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Stability Analysis of Dynamic Output Controllers under Aperiodic Sampling and Input Saturation*

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Abstract— **This paper addresses stability issues of sampled data controllers. Considering a continuous-time linear plant and a linear discrete-time dynamic output feedback control law designed from a classical periodic sampling paradigm, the main goal is to assess the effects of aperiodic sampling on the closed-loop stability. This aperiodic sampling models for instance the communication delays and package losses through a network. In addition, the effects of control signal saturation on the stability and the maximal admissible sampling interval are also taken into account . In this context, based on the use of a looped functional, linear matrix inequalities (LMI) are derived to ensure the global asymptotic stability of the origin for the aperiodic sampled-data closed-loop system, provided a bound on the maximal sampling interval is given. An optimization problem in order to evaluate the maximal admissible value for the interval between two sampling instants is then associated to the LMI conditions.**

Keywords— **Sampled-data systems, aperiodic sampling, saturating input, dynamic output controller.**

I. INTRODUCTION

A large attention has been paid to the stability analysis and stabilization of systems with input saturation (see, for example, [16], [23]). In this context, many methods concerning the guarantee of global or local asymptotic stability of the closed-loop systems under saturating control laws have been proposed, both in continuous and discrete-time frameworks. Most of the results concerning discrete-time systems consider the classical paradigm of periodic sampling. However, due to the emergence of the new technologies, these stabilizing control laws are nowadays implemented on embedded devices, which have computation constraints and limitations. Moreover, these devices are potentially integrated in networks (networked control), with inherent communication issues (delays, package losses, etc.). These facts imply that the control laws operate non necessarily under periodic sampling. Thus it becomes clear the need for new methods of stability analysis and stabilization to cope with the aperiodic sampling paradigm.

Issues regarding the implementation of sampled data systems and the effects of sampling on the stability of the closed-loop system has been intensively studied since the the 80's (see, for instance, [4], [1] and also [14], [5], [19], [18], [24] in the context of sampled-data systems). Nevertheless most of the results consider a periodic sampling policy, which has been shown non suitable to cope with communication delays and packages losses issues in networked control systems. Mainly motivated by this fact, more recently the stability analysis of sampled-data systems has benefited from a second wave of theoretical developments. The results in this context allow to consider the aperiodic sampling case in a formal way and basically rely upon two efficient approaches of modeling. The first one is based on an uncertain discrete-time model, which is embedded onto a polytopic model from the use of exponential matrices ([7], [6]). The second approach regards the modeling of the sampling effects considering a particular time-varying delay on the control signal. This approach, initially proposed in [10] and further improved in [17] and [9] is based on the use of Lyapunov-Krasovskii functionals for time-delay systems. More recently, an alternative approach to the use of conventional Lyapunov-Krasovskii functionals has been proposed in [20], [21] and adapted to the case of impulsive systems in [3], [2]. This alternative method considers the application of looped-functionals, which in general provide less conservative stability conditions. It should however be highlighted that the aforementioned methods regard basically state feedback control laws. Then there is space for developments concerning closed-loop systems controlled by dynamic controllers implemented digitally.

In the present paper, the problem of assessing stability of a continuous-time plant controlled by a discrete-time dynamic output feedback control law with saturation constraints is addressed. Considering the periodic sampling case, this problem has been studied, for instance, in [8]. In [15], the authors extended the design of discrete-time anti-windup loops to the sampled-data case with a constant period by using the approach proposed in [4], in order to transpose the problem into a discrete-time framework. Here, we are interested by the aperiodic sampling case. In this context, the proposed approach is based on the impulsive system modeling as suggested in [11] and the use of looped functionals. The resulting stability conditions are expressed in terms of Linear Matrix Inequalities, which allows to assess the global asymptotic stability of the origin for the aperiodic sampled-data closed-loop system, provided a bound on the maximal sampling interval is given. An optimization problem to evaluate the maximal admissible value for the interval between two sampling instants, for which the stability can be guaranteed, is then associated to the LMI conditions. The

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proposed method can be seen as an alternative to the use of the hybrid systems framework.

The paper is organized as follows. Section II presents the problem formulation. Section III is devoted to the proposition of stability analysis conditions. In section IV, these conditions are cast in an optimization problem to evaluate the maximal admissible intersampling interval. A numerical example illustrating the application of the results is presented in Section V. Section VI ends the paper with some concluding remarks and discussions regarding possible further developments.

Notation. Throughout the article, the sets $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ and $\mathbb{S}^n$ denote respectively the set of positive integers, positive scalars, n-dimensional vectors, $n\times n$ matrices and symmetric matrices of $\mathbb{R}^{n\times n}$. For a given positive scalar $\mathcal{T}_2$, $\mathbb{K}$ is defined as the set of continuous functions from an interval $[0,\ T]$ to $\mathbb{R}^n$, where T is a positive scalar less than $\mathcal{T}_2$. The notations $|\cdot|$ and $\|\cdot\|$ stand for the absolute value of a scalar and for the Euclidean norm of a vector, respectively. The superscript $'$ stands for matrix or vector transposition. $P>0$ for $P\in\mathbb{S}^n$ means that P is a symmetric positive definite matrix. For any positive integer $j\leq n$, any vector $x\in\mathbb{R}^n$ and any matrix $A\in\mathbb{R}^{n\times n}$, A_j, x_j and $\mathrm{He}\{A\}>0$ refer to the jth line of matrix A, the jth component of vector x and $A+A'>0$, respectively. The symbols I and 0 represent the identity and zero matrices of appropriate dimensions. $Co\{\cdot\}$ denotes a convex hull. Each component of the vector-valued saturation function is defined by $sat(v_i)=sign(v_i)\min\{u_{0i},v_i\}$, $\quad i=1,\dots,m$, where u_{0i} denotes the bound on the ith control input.

II. PROBLEM FORMULATION

In this paper, we consider the interconnection between a continuous plant and a discrete-time dynamic output feedback controller. The connection between these two systems is performed in a sample-data way. At a sampling instant, the plant output value is sampled and sent to the input of the discrete-time controller. This value is used to update the controller state and, at the same time, to compute the new control value to be injected in the input of the continuous-time plant. This control value is kept constant until the next sampling time through a zero order hold (ZOH). The whole scheme is depicted in Figure 1.

We suppose that the discrete-time controller has been designed considering a periodic sampling paradigm, from one of the following two classical approaches in the literature ([4], [1]):

- the discretization of a continuous-time controller, using Euler or Tustin approximations.
- the direct synthesis in discrete-time domain considering the discretization of the plant plus the ZOH.

Moreover, we consider that the design has been performed disregarding the effects of the control saturation. Then, in order to mitigate the effects of control saturation on the performance as well as stability of the closed-loop system, a static anti-windup term may be added to the pre-computed controller. The design of this anti-windup can be carried out, for instance, from the techniques proposed in [12] or [13].

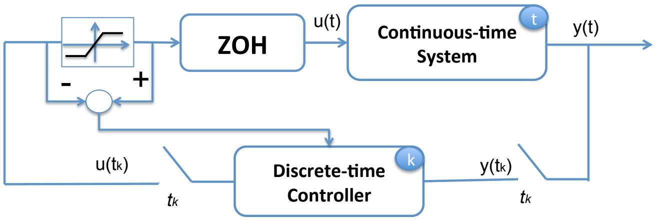

Fig. 1. Closed-loop system.

Considering the aforementioned assumptions and the scheme in Figure 1, the continuous-time plant is described by the following linear model:

$$\begin{cases} \dot{x}_p(t) &= A_p x_p(t) + B_p sat(u(t_k)) \\ y(t) &= C_p x_p(t), \end{cases} \qquad \forall t\in[t_k,\ t_{k+1}),\ k\in\mathbb{N}, \tag{1}$$

where $x_p\in\mathbb{R}^{n_p}$, $u\in\mathbb{R}^m$ and $y\in\mathbb{R}^p$ represent the state, the input and the output of the plant, respectively. Matrices A_p, B_p, C_p have appropriate dimensions and are supposed to be constant. t_k denotes the kth sampling instant and due to the presence of a ZOH, the control input is kept constant with value $u(t_k)$, $\forall t\in t\in[t_k,\ t_{k+1})$. Hence, the sequence $\{t_k\}_{k\in\mathbb{N}}$ is an increasing sequence of positive scalars such that $\bigcup_{k\in\mathbb{N}}[t_k,\ t_{k+1})=[0,\ +\infty)$. We assume that there exist two positive scalars $\mathcal{T}_1\leq\mathcal{T}_2$ such that the difference between two successive sampling instants $T_k=t_{k+1}-t_k$ satisfies

$$\forall k\in\mathbb{N},\quad 0\leq\mathcal{T}_1\leq T_k\leq\mathcal{T}_2. \tag{2}$$

Furthermore, in order to avoid Zeno solution we assume that $\{t_k\}_{k\in\mathbb{N}}$ has no accumulation point. Note that the intersampling time T_k can be variable, which allows to model an aperiodic sampling. The particular case of periodic sampling corresponds to $T_k=\mathcal{T}_1=\mathcal{T}_2=T$, $\forall k\in\mathbb{N}$.

At the instant $t=t_k$ we consider that the plant output is sampled and sent to a discrete-time dynamic output feedback controller, given by:

$$\begin{cases} x_c(t_{k+1}) &= A_c x_c(t_k) + B_c y(t_k) + E_c\psi(u(t_k)), \\ u(t_k) &= C_c x_c(t_k) + D_c y(t_k), \end{cases} \qquad \forall k\in\mathbb{N} \tag{3}$$

where $x_c\in\mathbb{R}^{n_c}$, $y\in\mathbb{R}^p$ and $u\in\mathbb{R}^m$ are the state, the input and the output of the controller, respectively. A_c, B_c, C_c, D_c, E_c are assumed to be constant and of appropriate dimensions. Matrix E_c denotes an static anti-windup gain, and $\psi(u(t_k))$ is a vector-valued decentralized deadzone nonlinearity defined as follows:

$$\psi(u(t_k)) = sat(u(t_k)) - u(t_k). \tag{4}$$

From the previous assumptions, the closed-loop system dynamics can be represented by an impulsive system described as follows:

$$\left\{\begin{array}{ll}\dot{x}_p(t) &= A_px_p(t) + B_pD_cC_px_m(t) \\ &\quad +B_pC_cx_c(t) + B_p\psi(u(t)), \\ \dot{x}_m(t) &= 0, \\ \dot{x}_c(t) &= 0,\end{array}\right. \quad \forall t \in \mathbb{R}\backslash \{t_k\}_{k\in\mathbb{N}}$$

$$\left\{\begin{array}{ll}x_p(t^+) &= x_p(t), \\ x_m(t^+) &= x_p(t), \\ x_c(t^+) &= A_cx_c(t) + B_cC_px_m(t) \\ &\quad +E_c\psi(u(t)),\end{array}\right. \quad \forall t \in \{t_k\}_{k\in\mathbb{N}} \tag{5}$$

Actually at the sampling time $t = t_k$, the variables are impulsively reset, i.e. at the instant $t^+ = \lim_{\tau\to 0^+} t + \tau$ the variables assume instantaneously new values. Of course, since the plant dynamics evolves continuously we have that $x_p(t^+) = x_p(t)$. On the other hand, x_m is an auxiliary variable that couples the continuous and the discrete-time dynamics. Note that the output of the continuous system at $t = t_k$ is actually the discrete-time input of the controller, i.e. $y(t_k) = C_px_p(t_k) = C_px_m(t_k)$, which by definition is kept constant between two sampling instants, i.e. $\dot{x}_m(t) = 0$. Moreover, the fact that $\dot{x}_c(t) = 0$ ensures that the value of the control input is also kept constant in the intersampling time.

Considering the nonlinear closed-loop system given by (5) we are interested in analyzing both the influence of the saturation and of the aperiodic sampling on the asymptotic stability of the closed-loop system. In particular, we consider the case where matrix A_p is Hurwitz. In this case, we focus on the analysis of the global asymptotic stability of the origin ([23]). Hence, from the considerations above, we are concerned by the problems stated as follows:

P1. Given $\mathcal{T}_1$ and $\mathcal{T}_2$, such that $\mathcal{T}_1 < \mathcal{T}_2$, provide conditions guaranteeing the global asymptotic stability of the closed-loop system (6);

P2. Evaluate the maximal admissible value of $\mathcal{T}_2$, for which the global asymptotic stability of the closed-loop system (6) is ensured.

III. STABILITY CONDITIONS

A. Closed-loop System Modeling

In order to tackle problems P1 and P2, instead of considering the classical hybrid framework to study mixed continuous and discrete dynamics as defined in [11], we use an alternative direction by adapting the technique based on a looped functional as developed in [3]. Hence, we consider the entire state trajectory of the closed-loop system as a sequence of functions $(x_p(t_k + \tau), x_m(t_k + \tau), x_c(t_k + \tau))$, for all $\tau \in [0,\ T_k]_{k\in\mathbb{N}}$. In order to express this notion, we define the lifting signals $\chi_{p,k} = x_p(t_k + \tau)$, $\chi_{m,k} = x_m(t_k + \tau)$, $\chi_{c,k} = x_c(t_k + \tau)$ and $\psi_k = \psi(t_k + \tau)$ with $\chi_{p,k}(0) = \lim_{s\to t_k} x_p(s)$, $\chi_{m,k}(0) = \lim_{s\to t_k} x_m(s)$, $\chi_{c,k}(0) = \lim_{s\to t_k} x_c(s)$ and $\psi_k(0) = \lim_{s\to t_k} \psi(s)$. We also define the augmented state $\chi_k = \left[\begin{array}{ccc}\chi'_{p,k} & \chi'_{m,k} & \chi'_{c,k}\end{array}\right]' \in \mathbb{R}^n$, with $n = 2n_p + n_c$. Then the closed-loop system can be written as:

$$\left\{\begin{array}{lcl}\dot{\chi}_{p,k}(\tau) &=& A_p\chi_{p,k}(\tau) + B_pD_cC_p\chi_{m,k}(\tau) \\ && +B_pC_c\chi_{c,k}(\tau) + B_p\psi_k(\tau), \\ \dot{\chi}_{m,k}(\tau) &=& 0, \\ \dot{\chi}_{c,k}(\tau) &=& 0, \\ \dot{\psi}_k(\tau) &=& 0, \\ && \\ \chi_{p,k+1}(0) &=& \chi_{p,k}(T_k), \\ \chi_{m,k+1}(0) &=& \chi_{p,k}(T_k), \\ \chi_{c,k+1}(0) &=& A_c\chi_{c,k}(T_k) + B_cC_p\chi_{m,k}(T_k) \\ && +E_c\psi_k(T_k), \\ \psi_{k+1}(0) &=& \psi(C_c\chi_{c,k}(T_k) + D_cC_p\chi_{m,k}(T_k)).\end{array}\right. \tag{6}$$

B. Main result

In order to develop the conditions regarding the global asymptotic stability of the closed-loop system (6) by using a looped functional, let us first recall the following Lemma.

Lemma 1: ([3]) Consider system (6) and let $\mathcal{T}_1$ and $\mathcal{T}_2$, $\mathcal{T}_1 < \mathcal{T}_2$, be two positive scalars and $V : \mathbb{R}^n \to \mathbb{R}^+$ be a quadratic function for which there exist real scalars $0 < \mu_1 < \mu_2$ such that

$$\forall x \in \mathbb{R}^n, \quad \mu_1|x|^2 \leq V(x) \leq \mu_2|x|^2. \tag{7}$$

Then, the two following statements are equivalent.

(i) For all $k \in \mathbb{N}$, $T_k \in [\mathcal{T}_1,\ \mathcal{T}_2]$, the increment of the Lyapunov function satisfies

$$\Delta V(k) < 0;$$

where $\Delta V(k) = V(\chi_{k+1}(0)) - V(\chi_k(0))$

(ii) There exists a continuous functional $\mathcal{V}_0 : [0,\ \mathcal{T}_2]\times\mathbb{K} \to \mathbb{R}$ which satisfies for all $z \in \mathbb{K}$

$$\forall T_k \in [\mathcal{T}_1,\ \mathcal{T}_2] \quad \mathcal{V}_0(T_k, z) = \mathcal{V}_0(0, z). \tag{8}$$

and such that, for all $k \in \mathbb{N}$, $T_k \in [\mathcal{T}_1,\ \mathcal{T}_2]$ and $\tau \in [0\ T_k]$ and

$$\dot{\mathcal{W}}(\tau, \chi_k) = \frac{\Lambda_k}{T_k} + \frac{d}{d\tau}[V(\chi_k(\tau)) + \mathcal{V}_0(\tau, \chi_k)] < 0, \tag{9}$$

with $\Lambda_k = V(\chi_{k+1}(0)) - V(\chi_k(T_k))$.

Moreover if one of these two statements holds, then, system (6) is asymptotically stable for all T_k satisfying (2).

From Lemma 1, the following theorem can be stated for the stability analysis of system (6).

Theorem 1: For given positive scalars $\mathcal{T}_1 < \mathcal{T}_2$, assume that there exist symmetric positive definite matrices P, R, symmetric matrices X, S_1, a symmetric positive definite diagonal matrix U and matrices S_2 and N of appropriate dimensions, that satisfy, for $i = 1, 2$

$$\Psi_1(\mathcal{T}_i) = \left[\begin{array}{cc}\Pi_1 + \mathcal{T}_i(\Pi_3 + \Pi_4) & \mathcal{T}_iN \\ * & -\mathcal{T}_iR\end{array}\right] < 0, \tag{10}$$

$$\Psi_2(\mathcal{T}_i) = \Pi_1 + \mathcal{T}_i(\Pi_2 - \Pi_3 + \Pi_4) < 0, \tag{11}$$

with

$$\begin{array}{rcl}
\Pi_1 & = & M^{+\prime}PM^{+} - M^{-\prime}PM^{-} + \Pi_{10} \\
& & -\text{He}\left\{M_5'UM_5 + M_5'UK\begin{bmatrix} M_3 \\ M_4 \end{bmatrix}\right\}, \\
\Pi_{10} & = & M_{12}'S_1M_{12} + \text{He}\{M_{12}'S_2M_2 + NM_{12}\}, \\
\Pi_2 & = & M_c'RM_c + \text{He}\{M_c'(S_1M_{12} + S_2M_2)\}, \\
\Pi_3 & = & \begin{bmatrix} M_2 \\ M_5 \end{bmatrix}' X \begin{bmatrix} M_2 \\ M_5 \end{bmatrix}, \\
\Pi_4 & = & \text{He}\left\{\begin{bmatrix} M_c \\ 0 \\ 0 \end{bmatrix}' P \begin{bmatrix} M_1 \\ M_3 \\ M_4 \end{bmatrix}\right\},
\end{array} \tag{12}$$

where[1]

$$\begin{array}{l}
M_1 = \begin{bmatrix} I & 0 & 0 & 0 & 0 \end{bmatrix}, \\
M_2 = \begin{bmatrix} 0 & I & 0 & 0 & 0 \end{bmatrix}, \\
M_3 = \begin{bmatrix} 0 & 0 & I & 0 & 0 \end{bmatrix}, \\
M_4 = \begin{bmatrix} 0 & 0 & 0 & I & 0 \end{bmatrix}, \\
M_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & I \end{bmatrix}, \\
M_{12} = M_1 - M_2, \\
M_c = \begin{bmatrix} A_p & 0 & B_pD_cC_p & B_pC_c & B_p \end{bmatrix}, \\
K = \begin{bmatrix} D_cC_p & C_c \end{bmatrix}, \\
M^{+} = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & B_cC_p & A_c & E_c \end{bmatrix}, \\
M^{-} = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix}.
\end{array}$$

Then, the origin of the closed-loop system given by (6) is globally asymptotically stable for any aperiodic sampling satisfying (2).

Proof: Let a quadratic Lyapunov function candidate be defined, for any χ_k in $\mathbb{R}^n$, by $V(\chi_k) = \chi_k'P\chi_k$, where P is a symmetric positive definite matrix from $\mathbb{S}^n$. Thus, the function V satisfies (7).

Consider now a functional $\mathcal{V}_0$ defined for all $\tau \in [0,\ T_k]$, as follows:

$$\begin{array}{rcl}
\mathcal{V}_0(\tau,\chi_k) & = & \frac{\tau}{T_k}(\chi_{p,k}(\tau) - \chi_{p,k}(T_k))'S_1(\chi_{p,k}(\tau) - \chi_{p,k}(T_k)) \\
& + & 2\frac{\tau}{T_k}(\chi_{p,k}(\tau) - \chi_{p,k}(T_k))'S_2\chi_{p,k}(T_k) \\
& + & \frac{(T_k-\tau)}{T_k}\tau\begin{bmatrix} \chi_{p,k}(T_k) \\ \psi_k(0) \end{bmatrix}' X \begin{bmatrix} \chi_{p,k}(T_k) \\ \psi_k(0) \end{bmatrix} \\
& - & \frac{\tau}{T_k}\int_\tau^{T_k} \dot{\chi}_{p,k}'(\theta)R\dot{\chi}_{p,k}(\theta)d\theta,
\end{array}$$

with $S_1 \in \mathbb{S}^{n_p}$, $S_2 \in \mathbb{R}^{n_p \times n_p}$, $X \in \mathbb{S}^{n_p+m}$, $R \in \mathbb{S}^{n_p}$ such that $R > 0$. Since $\mathcal{V}_0(\tau,\chi_k)$ is equal to zero at $\tau = 0$ and $\tau = T_k$, it satisfies condition (8). Moreover, it is continuous at all sampling instants and differentiable over $[0,\ T_k)$.

Then, from Lemma 1, if we prove that $\mathcal{W}(\tau,\chi_k)$ defined from $V(x)$ and $\mathcal{V}_0$ above, is such that $\dot{\mathcal{W}}(\tau,\chi_k) < 0$ along the trajectories of the system (6), we can conclude that $\Delta V(k) < 0$, which ensures that the trajectories converge uniformly asymptotically to the origin.

On the other hand, note that $\mathcal{W}$ depends on the dead-zone nonlinearity ψ_k. This nonlinearity verifies globally the sector condition ([23]):

$$\mathcal{W}_s(\chi_k,\psi_k) := 2\psi_k'(0)U\left[\psi_k(0) + K\begin{bmatrix} \chi_{m,k}(0) \\ \chi_{c,k}(0) \end{bmatrix}\right] < 0,$$

for any diagonal and positive definite matrix U. In order to include this constraint into the stability analysis, define the functional

$$\dot{\mathcal{W}}_g(\tau,\chi_k) \ := \ \frac{\Lambda_k}{T_k} + \dot{V}(\chi_k) + \dot{\mathcal{V}}_0(\tau,\chi_k) - \frac{\mathcal{W}_s(\chi_k,\psi_k)}{T_k}. \tag{13}$$

Then provided that $\dot{\mathcal{W}}_g(\tau,\chi_k) < 0$, it follows that $\dot{\mathcal{W}}(\tau,\chi_k) < 0$.

In order to express an upper bound on the functional $\dot{\mathcal{W}}_g(\tau,\chi_k)$, we define the extended vector

$$\zeta_k(\tau) = [\chi_{p,k}(\tau)'\ \chi_{p,k}(T_k)'\ \chi_{m,k}'(0)\ \chi_{c,k}(0)'\ \psi_k(0)']',$$

We consider now the computation of each term of $\dot{\mathcal{W}}_g(\tau,\chi_k)$ separately. Let us first focus on Λ_k. From its definition, it follows that

$$\begin{array}{rcl}
\Lambda_k & = & \begin{bmatrix} \chi_{p,k+1}(0) \\ \chi_{m,k+1}(0) \\ \chi_{c,k+1}(0) \end{bmatrix}' P \begin{bmatrix} \chi_{p,k+1}(0) \\ \chi_{m,k+1}(0) \\ \chi_{c,k+1}(0) \end{bmatrix} \\
& - & \begin{bmatrix} \chi_{p,k}(T_k) \\ \chi_{m,k}(T_k) \\ \chi_{c,k}(T_k) \end{bmatrix}' P \begin{bmatrix} \chi_{p,k}(T_k) \\ \chi_{m,k}(T_k) \\ \chi_{c,k}(T_k) \end{bmatrix}.
\end{array}$$

Moreover, from equation (6), the following holds:

$$\begin{array}{rcl}
\begin{bmatrix} \chi_{p,k+1}(0) \\ \chi_{m,k+1}(0) \\ \chi_{c,k+1}(0) \end{bmatrix} & = & \begin{bmatrix} \chi_{p,k}(T_k) \\ \chi_{p,k}(T_k) \\ A_c\chi_{c,k}(T_k) + B_cC_p\chi_{m,k}(T_k) \\ + E_c\psi_k(T_k) \end{bmatrix} \\
& = & M^{+}\zeta_k(\tau),
\end{array}$$

and

$$\begin{bmatrix} \chi_{p,k}(T_k) \\ \chi_{m,k}(T_k) \\ \chi_{c,k}(T_k) \end{bmatrix} = \begin{bmatrix} \chi_{p,k}(T_k) \\ \chi_{m,k}(0) \\ \chi_{c,k}(0) \end{bmatrix} = M^{-}\zeta_k(\tau).$$

This implies that

$$\Lambda_k \ = \ \zeta_k(\tau)'\left[M^{+\prime}PM^{+} - M^{-\prime}PM^{-}\right]\zeta_k(\tau). \tag{14}$$

Consider now the second term of $\dot{\mathcal{W}}_g$, which is given by $\frac{d}{d\tau}V(\chi_k(\tau)) = 2\dot{\chi}_k(\tau)'P\chi_k(\tau)$. This term can also be expressed using the extended vector ζ_k by noting that

$$\dot{\chi}_k(\tau) = \begin{bmatrix} \dot{\chi}_{p,k}(\tau) \\ \dot{\chi}_{m,k}(\tau) \\ \dot{\chi}_{c,k}(\tau) \end{bmatrix} = \begin{bmatrix} M_c \\ 0 \\ 0 \end{bmatrix}\zeta_k(\tau),$$

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[1] Note that matrices M_i are not of the same dimension.

and

$$\chi_k(\tau) = \begin{bmatrix} \chi_{p,k}(\tau) \\ \chi_{m,k}(\tau) \\ \chi_{c,k}(\tau) \end{bmatrix} = \begin{bmatrix} \chi_{p,k}(\tau) \\ \chi_{m,k}(0) \\ \chi_{c,k}(0) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_3 \\ M_4 \end{bmatrix} \zeta_k(\tau).$$

Then it follows that:

$$\frac{dV(\chi_k(\tau))}{d\tau} = \zeta_k(\tau)' \left[\text{He} \left\{ \begin{bmatrix} M_c \\ 0 \\ 0 \end{bmatrix}' P \begin{bmatrix} M_1 \\ M_3 \\ M_4 \end{bmatrix} \right\} \right] \zeta_k(\tau).$$

Regarding the sector condition, the same formulation can be obtained by noting that

$$\begin{array}{rcl} \psi_k(0) & = & M_5 \zeta_k(\tau), \\ \begin{bmatrix} \chi_{m,k}(0) \\ \chi_{c,k}(0) \end{bmatrix} & = & \begin{bmatrix} M_3 \\ M_4 \end{bmatrix} \zeta_k(\tau), \end{array}$$

which leads to

$$\mathcal{W}_s = \zeta_k(\tau)' \left[\text{He} \left\{ M_5' U \left(M_5 + K \begin{bmatrix} M_3 \\ M_4 \end{bmatrix} \right) \right\} \right] \zeta_k(\tau). \tag{15}$$

Finally, we will address the computation of $\dot{\mathcal{V}}_0$:

$$\begin{aligned} \dot{\mathcal{V}}_0(\tau, \chi_k) = {} & T_k^{-1} \Big(2\tau \dot{\chi}_{p,k}'(\tau) S_1 (\chi_{p,k}(\tau) - \chi_{p,k}(T_k)) \\ & + 2\tau \dot{\chi}_{p,k}'(\tau) S_2 \chi_{p,k}(T_k) + \tau \dot{\chi}_{p,k}'(\tau) R \dot{\chi}_{p,k}(\tau) \\ & + (\chi_{p,k}(\tau) - \chi_{p,k}(T_k))' S_1 (\chi_{p,k}(\tau) - \chi_{p,k}(T_k)) \\ & + 2(\chi_{p,k}(\tau) - \chi_{p,k}(T_k))' S_2 \chi_{p,k}(T_k) \\ & + (T_k - 2\tau) \begin{bmatrix} \chi_{p,k}(T_k) \\ \psi_k(0) \end{bmatrix}' X \begin{bmatrix} \chi_{p,k}(T_k) \\ \psi_k(0) \end{bmatrix} \\ & - \textstyle\int_\tau^{T_k} \dot{\chi}_{p,k}'(\theta) R \dot{\chi}_{p,k}(\theta) d\theta \Big). \end{aligned}$$

Thanks to a classical bounding technique, one gets that for any matrix N in $\mathbb{R}^{(3n_p+2n_c)\times n_p}$, the following inequality holds:

$$\begin{array}{rcl} -\int_\tau^{T_k} \dot{\chi}_k(\theta)' R \dot{\chi}_k(\theta) d\theta & \geq & 2\zeta_k(\tau)' N (\chi_k(\tau) - \chi_k(T_k)) \\ & & +(T_k - \tau) \zeta_k(\tau)' N R^{-1} N' \zeta_k(\tau). \end{array}$$

Then using the extended vector ζ_k it follows that:

$$\begin{array}{rcl} \dot{\mathcal{V}}_0(\tau, \chi_k) & \leq & \zeta_k(\tau)' \left[\Pi_{10} + \tau \Pi_2 + (T_k - 2\tau) \Pi_3 \right. \\ & & \left. + \tau N R^{-1} N' \right] \zeta_k(\tau). \end{array} \tag{16}$$

Hence, combining the previous expressions, it follows that:

$$\begin{aligned} \dot{\mathcal{W}}_g(\tau, \chi_k) \leq {} & \tfrac{1}{T_k} \zeta_k(\tau)' [\Pi_1 + \tau \Pi_2 + T_k \Pi_4 \\ & + \tau N R^{-1} N' + (T_k - 2\tau) \Pi_3] \zeta_k(\tau). \end{aligned} \tag{17}$$

with Π_1, Π_2, Π_3 and Π_4 as defined in (12). A sufficient condition to ensure that $\dot{\mathcal{W}}_g < 0$ consists therefore in guaranteeing that:

$$\begin{aligned} & \Pi_1 + \tau \Pi_2 + (T_k - 2\tau) \Pi_3 + T_k \Pi_4 \\ & \qquad + (T_k - \tau) N R^{-1} N' < 0. \end{aligned} \tag{18}$$

Note now that the inequality (18) is affine with respect to the variable τ in $[0,\ T_k]$. Then, by convexity, it suffices to ensure that (18) is negative for $\tau = 0$ and $\tau = T_k$ (see [17] for more details). Thus $\dot{\mathcal{W}}_g < 0$ if both inequalities

$$\Pi_1 + T_k(\Pi_3 + \Pi_4 + N R^{-1} N') < 0,$$
$$\Pi_1 + T_k(\Pi_2 - \Pi_3 + \Pi_4) < 0,$$

are satisfied.

Applying the same argument on T_k for the interval $[\mathcal{T}_1,\ \mathcal{T}_2]$ and by using the Schur's complement, conditions $\Psi_1(\mathcal{T}_i) < 0$ and $\Psi_2(\mathcal{T}_i) < 0$, for $i = 1, 2$, given in (10) and (11), are obtained. ∎

Theorem 1 provides a sufficient condition for ensuring the global asymptotic stability of the origin for the closed-loop system given by (6) for any aperiodic sampling satisfying (2).

Remark 1: It is worth mentioning that the functional $\mathcal{V}_0$ used in the proof of Theorem 1 differs from the usual Lyapunov-Krasovskii ones. Indeed from Lemma 1, the components of $\mathcal{V}_0$ are not needed to be positive definite. This is the reason why the last term is allowed to be defined with the positive definite matrix R.

IV. OPTIMIZATION ISSUES

Considering $\mathcal{T}_1$ and $\mathcal{T}_\in$ given, the conditions in Theorem 1 are actually LMIs in variables P, R, X, S_1, S_2, U and N. In this case the global stability of the origin can be easily checked by solving a feasibility LMI problem. In other words, problem P1 is solved if the LMIs (10) and (11) are feasible.

On the other hand, a practical problem of interest may be to evaluate, for a given $\mathcal{T}_1$, an upper bound for the maximal admissible value of $\mathcal{T}_2$ for which the global asymptotic stability of the closed-loop system under aperiodic sampling can be ensured. Such a problem corresponds to problem P2 and can be formulated as follows:

$$\begin{array}{c} \max\ \mathcal{T}_2 \\ \text{subject to} \\ (10), (11) \end{array} \tag{19}$$

Hence, considering $\mathcal{T}_1$ given, problem (19) can be solved as a feasibility LMI problem by iteratively increasing $\mathcal{T}_2$ and testing the feasibility of LMIs (10) and (11). Note that a good way to fix $\mathcal{T}_1$ is to choose $\mathcal{T}_1$ equals to the sampling period used to discretize the controller.

V. NUMERICAL EXAMPLE

Let us consider the MIMO continuous-time system given by:

$$A_p = \begin{bmatrix} -0.1 & 0.0 \\ 0.0 & -0.1 \end{bmatrix}, B_p = \begin{bmatrix} 1.5 & 4.0 \\ 1.2 & 3.0 \end{bmatrix}, C_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for which a continuous-time globally stabilizing controller for the saturated system with $u_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has been provided in [23] (chap. III, p.153). An Euler discretized version of the controller, obtained for a sampling period equal to 0.5 is given as follows:

$$A_c = \begin{bmatrix} 0.55 & 0 \\ 0 & 0.55 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0.2 & 0 \\ 0 & -0.2 \end{bmatrix},$$
$$C_c = \begin{bmatrix} 4 & 5.33 \\ -1.6 & -2 \end{bmatrix}, \quad D_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let us fix $\mathcal{T}_1 = 0.5$, i.e corresponding to the discretization period of the controller. Let us then consider the problem of evaluating the maximum admissible value of $\mathcal{T}_2$, by considering a line search on parameter $\mathcal{T}_2$. Consider first the case without anti-windup correction ($E_c = 0$). The solution to Problem (19) is given by:

$$E_c = 0 : \quad \mathcal{T}_2 = 2.30$$

Consider now an anti-windup gain defined as follows:

$$E_c = \begin{bmatrix} -0.61 & -0.58 \\ 0.49 & -0.20 \end{bmatrix}$$

With such an anti-windup, the solution to Problem (19) is given by:

$$E_c \neq 0 : \quad \mathcal{T}_2 = 2.64$$

These values may be compared with the solution given without saturation. With this aim, matrices in (12) should be modified by removing all the terms related to ψ_k, i.e. by removing the fifth column and line respectively. In this case, the solution to the problem without saturation, is given by:

$$\text{without saturation :} \quad \mathcal{T}_2 = 3.19$$

This illustrates that the saturation deteriorates the maximal admissible bound for the admissible intersampling time. On the other hand, the anti-windup allows to partially recover the admissible value of $\mathcal{T}_2$ obtained for the closed-loop system when the control signal is unbounded (i.e. without saturation).

VI. CONCLUSION

This paper provided theoretical conditions for the stability analysis of a class of sampled data control system under aperiodic sampling and input saturation. The closed-loop system is composed of a linear continuous plant connected to a sampled-data dynamic output controller in connection through a saturated input. Thanks to the use of an adequate looped functional, LMIs were proposed to assess the global asymptotic stability of the origin for the closed-loop system under aperiodic sampling. In order to evaluate the maximal admissible value for the intersampling time, a convex optimization problem was then associated to the LMI conditions.

Several directions of research could be associated to this current work. Among them, note that it should be interesting to provide local stability analysis conditions, associated to the characterization of regions of stability for the sampled data closed-loop system. Moreover, the design of the anti-windup term should be addressed in the future. Another direction of future research concerns the reduction of the conservatism of the stability conditions by applying recent tools such as the improved Wirtinger-based inequality provided in [22].

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