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Stabilization of Infinite Dimensional Systems by State Feedback with Positivity Constraints

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Abstract— **This paper deals with the stabilization of infinite dimensional linear systems by state feedback with a positivity constraint on the state components. It is assumed that the open loop system is positive and unstable. The positive stabilization problem consists mainly of a) analyzing necessary and/or sufficient conditions on the system parameters that guarantee its positive stabilizability, i.e. the existence of a stabilizing state feedback operator preserving positivity, and b) developing useful methods for getting such a state feedback law. The main part of this contribution is the description and the analysis of a specific method for computing a positively stabilizing state feedback, under suitable conditions on the nominal system. The main feature of this method is the fact that it guarantees that the closed loop dynamics are nonnegative by designing a control law such that the resulting input trajectory is nonnegative.**

I. INTRODUCTION

Positive linear systems are linear dynamical systems whose state trajectories are nonnegative for every nonnegative initial state and for all nonnegative input functions. Equivalently a linear dynamical system is positive whenever the nonnegative orthant of the state-space is invariant under the corresponding state transition map (positive invariance). Positive systems are of great practical importance, as the nonnegativity property occurs quite frequently in practical applications where the state variables correspond to quantities that do not have real meaning unless they are nonnegative, e.g. human population, cells number, concentration, etc, see e.g. [11] and the references therein. The importance of the positivity property for infinite-dimensional (state-space) systems has been notably revealed by the stocking and industrial systems which involve chemical reactions and heat exchangers, e.g. distributed parameter models of tubular reactors, [12].

For finite dimensional systems, different approaches and many issues concerning positive system analysis and control have been studied by several authors: e.g. reachability, stabilization and optimal control, see e.g. [11], [4], [5] and references therein. As positive linear systems are defined on cones and not on linear spaces, the theory of infinite-dimensional positive systems is more complicated than the theory of standard semigroup state-space systems, see e.g. [10], [7]. The characterization of generators of positive C_0-semigroups without any metric condition is in general a difficult problem. In particular, the positive off-diagonal property is known to characterize positive C_0-semigroups on an ordered Banach space, provided that the positive cone of the latter has a nonempty interior. In [1] - [2], algebraic conditions (among which the positive off-diagonal property) on the infinitesimal generator of a positive C_0-semigroup were studied in the case of a space whose positive cone may have an empty interior.

This paper deals with the positive stabilization problem, i.e. the stabilization of infinite dimensional linear systems by state feedback with a positivity constraint on the state components. In this framework, it is basically assumed that the open loop system is positive and unstable. Therefore the positive stabilization problem consists mainly of a) analyzing necessary and/or sufficient conditions on the system parameters that guarantee its positive stabilizability, i.e. the existence of a stabilizing state feedback operator preserving positivity, i.e. such that the resulting closed loop system is positive, and b) developing useful methods for getting such a state feedback law. A method for solving this problem was developed in [1] - [2]: this method is based on the decomposition of the state space as the direct sum of a stable (infinite-dimensional) subspace and an unstable (finite-dimensional) subspace corresponding to the structure of the spectrum of the dynamics generator.

The main part of this contribution is the description and the analysis of a specific method for computing a positively stabilizing state feedback, under suitable conditions on the nominal system. The main feature of this method is the fact that it guarantees that the closed loop dynamics are nonnegative by designing a control law such that the resulting input trajectory is nonnegative. This method is based on a finite-dimensional spectrum assignment technique and on the resolution of a specific algebraic (invariance) equation, which lead to the stability and to the positivity of the closed-loop system, respectively. The main results are illustrated by a numerical example.

II. PROBLEM STATEMENT AND MAIN RESULT

A. Problem Statement

The problem that is studied in this paper is the positive stabilization problem for a specific class of distributed parameter systems. More precisely, we consider infinite dimensional

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(state-space) systems described by abstract linear differential equations of the following form:

$$\left\{ \begin{array}{lcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ x(0) & = & x_0 \in D(A) \,, \end{array} \right. \tag{1}$$

where the linear operator A is a Riesz spectral operator on an ordered (separable) Hilbert space X with positive cone X_+, which is the infinitesimal generator of a positive unstable C_0-semigroup $T(t)$ on X, [8]-[10], such that there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\text{for all } \; t \geq 0, \; \|T(t)\| \leq Me^{\omega t}, \tag{2}$$

the operator $B \in \mathcal{L}(\mathbb{R}^m, X)$ is positive, i.e. B is a bounded linear operator from $\mathbb{R}^m$ to X such that

$$B(\mathbb{R}^m_+) \subset X_+ \,, \tag{3}$$

where $\mathbb{R}_+$ denotes the set of nonnegative real numbers and the input $u(\cdot)$ is assumed to be any locally square integrable function, such that $u(\cdot) \in L^2([0,T], \mathbb{R}^m)$ for all $T > 0$.

It follows from these assumptions that the system (1), i.e. the pair (A, B), is **positive**: for every initial state $x_0 \in X_+$ and for every (admissible) nonnegative input u, the corresponding state trajectory $x(\cdot)$ (interpreted as the mild solution of (1)) remains in X_+, i.e. for all $t \geq 0$, $x(t) \in X_+$ (see e.g. [1] and references therein).

In addition, the following conditions are assumed to hold:
(C0) The operator A admits a Riesz basis of eigenvectors $(\phi_n)_{n\geq 1}$;
(C1) the system (1), i.e. the pair (A, B), is (exponentially) stabilizable;
(C2) the C_0-semigroup $T(t)$ has a compact resolvent, i.e. the resolvent operator $R(\lambda, A) := (\lambda I - A)^{-1} \in \mathcal{L}(X)$ is compact for all $\lambda > \omega$.

Observe that condition (C0) holds e.g. if the operator A is a Riesz spectral operator or if it is similar to a normal operator (see e.g. [9]). It follows from this condition that $(\phi_n)_{n\geq 1}$ is also a Riesz basis of eigenvectors of the operator $K = \lambda R(\lambda, A)$, for $\lambda > \omega$.
Condition (C1) implies that the unstable part of the dynamics is a (totally) unstable finite-dimensional system corresponding to all the eigenvalues of the operator A that belong to the closed right-half plane, [8]. In the sequel, we will assume that the number of such eigenvalues (counting multiplicities) is m.

Definition 1: The system (1), i.e. the pair (A, B), is said to be **partially positively stabilizable** if there exist a subset $S \subset X$ and a state feedback control law $u = Fx$, where the feedback operator F is in $\mathcal{L}(X, \; \mathbb{R}^m)$, such that the resulting closed-loop system is stable and positive on S, i.e. the C_0-semigroup $T_F(t)$ generated by $A + BF$ is exponentially stable and $T_F(t)$ is **positive on the subset** S, i.e. the set $X_+ \cap S$ is $T_F(t)$-invariant:

$$T_F(t)(X_+ \cap S) \subset X_+ \cap S \,; \tag{4}$$

hence, in particular, (A, B) is stabilizable.

The **positive stabilization problem** consists of finding sufficient and/or necessary conditions for the (partial) positive stabilizability of a given system of the form (1), leading hopefully to a computational method for designing a positively stabilizing feedback, i.e. a feedback operator $F \in \mathcal{L}(X, \; \mathbb{R}^m)$ such that (i) the C_0-semigroup $T_F(t)$ generated by $A + BF$ is exponentially stable, and (ii) $T_F(t)$ is positive on a subset S, or equivalently condition (4) holds.

B. Main Result

The main idea of the approach that is followed in this paper is to use a finite spectrum assignment technique yielding a stable closed-loop system and guaranteeing the nonnegativity of the input, thereby leading to positive state trajectories. This approach, which is analyzed in detail in the next section, leads to the following result:

Theorem 1: Consider an infinite-dimensional system described by (1)-(3) and assume that conditions (C0), (C1) and (C2) hold. Under these conditions :
a) Consider a feedback operator $F \in \mathcal{L}(X, \; \mathbb{R}^m)$ of full rank m. Then the set $\mathcal{C}(F)$ given by

$$\mathcal{C}(F) := F^{-1}(\mathbb{R}^m_+) := \{ \, x \in X \; : \; Fx \in \mathbb{R}^m_+ \, \}$$

is $T_F(t)$-invariant, i.e. $T_F(t) \; \mathcal{C}(F) \subset \mathcal{C}(F)$, if and only if there exists a Metzler matrix H such that

$$F(A + BF) - HF = 0 \quad \text{on} \;\; D(A) \tag{5}$$

i.e.

$$F(A + BF)x = HFx \quad \text{for all} \;\; x \in D(A) \; .$$

b) If there exists a stabilizing feedback operator $F \in \mathcal{L}(X, \; \mathbb{R}^m)$ of full rank m, such that condition (5) holds for some Metzler matrix H, then the system (1), i.e. the pair (A, B), is partially positively stabilizable and, in particular, for every initial state $x_0 \in X_+ \cap \mathcal{C}(F)$ and for all $t \geq 0$,

$$x(t) = T_F(t)x_0 \in X_+ \cap \mathcal{C}(F)$$

such that, for some constants $\mu \geq 1$ and $\sigma > 0$,

$$\|x(t)\| \;\leq\; \mu \, e^{-\sigma t}, \qquad \text{for all} \;\; t \geq 0.$$

Remark 1: a) A Metzler matrix is a square matrix whose off-diagonal entries are nonnegative, see e.g. [11].
b) The fact that the set $\mathcal{C}(F)$ is $T_F(t)$-invariant implies that, for any initial state x_0 in $X_+ \cap \mathcal{C}(F)$, the corresponding input trajectory $u(\cdot) = Fx(\cdot)$ (generated by the feedback F) is nonnegative, hence the set $X_+ \cap \mathcal{C}(F)$ is also $T_F(t)$-invariant (since the open-loop system is positive).
c) Part b) of Theorem 1 follows directly from its Part a) and from the previous remark. In the next section, we will therefore focus on the proof of the first part of Theorem 1.

III. ANALYSIS

This section is devoted to the proof of Theorem 1a , which can be seen as a straightforward corollary of the following result.

Theorem 2: Consider a C_0-semigroup $S(t)$ of bounded linear operators on a Hilbert space X, whose infinitesimal generator is the operator $\mathcal{A}$, such that

$$\text{for all } \; t \geq 0, \; \|S(t)\| \leq Me^{\omega t}, \tag{6}$$

for some constants $M \geq 1$ and $\omega \in \mathbb{R}$. Assume that A admits a Riesz basis of eigenvectors $(\phi_n)_{n\geq 1}$ and that $S(t)$ has a compact resolvent, i.e. the resolvent operator $R(\lambda, \mathcal{A}) := (\lambda I - \mathcal{A})^{-1} \in \mathcal{L}(X)$ is compact for all $\lambda > \omega$. Consider any given bounded linear operator $F \in \mathcal{L}(X, \; \mathbb{R}^m)$ of full rank m. Then the set

$$\mathcal{C}(F) := F^{-1}(\mathbb{R}^m_+)$$

is $S(t)$-invariant, i.e. $S(t)\, \mathcal{C}(F) \subset \mathcal{C}(F)$, if and only if there exists a Metzler matrix H such that

$$F\mathcal{A} - HF = 0 \quad \text{on} \quad D(\mathcal{A}) \; . \tag{7}$$

The proof of Theorem 2 is detailed in Subsection III.B. It is based on auxiliary results concerning discrete time systems, that are developed in Subsection III.A.

Under the conditions of Theorem 2, consider the family $(\Sigma_\lambda)_{\lambda>\omega}$ of discrete-time infinite-dimensional systems

$$(\Sigma_\lambda) \left\{ \begin{array}{l} x(k+1) = \lambda R(\lambda, \mathcal{A}) x(k) \\ x(0) = x_0 \; . \end{array} \right. \tag{8}$$

Proposition 1: If the set $\mathcal{C}(F)$ is $S(t)$-invariant, then for all $\lambda > \max\{0, \omega\}$, $\mathcal{C}(F)$ is invariant with respect to the system (Σ_λ), i.e.

$$\lambda R(\lambda, \mathcal{A})\, \mathcal{C}(F) \subset \mathcal{C}(F) \; .$$

Proof: If x belongs to $\mathcal{C}(F)$, then for all $t \geq 0$, $S(t)x$ is also in $\mathcal{C}(F)$, i.e. $FS(t)x \geq 0$. By using the fact that the resolvent operator $R(\lambda, \mathcal{A})$ is the Laplace transform (interpreted as a Bochner integral, i.e. in the strong sense), of the semigroup $S(t)$, it follows that $FR(\lambda, \mathcal{A})x \geq 0$ for $\lambda > \omega$, hence the conclusion.

A. Invariance of Discrete Time Systems

In order to study the invariance properties of discrete time systems of the form (8), in this section we consider a more general class of discrete-time infinite-dimensional systems, namely

$$\Sigma^d \left\{ \begin{array}{l} x(k+1) = K\, x(k) \\ x(0) = x_0 \in X \; , \end{array} \right. \tag{9}$$

where the bounded linear operator $K \in \mathcal{L}(X)$ is assumed to be compact and to have a Riesz basis of eigenvectors $(\phi_n)_{n\geq 1}$.

For every $N \geq 1$, let X_N denote the K-invariant finite-dimensional linear subspace of X defined by

$$X_N := \text{span}\{\phi_i \; : \; i = 1, 2, \cdots, N\}$$

and let the operator $\Gamma_N := \text{Proj}_{X_N}$ denote the orthogonal projection on X_N. Observe that the operator $K_N := K\, \Gamma_N$ has a finite rank and that the sequence (K_N) converges strongly towards K in $\mathcal{L}(X)$, i.e.

$$\lim_{N\to\infty} \|(K - K_N)x\| = 0 \;\; , \;\; x \in X.$$

Now we can define the sequence (Σ^d_N) of discrete-time finite-dimensional systems:

$$\Sigma^d_N \left\{ \begin{array}{l} x_N(k+1) = K_N\, x_N(k) \\ x_N(0) \in X_N \; . \end{array} \right. \tag{10}$$

Observe that such system is well-defined on X_N. Indeed $K_N \in \mathcal{L}(X)$ is such that $K_N(X_N) \subset X_N$.

Proposition 2: If the set $\mathcal{C}(F)$ is invariant with respect to the system Σ^d, i.e. $K\, \mathcal{C}(F) \subset \mathcal{C}(F)$, then for all $N \geq 1$, the set

$$\mathcal{C}_N(F) := \mathcal{C}(F) \cap X_N := \{\; x \in X_N \;\; : \;\; Fx \in \mathbb{R}^m_+ \;\}$$

is invariant with respect to the system Σ^d_N, i.e.

$$K_N\, \mathcal{C}_N(F) \subset \mathcal{C}_N(F) \; .$$

Proof: It suffices to observe that, for any $x \in \mathcal{C}_N(F)$, $K_N x \in X_N$ and $FK_N x = FKx \in \mathbb{R}^m_+$.

Now we are in a position to state and prove the main result of this subsection.

Proposition 3: The set $\mathcal{C}(F)$ is invariant with respect to the system Σ^d, i.e. $K\; \mathcal{C}(F) \subset \mathcal{C}(F)$, if and only if there exists a nonnegative matrix H such that

$$FK = HF \quad \;\; \text{on } X. \tag{11}$$

The proof of the necessity of Condition (11) in Proposition 3 is based on the two following lemmas.

Lemma 1: If the set $\mathcal{C}_N(F)$, $N \geq 1$, is invariant with respect to the system Σ^d_N, then the null space $\mathcal{N}(F_N)$ of the feedback operator $F_N := F\, \Gamma_N$ is K_N-invariant, i.e.

$$K_N(\mathcal{N}(F_N)) \subset \mathcal{N}(F_N) \; . \tag{12}$$

Proof: For the sake of a contradiction, assume that there exists some $x_0 \in X_N$ such that $F_N x_0 = 0$ and $F_N K_N x_0 \neq 0$, or equivalently $\alpha := (F_N K_N x_0)_{i_0} \neq 0$ for some $i_0 \in \{1, \cdots, m\}$. Observe that $Fx_0 = F_N x_0 = 0$, thus x_0 belongs to $\mathcal{C}_N(F)$. It follows that the component α should be positive and that the state $x_1 := -\alpha x_0$ belongs also to $\mathcal{C}_N(F)$, whereas $(FK_Nx_1)_{i_0} = -\alpha^2 < 0$. This contradicts the fact that K_Nx_1 should be in $\mathcal{C}_N(F)$.

Lemma 2: [6, Lemma 2 , p. 1716] Consider $M \in \mathbb{R}^{n\times n}$ and $L \in \mathbb{R}^{m\times n}$, where $m \leq n$. Assume that L is surjective, i.e. $\text{rank } L = m$ and $\mathcal{N}(L)$ is M-invariant, i.e. $M(\mathcal{N}(L)) \subset \mathcal{N}(L)$. Then there exist a matrix $H \in \mathbb{R}^{m\times m}$ such that $LM = HL$.

Proof of Proposition 3: Sufficiency is straightforward.
Necessity: Since the operator F is surjective and the set $\cup\{X_N : N \geq 1\}$ is dense in X, where (X_N) is a monotone increasing sequence of linear subspaces of X, by [3, Lemma 3.1], there exists N_0 such that $F(X_{N_0}) = \mathbb{R}^m$, therefore for all $N \geq N_0$, $F(X_N) = \mathbb{R}^m$. Moreover, by Proposition 2, for all $N \geq 1$, the set $\mathcal{C}_N(F)$ is invariant with respect to the system Σ_N^d; thus, by Lemma 1, the null space $\mathcal{N}(F_N)$ is K_N-invariant. Thus, by Lemma 2, for all $N \geq \max\{N_0, m\}$, there exists a matrix $H_N \in \mathbb{R}^{m\times m}$ such that

$$F_NK_N = H_NF_N \text{ on } X . \tag{13}$$

Now observe that, for all $N \geq \max\{N_0, m\}$, the matrix H_N is nonnegative. Indeed, if this was not the case, there would exist $y_0 \in \mathbb{R}_+^m$ such that, for some $i_0 \in \{1, \cdots, m\}$, $(H_Ny_0)_{i_0} < 0$. By the surjectivity of F on X_N, it follows that $y_0 = Fx_0 = F_Nx_0$ for some $x_0 \in X_N$, such that

$$(H_Ny_0)_{i_0} = (H_NF_Nx_0)_{i_0} = (F_NK_Nx_0)_{i_0} < 0 .$$

Hence x_0 is in $\mathcal{C}_N(F)$ whereas K_Nx_0 is not. This contradicts the invariance property of the set $\mathcal{C}_N(F)$ with respect to the system Σ_N^d.
Consider any vector $y \in \mathbb{R}^m$. For all $N \geq \max\{N_0, m\}$, by the surjectivity of F_N and by the definition of X_N, there exists $x \in X_{\max\{N_0,m\}} \subset X_N$ (hence x is independent of N) such that $y = Fx = F_Nx$. It follows by identity (13) that,

$$H_Ny = H_NFx = H_NF_Nx = F_NK_Nx = FK_Nx .$$

Thanks to the convergence of the sequence (K_N) towards the operator K, the sequence H_Ny is convergent in $\mathbb{R}^m$. Let's define the matrix operator $H \in \mathbb{R}^{m\times m}$ by

$$Hy := \lim_{N\to\infty} H_Ny .$$

Obviously H is nonnegative thanks to the nonnegativity of H_N for N sufficiently large. In addition, by the convergence of (F_N) towards F, it follows from identity (13) that (11) holds.

B. *Proof of Theorem 2*

Sufficiency: Using the density of $D(\mathcal{A})$ in X, it follows from (7) that, for every $x_0 \in X$, the function $u : \mathbb{R}_+ \to \mathbb{R}^m : t \mapsto u(t) := FS(t)x_0$ is the solution of the finite-dimensional Cauchy problem:

$$\dot{u}(t) = Hu(t), \;\; u(0) = Fx_0 ,$$

or equivalently $u(t) = e^{Ht}Fx_0$, where H is a Metzler matrix. Thus, if in addition x_0 is in $\mathcal{C}(F)$, i.e. $Fx_0 \in \mathbb{R}_+^m$, then for all $t \geq 0$, $u(t) \in \mathbb{R}_+^m$, i.e. $S(t)x_0 \in \mathcal{C}(F)$; see e.g. [11]. This shows that the set $\mathcal{C}(F)$ is $S(t)$-invariant.

Necessity: By Proposition 1, for all $\lambda > \max\{0, \omega\}$, $\mathcal{C}(F)$ is invariant with respect to the system (Σ_λ); it follows by Proposition 3 applied to $K = \lambda R(\lambda, \mathcal{A})$, that there exists a nonnegative matrix H_λ such that

$$F\lambda R(\lambda, \mathcal{A}) = H_\lambda F \quad \text{on } X. \tag{14}$$

Now consider the (bounded linear) Yosida approximant of $\mathcal{A}$ (see e.g. [10]):

$$\mathcal{A}_\lambda := \lambda \mathcal{A} R(\lambda, \mathcal{A}) = \lambda^2 R(\lambda, \mathcal{A}) - \lambda I .$$

Observe that Equation (14) yields the identity:

$$F\mathcal{A}_\lambda = \lambda(H_\lambda - I)F . \tag{15}$$

Moreover, by [10, Lemma II.3.4, p. 65], for all $x \in D(\mathcal{A})$,

$$\lim_{\lambda\to\infty} \mathcal{A}_\lambda x = \mathcal{A}x . \tag{16}$$

Besides, since the operator F is onto and $D(\mathcal{A})$ is a dense subspace of X, by using [3, Lemma 3.1], $F(D(\mathcal{A})) = \mathbb{R}^m$. Therefore every $y \in \mathbb{R}^m$ can be written as $y = Fx$ for some $x \in D(\mathcal{A})$. Using this fact, it follows from (15) that, for all $y \in \mathbb{R}^m$, the following limit

$$Hy := \lim_{\lambda\to\infty} \lambda(H_\lambda - I)y \tag{17}$$

exists (in $\mathbb{R}^m$). In addition, identity (7) holds. Indeed, for all $x \in D(\mathcal{A})$,

$$HFx = \lim_{\lambda\to\infty} \lambda(H_\lambda - I)Fx = \lim_{\lambda\to\infty} F\mathcal{A}_\lambda x = F\mathcal{A}x .$$

It remains to be shown that the matrix H given by (17) is a Metzler matrix. Recall that, for $\lambda > \max\{0, \omega\}$, the matrix H_λ is nonnegative, hence $\lambda(H_\lambda - I)$ is a Metzler matrix. It follows by (17) that so is the matrix H.

IV. Design

A. *Computational Method*

Consider the system (1), where A is a Riesz spectral operator which is the infinitesimal generator of a positive unstable C_0-semigroup $T(t)$ on an ordered separable Hilbert space X and B is a bounded linear operator from $\mathbb{R}^m$ in X which is assumed to be positive, i.e. such that (3) holds.
In this section, we assume that the pair (A, B) is exponentially stabilizable (hence the unstable part of system (1) is finite-dimensional) and in addition that the operator A has exactly m unstable eigenvalues (counting multiplicities), where m is the number of inputs.
In order to find a feedback operator $F \in \mathcal{L}(X, \mathbb{R}^m)$, which is necessarily nonpositive (otherwise $A + BF$ would be unstable), such that the control law $u(t) = Fx(t)$ is positive for any initial condition x_0 in $\mathcal{C}(F)$ and stabilizes the system

(1), we consider a stable Metzler matrix $H \in \mathbb{R}^{m\times m}$ such that

$$\sigma(A) \cap \sigma(H) = \varnothing \,, \tag{18}$$

$$\sigma(H) = \{\gamma_i \in \mathbb{C} \;\mid\; Re(\gamma_i) < 0, \; 1 \le i \le m\} \,, \tag{19}$$

with corresponding eigenvectors θ_i, $1 \le i \le m$, such that

$$\theta_i, i = 1, \cdots, m, \text{ are linearly independent.} \tag{20}$$

Now let $F \in \mathcal{L}(X, \mathbb{R}^m)$ be a stabilizing feedback such that

$$\sigma(A+BF) = \sigma(H) \cup (\sigma(A) \cap \mathring{\mathbb{C}}_-) \tag{21}$$

Let $(\xi_i)_{i\ge1}$ be a set of eigenvectors of $A+BF$. Assume that F is solution of equation (5) (The existence of such F is established in Proposition 4 below). Consider $\gamma_i \in \sigma(H)$, with the associated eigenvectors θ_i and ξ_i such that $H\theta_i = \gamma_i\theta_i$ and

$$(A+BF)\xi_i = \gamma_i\xi_i \,, \tag{22}$$

for $i = 1,..,m$. It follows from equation (5) that, for $i = 1,..,m$,

$$H(F\xi_i) = \gamma_i(F\xi_i) \,, \tag{23}$$

i.e. $F\xi_i$ is an eigenvector of H corresponding to the eigenvalue γ_i. Hence without loss of generality,

$$F\xi_i = \theta_i \quad \text{for} \quad i = 1, \cdots, m. \tag{24}$$

By plugging (24) in the identity (22), one gets

$$\xi_i = R(\gamma_i, A)B\theta_i, \quad i = 1,..,m. \tag{25}$$

Now consider any stable eigenvalue $\lambda_j \in \sigma(A) \cap \mathring{\mathbb{C}}_-$ and ξ_j an associated eigenvector of $(A+BF)$. Then, for all $j \ge m+1$, $F(A+BF)\xi_j = HF\xi_j$, or equivalently $H(F\xi_j) = \lambda_j(F\xi_j)$. Since λ_j is not an eigenvalue of the matrix H, this implies that

$$F\xi_j = 0, \quad \text{for all} \;\; j \ge m+1. \tag{26}$$

Hence, the eigenvectors ξ_j of $(A+BF)$ for $j \ge m+1$ are also eigenvectors of A.

Proposition 4: [2] For a given stable Metzler matrix $H \in \mathbb{R}^{m\times m}$ satisfying (18)-(20), there exists a unique solution F to equation (5) such that (21) holds if $(\xi_i)_{i\ge1}$ is a basis of X. In addition, F is determined by the conditions (24), i.e. (25), and (26).

Since A is a Riesz spectral operator, its resolvent is given, for all $\lambda \in \rho(A)$ and $x \in X$, by

$$R(\lambda, A)x = \sum_{k=1}^{\infty} \frac{1}{\lambda - \lambda_k} \langle x, \psi_k \rangle \varphi_k \,,$$

where $(\varphi_k)_{k\ge1}$ is a Riesz basis of eigenvectors of A and $(\psi_k)_{k\ge1}$ is a corresponding dual Riesz basis. By (24)-(25) and (26), the following relations hold:

$$\left\{ \begin{array}{lcl} FR(\gamma_i, A)B\theta_i & = & \theta_i, \;\; 1 \le i \le m \\ F\varphi_i & = & 0, \;\; i \ge m+1. \end{array} \right. \tag{27}$$

Then one can determine F by solving the following system of m equations and m unknowns $F\varphi_i, \; i = 1, m$:

$$\sum_{k=1}^{\infty} \frac{1}{\gamma_i - \lambda_k} \langle B\theta_i, \psi_k \rangle F\varphi_k = \theta_i, \;\; 1 \le i \le m. \tag{28}$$

Observe that this method of positive stabilization works provided that the initial state x_0 be in the set $X_+ \cap \mathcal{C}(F)$; see Theorem 1. This requires that this set be not empty. The main problem that remains is how to determine F such that the latter condition is satisfied.

The design method derived in this subsection is based on the crucial assumption that there are as many inputs as unstable eigenvalues (counting multiplicities) of A. It can be readily extended to the case where the number of inputs is larger than the number of unstable eigenvalues.

B. *Illustrative Example*

Consider the following example, which is similar to the one studied in [1], i.e. the heat diffusion model in the separable Hilbert state space $X = L^2(0,1)$, described by the following linear partial differential equation with Dirichlet boundary conditions:

$$\left\{ \begin{array}{lcl} \dfrac{\partial x}{\partial t} & = & \dfrac{1}{\pi^2}\dfrac{\partial^2 x}{\partial \xi^2} + 4x + b_1u_1(t) + b_2u_2(t), \\ x(0,t) & = & x(1,t) \;=\; 0. \end{array} \right. \tag{29}$$

This model can be described by an abstract differential equation of the form (1), where A is defined on its domain

$$D(A) = \{h \in X \;\;|\;\; \frac{\partial h}{\partial \xi}, \frac{\partial^2 h}{\partial \xi^2} \in Z \;\text{ and } h(0) = h(1) = 0\}$$

by

$$Ah = (\frac{1}{\pi^2})\frac{d^2h}{d\xi^2} + 4h$$

and B is the bounded linear operator on $\mathbb{R}^2$ defined by

$$B \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = b_1u_1 + b_2u_2 \;\text{ where }\; b_1, b_2 \in L^2(0,1).$$

The operator A has a pure point spectrum $\sigma(A)$ which consists of simple eigenvalues $\lambda_k = 4 - k^2, \; k \ge 1$, and the corresponding eigenvectors $\varphi_k(\xi) = \sqrt{2}\sin(k\pi\xi), \;\; k \ge 1$, form an orthonormal basis of $L^2(0,1)$, see e.g. [8]. So A is the Riesz spectral operator given by

$$Az = \sum_{k=0}^{\infty} \lambda_k \langle z, \varphi_k \rangle \varphi_k, \quad \text{for } z \in D(A) \tag{30}$$

and A is the infinitesimal generator of the following C_0-semigroup:

$$T(t)z_0 = \sum_{k=1}^{\infty} 2e^{\lambda_k t} \langle z_0, \sin k\pi(\cdot) \rangle \sin k\pi(\cdot).$$

It is also known that $(T(t))_{t\ge0}$ is a positive C_0-semigroup, i.e.

$$T(t)L^2_+(0,1) \subset L^2_+(0,1), \quad t \ge 0,$$

where the positive cone of the state space is given by

$$L^2_+(0,1) = \{h \in L^2(0,1) \mid h(x) \geq 0 \text{ almost everywhere}\}.$$

Now let H be a Metzler matrix in $\mathbb{R}^{2\times 2}$ with spectrum $\sigma(H) = \{\gamma_1, \gamma_2\} \subset \mathring{\mathbb{C}}_-$ and corresponding eigenvectors θ_1, θ_2. According to the above reasoning, it is clear that $F\varphi_k = 0, \quad$ for all $k \geq 3$. Since $(\varphi_k)_{k\geq 1}$ is a basis of $L^2(0,1)$, the feedback F is completely determined by the knowledge of $F\varphi_1$ and $F\varphi_2$. With the aim to compute these two vectors of $\mathbb{R}^2$, one can use the equations

$$\begin{aligned} FR(\gamma_1, A)B\theta_1 &= \theta_1 \\ FR(\gamma_2, A)B\theta_2 &= \theta_2 \end{aligned}$$

i.e.

$$\begin{aligned} \frac{1}{\gamma_1 - 3}\langle B\theta_1, \varphi_1\rangle F\varphi_1 + \frac{1}{\gamma_1}\langle B\theta_1, \varphi_2\rangle F\varphi_2 &= \theta_1 \\ \frac{1}{\gamma_2 - 3}\langle B\theta_2, \varphi_1\rangle F\varphi_1 + \frac{1}{\gamma_2}\langle B\theta_2, \varphi_2\rangle F\varphi_2 &= \theta_2 \end{aligned}$$

A straightforward computation yields the following solution to this system of linear equations:

$$\begin{cases} F\varphi_1 &= -\gamma_1 \dfrac{\langle B\theta_2, \varphi_2\rangle}{C}\theta_1 + \gamma_2 \dfrac{\langle B\theta_1, \varphi_2\rangle}{C}\theta_2, \\ F\varphi_2 &= \dfrac{\gamma_1\gamma_2}{\gamma_2 - 3}\dfrac{\langle B\theta_2, \varphi_1\rangle}{C}\theta_1 - \dfrac{\gamma_1\gamma_2}{\gamma_1 - 3}\dfrac{\langle B\theta_1, \varphi_1\rangle}{C}\theta_2, \\ F\varphi_k &= 0, \quad k \geq 3. \end{cases} \tag{31}$$

where

$$C = \frac{\gamma_2}{\gamma_2 - 3}\langle B\theta_2, \varphi_1\rangle\langle B\theta_1, \varphi_2\rangle - \frac{\gamma_1}{\gamma_1 - 3}\langle B\theta_1, \varphi_1\rangle\langle B\theta_2, \varphi_2\rangle.$$

Note that the matrix H (i.e. γ_i and θ_i, $i = 1, 2$) should be chosen such that $C \neq 0$.

For any initial state $x_0 = \sum_{k=0}^{\infty} \alpha_k \varphi_k \in L^2_+(0,1)$, the corresponding value of the input is given by

$$u_0 = Fx_0 = \alpha_1 F\varphi_1 + \alpha_2 F\varphi_2\ . \tag{32}$$

Therefore the stabilizing feedback F will be such that the state trajectories of the corresponding closed-loop system stay in the set $\mathcal{C}(F)$, provided that the initial state x_0 be such that $Fx_0 \geq 0$, or equivalently

$$\begin{bmatrix} (F\varphi_1)_1 & (F\varphi_2)_1 \\ (F\varphi_1)_2 & (F\varphi_2)_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \geq 0\ ,$$

where the components of the vectors $F\varphi_1$ and $F\varphi_2$ can be computed by using (31), where the different terms can be computed in terms of the components of the vectors $\theta_1 = \begin{bmatrix} \theta_{11} \\ \theta_{12} \end{bmatrix}$ and $\theta_2 = \begin{bmatrix} \theta_{21} \\ \theta_{22} \end{bmatrix}$ and the two first components of the vectors b_1 and b_2 (defining the control matrix B) with respect to the basis $(\varphi_k)_{k\geq 1}$, i.e. $\langle b_1, \varphi_1\rangle, \langle b_1, \varphi_2\rangle, \langle b_2, \varphi_1\rangle$ and $\langle b_2, \varphi_2\rangle$. [2]

V. CONCLUDING REMARKS

In the approach developed in this paper, the focus is on a positive stabilization method which is based on (finite) spectrum assignment and on the use of a nonnegative input. This approach seems to be promising. The main challenge is the design of a numerical method for computing a positively stabilizing feedback F such that the feasible invariant set $X_+ \cap \mathcal{C}(F)$ be nonempty and in particular such that the initial state be feasible.

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