotically to $\mathcal{M}$ and $\mathcal{S}_{\rm m}$ respectively.

To illustrate the set sequences in a numerical example, we set $\delta = 0.05, \gamma = 1, \beta = 2$. In Figure 1, the extreme points of the elements of the set sequences $\{\mathcal{Y}_i\}_{i\in\mathbb{N}_{[1,99]}}$ and $\{\mathcal{Z}_i\}_{i\in\mathbb{N}_{[1,99]}}$ are shown in blue dots. Moreover, the extreme points of the starting sets $\mathcal{Y}_0 = \{0\}$, $\mathcal{Z}_0 := \mathbb{R}_{[-2,2]}$ $\mathcal{S}_0 := \mathcal{Y}_1 = \mathbb{R}_{[-0.0476,0.0476]}$, are shown in red circles while the extreme points of the sets $\mathcal{Y}_{100}$, $\mathcal{Z}_{100}$ and $\mathrm{closure}(\mathcal{M})$ are shown in red squares.

IV. Set mappings

In this section, we define the set mappings utilized in this article to construct convergent sequences of controlled invariant sets which are regions of stabilizability.

The *one-step admissible inverse reachability mapping* $\mathcal{C}(\cdot,\cdot,\cdot) : \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^n$, or simply inverse reachability mapping, associated to the system (4), is

$$\mathcal{C}(\mathcal{S},\mathcal{X},\mathcal{U}) := \{x \in \mathcal{X} : (\exists (y,u) \in \mathcal{S}\times\mathcal{U} : y = Ax+Bu)\}. \tag{7}$$

The *one-step admissible forward reachability mapping* $\mathcal{F}(\cdot,\cdot,\cdot) : \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^n$, or simply forward reachability mapping, associated to the system (4), is

$$\mathcal{F}(\mathcal{S},\mathcal{X},\mathcal{U}) := \{x \in \mathcal{X} : (\exists (y,u) \in \mathcal{S}\times\mathcal{U} : x = Ay+Bu)\}. \tag{8}$$

Let $f(\cdot) : \mathcal{X} \to \mathcal{U}$ be an admissible state-feedback control law. Then, the closed-loop system is of the form

$$x_{t+1} = Ax_t + Bf(x_t). \tag{9}$$

The one-step admissible forward reachability mapping $\mathcal{F}_f(\cdot,\cdot) : \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n$, or simply forward reachability mapping, associated to the closed-loop system (9), is

$$\mathcal{F}_f(\mathcal{S},\mathcal{X}) := \{x \in \mathcal{X} : (\exists y \in \mathcal{S} : x = Ay + Bf(y))\}. \tag{10}$$

From (8) and (10), it follows directly that for any admissible control law $u = f(x)$, the relation

$$\mathcal{F}_f(\mathcal{S},\mathcal{X}) \subseteq \mathcal{F}(\mathcal{S},\mathcal{X},\mathcal{U}) \tag{11}$$

holds. Next, we define the inverse directional reachability mapping. The *one-step inverse directional reachability mapping* $\mathcal{D}(\cdot,\cdot,\cdot) : \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^n$, associated to the system (4), is

$$\mathcal{D}(\mathcal{S},\mathcal{X},\mathcal{U}) := \{x \in \mathcal{X} : (\exists (y,u) \in \mathcal{S}^x \setminus \{x\} \times \mathcal{U} : y = Ax + Bu)\}. \tag{12}$$

Intuitively, for each element x of the set $\mathcal{D}(\mathcal{S},\mathcal{X},\mathcal{U})$, there exists an input vector $u \in \mathcal{U}$ such that the trajectory of the system in one step is included in the convex hull of that element and the set $\mathcal{S}$, and is different from x. This is a distinct difference from the set $\mathcal{C}(\mathcal{S},\mathcal{X},\mathcal{U})$, where the trajectory of the system must be transferred in one step in the set $\mathcal{S}$. The definitions of the reduced inverse reachability mapping and the reduced directional inverse reachability mapping for proper $\mathcal{C}$-polytopic sets follow.

Let $\mathcal{S} \subset \mathbb{R}^n$ be a proper $\mathcal{C}$-polytopic set, described in half-space and vertex representation (1) and (2) respectively. The *one-step reduced inverse reachability mapping* $\mathcal{C}_{\rm R}(\cdot,\cdot,\cdot) : \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^n$, associated to the system (4), is

$$\mathcal{C}_{\rm R}(\mathcal{S},\mathcal{X},\mathcal{U}) := \mathrm{conv}(\{v^i\}_{i\in\mathbb{N}_{[1,q]}}, \{y^i\}_{i\in\mathbb{N}_{[1,p]}}), \tag{13}$$

where each of the vectors $y^i \in \mathcal{C}(\mathcal{S},\mathcal{X},\mathcal{U})$, $i \in \mathbb{N}_{[1,p]}$, satisfies the relation

$$g_i^\top y^i \geq g_i^\top x, \tag{14}$$

for all $x \in \mathcal{C}(\mathcal{S},\mathcal{X},\mathcal{U})$.

Similarly, the *one-step reduced inverse directional reachability mapping* $\mathcal{D}_{\rm R}(\cdot,\cdot,\cdot) : \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^n$, associated to the system (4), is

$$\mathcal{D}_{\rm R}(\mathcal{S},\mathcal{X},\mathcal{U}) := \mathrm{conv}(\{v^i\}_{i\in\mathbb{N}_{[1,q]}}, \{z^i\}_{i\in\mathbb{N}_{[1,p]}}), \tag{15}$$

where each of the vectors $z^i \in \mathcal{D}(\mathcal{S},\mathcal{X},\mathcal{U})$, $i \in \mathbb{N}_{[1,p]}$, satisfies the relation

$$g_i^\top z^i \geq g_i^\top x, \tag{16}$$

for all $x \in \mathcal{D}(\mathcal{S},\mathcal{X},\mathcal{U})$.

The k-th iterated set mapping $\mathcal{C}^k(\cdot,\cdot,\cdot) : \mathcal{R}^n\times\mathcal{R}^n\times\mathcal{R}^m \to \mathcal{R}^n$, $k \in \mathbb{N}$, is defined as follows. For $k = 0$, it holds that $\mathcal{C}^0(\mathcal{S},\mathcal{X},\mathcal{U}) := \mathcal{S}$ by convention. For $k = 1$, $\mathcal{C}^1(\mathcal{S},\mathcal{X},\mathcal{U}) := \mathcal{C}(\mathcal{S},\mathcal{X},\mathcal{U})$, as defined in (7), while for $k \in \mathbb{N}_{\geq 2}$ it holds that $\mathcal{C}^k(\mathcal{S},\mathcal{X},\mathcal{U}) := \mathcal{C}(\mathcal{C}^{k-1}(\mathcal{S},\mathcal{X},\mathcal{U}),\mathcal{X},\mathcal{U})$. The k-iterated set mappings of (8), (10), (12), (13)-(14) and (15)-(16) are defined similarly.

V. Main results

In this section, several useful properties of the set-mappings defined in the previous section are established, along with the main convergence results of the induced set sequences. The results that follow concern properties of the set mappings (7), (12), (13)-(14) and (15)-(16).

Fact 1 *Let $\mathcal{S} \subset \mathbb{R}^n$ be a proper $\mathcal{C}$-polytopic set and $\xi \in \mathbb{R}^n$ a vector. Then,* (i) *the set $\mathcal{S}^\xi$ is a proper $\mathcal{C}$-polytopic set,* (ii) *for all $\xi \notin \mathcal{S}$ it holds that $\mathcal{S} \subset \mathcal{S}^\xi$, and* (iii) *for all $\xi \in \mathcal{S}$ it holds that $\mathcal{S} = \mathcal{S}^\xi$.*

Lemma 1 *Consider a controlled invariant proper $\mathcal{C}$-polytopic set $\mathcal{S} \subset \mathbb{R}^n$, which is also a region of stabilizability with respect to the system* (4) *and the state and input constraints* (5) *and* (6) *respectively. The following statements hold.*

i. $\mathcal{S} \subseteq \mathcal{C}(\mathcal{S},\mathcal{X},\mathcal{U})$.
ii. $\mathcal{S} \subseteq \mathcal{C}(\mathcal{F}(\mathcal{S},\mathcal{X},\mathcal{U}),\mathcal{X},\mathcal{U})$.
iii. $\mathcal{S} \subseteq \mathcal{D}(\mathcal{S},\mathcal{X},\mathcal{U})$.
iv. $\mathcal{C}(\mathcal{S},\mathcal{X},\mathcal{U}) \subseteq \mathcal{D}(\mathcal{S},\mathcal{X},\mathcal{U})$.
v. *Let $\mathcal{X}_1 \subset \mathbb{R}^n$, $\mathcal{X}_2 \subset \mathbb{R}^n$, be proper $\mathcal{C}$-sets such that $\mathcal{S} \subseteq \mathcal{X}_1 \subseteq \mathcal{X}_2$. Then, $\mathcal{C}(\mathcal{S},\mathcal{X}_1,\mathcal{U}) \subseteq \mathcal{C}(\mathcal{S},\mathcal{X}_2,\mathcal{U})$.*
vi. *Let set $\mathcal{S}$ be also a polytopic set. Then, $\mathcal{S} \subseteq \mathcal{C}_{\rm R}(\mathcal{S},\mathcal{X},\mathcal{U})$ and $\mathcal{S} \subseteq \mathcal{D}_{\rm R}(\mathcal{S},\mathcal{X},\mathcal{U})$.*
vii. *Let $f(\cdot) : \mathcal{X} \to \mathcal{U}$ be an admissible state-feedback control law for the system* (4)*. Then, $\mathcal{S} \subseteq \mathcal{C}(\mathcal{F}_f(\mathcal{S},\mathcal{X}),\mathcal{X},\mathcal{U})$.*

viii. *Let $\mathcal{S}_1 \subset \mathcal{X}$, $\mathcal{S}_2 \subset \mathcal{X}$, be proper $\mathcal{C}$-sets such that $\mathcal{S}_1 \subseteq \mathcal{S}_2$. Then, $\mathcal{F}(\mathcal{S}_1, \mathcal{X}, \mathcal{U}) \subseteq \mathcal{F}(\mathcal{S}_2, \mathcal{X}, \mathcal{U})$.*

ix. *Let $\mathcal{S}_1 \subset \mathcal{X}$, $\mathcal{S}_2 \subset \mathcal{X}$, be proper $\mathcal{C}$-sets such that $\mathcal{S}_1 \subseteq \mathcal{S}_2$. Then, $\mathcal{C}(\mathcal{S}_1, \mathcal{X}, \mathcal{U}) \subseteq \mathcal{C}(\mathcal{S}_2, \mathcal{X}, \mathcal{U})$.*

Most of the statements of Lemma 2 can be proven by hypothesis and the definitions of the set mappings (8), (10), (12), (13)-(14) and (15)-(16).

The next result establishes that the set mappings (7), (12) preserve the properties of controlled invariance and stabilizability.

Lemma 2 *Consider a controlled invariant proper $\mathcal{C}$-polytopic set $\mathcal{S} \subset \mathbb{R}^n$, which is also a region of stabilizability with respect to the system* (4) *and the state and input constraints* (5) *and* (6) *respectively. The following statements hold.*

i. *The set $\mathcal{C}(\mathcal{S}, \mathcal{X}, \mathcal{U})$ is a controlled invariant set and a region of stabilizability with respect to the system* (4) *and the constraints* (5)*,* (6)*.*

ii. *The set $\mathrm{conv}(\mathcal{D}(\mathcal{S}, \mathcal{X}, \mathcal{U}))$ is a controlled invariant set and a region of stabilizability with respect to the system* (4) *and the state and input constraints* (5)*,* (6)*.*

Proof sketch: (i) The statement can be proven using the definition of the one-step inverse reachability mapping. (ii) From Lemma 1(iii), it holds directly that $\mathcal{S} \subseteq \mathcal{D}(\mathcal{S}, \mathcal{X}, \mathcal{U})$, thus, it is sufficient to investigate the case when $\mathcal{S} \subset \mathcal{D}(\mathcal{S}, \mathcal{X}, \mathcal{U})$. First, it is proven that the sets $\mathcal{S}^x$, $x \in \mathcal{D}(\mathcal{S}, \mathcal{X}, \mathcal{U}) \setminus \mathcal{S}$, are controlled invariant and regions of stabilizability. This is done by constructing an admissible control strategy that drives any trajectory starting from $\mathcal{S}^x \setminus \mathcal{S}$ in the interior of $\mathcal{S}^x$ after a finite number of iterations, without leaving the set $\mathcal{S}^x$. This is sufficient to show that $\mathcal{S}^x$ is a region of stabilizability, see e.g. [23]. Next, it is shown that for any pair $(x_1, x_2) \in \mathcal{D}(\mathcal{S}, \mathcal{X}, \mathcal{U}) \times \mathcal{D}(\mathcal{S}, \mathcal{X}, \mathcal{U})$, the set $\mathrm{conv}(\mathcal{S}^{x_1}, \mathcal{S}^{x_2})$ is controlled invariant and a region of stabilizability. The result then follows. □

Consider the proper $\mathcal{C}$-polytopic set in half-space description $P(H)$, $H \in \mathbb{R}^{p_0 \times n}$,

$$\mathcal{S}_0 := \{x \in \mathbb{R}^n : Hx \leq \mathbf{1}_{p_0}\}. \tag{17}$$

The first two main theoretical results of the article are reported.

Theorem 1 *Let the proper $\mathcal{C}$-set $\mathcal{S}_0$ be a controlled invariant set and an admissible region of stabilizability with respect to the system* (4) *and the state and input constraints $\mathcal{X}$* (5) *and $\mathcal{U}$* (6) *respectively. Consider the set sequence $\mathcal{S}_{i+1} := \mathcal{C}(\mathcal{S}_i, \mathcal{X}, \mathcal{U})$. Then, $\mathcal{M} = \lim_{i\to\infty} \mathcal{S}_i$.*

Proof sketch: First, it is proven that the set sequence $\{\mathcal{S}_i\}_{i\in\mathbb{N}}$ is monotonically increasing and property-preserving using Lemma 1(i) and Lemma 2(i). Moreover, since the sequence is upper bounded, its limit, say $\mathcal{S}^\star \subseteq \mathcal{X}$, exists, see e.g. [24]. To prove that its limit coincides with $\mathcal{M}$, it suffices to prove by contradiction that $\mathcal{S}^\star$ necessarily contains any region of stabilizability. □

Theorem 2 *Let the proper $\mathcal{C}$-set $\mathcal{S}_0$ be a controlled invariant set and an admissible region of stabilizability with respect to the system* (4) *and the state and input constraints $\mathcal{X}$* (5) *and $\mathcal{U}$* (6) *respectively. Consider the set sequence $\mathcal{S}_{i+1} = \mathrm{conv}(\mathcal{D}(\mathcal{S}_i, \mathcal{X}, \mathcal{U}))$. Then, $\mathcal{M} = \lim_{i\to\infty} \mathcal{S}_i$.*

Proof sketch: As in Theorem 1, it can be easily verified that the set sequence is monotonically increasing, property-preserving, and has a well defined limit $\mathcal{S}^\star \subseteq \mathcal{X}$. To prove that its limit coincides with $\mathcal{M}$, it suffices to observe that each term of the sequence is lower bounded by the equivalent term of the sequence generated by application of the one-step inverse reachability mapping with the same initial condition. Thus, taking into account Theorem 1, the sequence converges to the maximal region of stabilizability. □

Let $\mathcal{S} \subset \mathbb{R}^n$ be a proper $\mathcal{C}$-polytopic set described in half-space representation $P(G)$ (1), where $G \in \mathbb{R}^{p\times n}$. For all $k \in \mathbb{N}_{[1,n]}$, we consider the index sets $\mathcal{I}_j^k$, $\mathrm{card}(\mathcal{I}_j^k) = k$, $j \in \mathbb{N}_{[1,c_k]}$, which generate all non-empty $(n-k)$-dimensional faces $\mathcal{T}_j(\mathcal{S}, k)$ of the set $\mathcal{S}$, defined by

$$\mathcal{T}_j(\mathcal{S}, k) := \{x \in \mathcal{S} : g_i^\top x = 1, i \in \mathcal{I}_j^k\}, \tag{18}$$

for all $(j,k) \in \mathbb{N}_{[1,c_k]} \times \mathbb{N}_{[1,n]}$. For each $k \in \mathbb{N}_{[1,n]}$, we denote the set which contains all non-empty $(n-k)$-dimensional facets of $\mathcal{S}$ with $\mathcal{P}_k(\mathcal{S})$, i.e.,

$$\mathcal{P}_k(\mathcal{S}) := \{\mathcal{T}_j(\mathcal{S}, k)\}_{j\in\mathbb{N}_{[1,c_k]}}. \tag{19}$$

The set that contains all the non-empty faces of the proper $\mathcal{C}$-polytopic set $\mathcal{S}$ is

$$\Omega(\mathcal{S}) := \{\mathcal{P}_k(\mathcal{S})\}_{k\in\mathbb{N}_{[1,n]}}. \tag{20}$$

The elements of $\Omega(\mathcal{S})$ together with the relation $\subseteq$ form a finite partially ordered set $(\Omega(\mathcal{S}), \subseteq)$, called the face lattice of $\mathcal{S}$ and describes the set uniquely [20]. The following standard result is of use.

Fact 2 *Consider two proper $\mathcal{C}$-polytopic sets $\mathcal{S}_1 \subset \mathbb{R}^n$, $\mathcal{S}_2 \subset \mathbb{R}^n$. Then, the sets satisfy the relation $\mathcal{S}_1 = \mathcal{S}_2$ if and only if $\Omega(\mathcal{S}_1) = \Omega(\mathcal{S}_2)$.*

The following two results associate the set sequences generated from the application of the one-step reduced inverse reachability mapping (13),(14) and the one-step reduced inverse directional reachability mapping (15),(16) with the elements of the sequences generated by application of the standard inverse reachability mapping (7). These intermediate results are required for proving Theorems 3 and 4.

Lemma 3 *Let $\mathcal{S} \subset \mathbb{R}^n$, $\mathcal{S}_0 \subset \mathbb{R}^n$, be proper $\mathcal{C}$-polytopic sets defined in* (1),(17)*, such that $\mathcal{S}_0 \subset \mathcal{S}$. Consider the vectors y^i, $i \in \mathbb{N}_{[1,p_0]}$, which satisfy the relations*

$$y^i = \arg\max_y \{h_i^\top x : x \in \mathcal{S}\}. \tag{21}$$

Then, there exists at least one index $i^\star \in \mathbb{N}_{[1,p_0]}$, such that

$$y^{i^\star} \in \Omega(\mathcal{S}) \setminus \Omega(\mathcal{S}_0). \tag{22}$$

Proof sketch: For all $i \in \mathbb{N}_{[1,p_0]}$, construct the sets $\mathcal{E}_i$, where $\mathcal{E}_i := \{x \in \mathbb{R}^n : Gx \leq 1_p,\ h_i^\top x \geq 1\}$. It can be proven that there exists $i^\star \in \mathbb{N}_{[1,p_0]}$ such that $\mathrm{interior}(\mathcal{E}_{i^\star}) \neq \emptyset$, and moreover, $y^{i^\star} \in \Omega(\mathcal{E}_{i^\star})$. Next, it is proven by contradiction that $y^{i^\star} \notin \Omega(\mathcal{S}_0)$, and consequently, $y^{i^\star} \in \Omega(\mathcal{S})$. □

Lemma 4 *Let $\mathcal{S}_0$ be a proper $\mathcal{C}$-polytopic set, which is controlled invariant with respect to the system* (4) *and the constraints* (5) *and* (6)*. Then, there exists a finite integer $k^\star \in \mathbb{N}$ such that*

$$C(\mathcal{S}_0, \mathcal{X}, \mathcal{U}) \subseteq \mathcal{C}_{\mathrm{R}}^{k^\star}(\mathcal{S}_0, \mathcal{X}, \mathcal{U}). \tag{23}$$

Proof sketch: The proof relies heavily on Lemma 3. In specific, the core of the proof is to show that each iterated mapping $\mathcal{C}_R^i(\mathcal{S}_0, \mathcal{X}, \mathcal{C}(\mathcal{S}_0, \mathcal{X}, \mathcal{U}))$, $i \in \mathbb{N}_{\geq 1}$, of the set $\mathcal{S}_0$, shares at least i faces with the set $\mathcal{C}(\mathcal{S}_0, \mathcal{X}, \mathcal{U})$. Then, the result follows by exploiting Lemma 1(v) and Fact 2. □

Lemma 5 *Let $\mathcal{S}_0$ be a proper $\mathcal{C}$-polytopic set, which is controlled invariant with respect to the system* (4) *and the constraints* (5) *and* (6)*. Then, there exists a finite integer $k^\star \in \mathbb{N}$ such that*

$$\mathcal{C}(\mathcal{S}_0, \mathcal{X}, \mathcal{U}) \subseteq \mathcal{D}_{\mathrm{R}}^{k^\star}(\mathcal{C}, \mathcal{X}, \mathcal{U}). \tag{24}$$

The proof of Lemma 5 follows the same steps with the proof of Lemma 4. The next results establish that the set sequences induced by the mappings (13)-(14), (15)-(16), starting from any proper $\mathcal{C}$-polytopic set $\mathcal{S} \subseteq \mathcal{X}$ which is controlled invariant and a region of stabilizability, converge to the maximal region of stabilizability $\mathcal{M}$.

Theorem 3 *Let the proper $\mathcal{C}$-set $\mathcal{S}_0$ be a controlled invariant set and an admissible region of stabilizability with respect to the system* (4) *and the state and input constraints $\mathcal{X}$* (5) *and $\mathcal{U}$* (6) *respectively. Consider the set sequence $\mathcal{S}_{i+1} := \mathcal{C}_R(\mathcal{S}_i, \mathcal{X}, \mathcal{U})$. Then, $\mathcal{M} = \lim_{i\to\infty} \mathcal{S}_i$.*

Proof sketch: First, it is verified that the set sequence is monotonically increasing, property-preserving, and has a well defined limit. The key idea of the proof is to extract a subsequence $\{\mathcal{S}_{k_i}\}_{i\in\mathbb{N}}$ whose elements satisfy the relations $\mathcal{S}_{k_i} \supseteq \mathcal{C}^i(\mathcal{S}_0, \mathcal{X}, \mathcal{U})$, for all $i \in \mathbb{N}$. Then, the statement can be proven by utilizing results in set series convergence analysis, e.g. [24], for series which are upper and lower bounded by the same limit. □

Theorem 4 *Let the proper $\mathcal{C}$-set $\mathcal{S}_0$ be a controlled invariant set and an admissible region of stabilizability with respect to the system* (4) *and the state and input constraints $\mathcal{X}$* (5) *and $\mathcal{U}$* (6) *respectively. Consider the set sequence $\mathcal{S}_{i+1} := \mathcal{D}_R(\mathcal{S}_i, \mathcal{X}, \mathcal{U})$. Then, $\mathcal{M} = \lim_{i\to\infty} \mathcal{S}_i$.*

The proof of Theorem 4 is similar to the one of Theorem 3.

Remark 1 *Apart from being regions of stabilizability, all elements of the set sequences in Theorems 1,2,3 and 4 preserve the controlled invariance property. Thus, the proposed set sequences can be used to tackle other relevant control engineering problems such as the constrained regulation problem of a preassigned set of initial conditions and problems that require directional spatial expansions of invariant sets.*

VI. Conclusions

Four novel convergent set sequences were introduced for linear discrete-time systems subject to input and state constraints. The set sequences converge to the maximal region of stabilizability. All the elements of the sequences preserve the property of controlled invariance and are regions of stabilizability. This makes their use appealing in several relevant constrained control problems, see e.g. [25]–[27].

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