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Optimal sensor placement and scheduling of hybrid PDEs arising in environmental and meteorological applications

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***Abstract*— We consider environmental and meteorological applications which undergo significant parametric changes that alter the behavior of the system. For the transport process under study, which depends on the Péclet number, the behavior changes from an advection dominated to a diffusion dominated process. For each value of the Péclet parameter, a set of optimal sensor locations and number is found and the associated state estimator is subsequently designed. A supervisory scheme is then utilized to schedule the sensors corresponding to the current value of the Péclet number by first turning off all the sensors associated with a different value of the Péclet number and then activate the sensors that are optimal for the current value of the Péclet number. Simulation results are included to provide further insight on parameter-dependent sensor and observer scheduling for a representative 1D environmental process.**

***Index Terms*— Distributed parameter systems; hybrid PDEs; optimal sensor selection; sensor scheduling; gain optimization.**

I. Physical modeling and problem description

Many environmental processes associated with weather prediction and monitoring are modeled by advection-diffusion PDEs in bounded domains, infinite domains of semi-infinite domains, [1], [2], [3], [4].

A representative PDE that captures the salient features of such environmental processes is the diffusion-advection equation restricted on the real line and given by the 1D PDE with source/sink term

$$\frac{\partial}{\partial t}\bar{u}(t,z) = D\frac{\partial^2}{\partial z^2}\bar{u}(t,z) - \upsilon\frac{\partial}{\partial z}\bar{u}(t,z) + \bar{f}(u) + \bar{d}(z)w(t). \quad (1)$$

Many choices for the boundary conditions are considered: *Dirichlet BCs* where the concentration is specified on a boundary and *Fourier BCs* at an inlet boundary when considering a problem defined on a semi-infinite domain $\xi > 0$ with inlet boundary at $\xi = 0$. Following [4], in the Fourier BCs case the flux is required to be continuous across the boundary.

To better present the concepts discussed in this work, we focus on the non-dimensionalized PDE form

$$\frac{\partial}{\partial t}u(t,\xi) = \frac{1}{Pe}\frac{\partial^2}{\partial \xi^2}u(t,\xi) - \frac{\partial}{\partial \xi}u(t,\xi) + f(u) + d(\xi)w(t) \quad (2)$$

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where Pe denotes the Péclet number, $u(t,\xi)$ the state at spatial location ξ and time t, $f(u)$ is nonlinear term and $d(\xi)w(t)$ the disturbance; in the latter, the spatial function $d(\xi)$ represents the spatial distribution of disturbances and the function $w(t)$ its temporal input. The Péclet number Pe plays an important role in the quantitative and qualitative bevavior of (2). The variation of the Péclet number renders the PDE time varying; a high Péclet number represents advective-dominated process and a low Péclet number represents diffusive-dominated process. This diurnal variation of the Péclet number renders (2) a switched dynamical system with a parameter (Pe) taking integer values over small set. For simplicity, it is assumed that the parameter set Θ consists of two elements

$$\Theta = \{Pe_a, Pe_d\},$$

a high value Pe_a associated with an advective-dominated process and a low value Pe_d associated with a diffusive-dominated process.

To enable the analysis, design and optimization associated with the sensor location and observer design, the above PDE is written in an abstract form as an evolution equation in a function space. Specifically, it is written as an evolution equation in the Hilbert space $H \subset L_2(\Omega)$

$$\dot{u}(t) = \mathcal{A}(Pe)u(t) + \mathcal{F}(u(t)) + \mathcal{D}w(t) \quad (3)$$

where the *parameter-dependent* spatial operator $\mathcal{A}(Pe)$ is defined as

$$\mathcal{A}(Pe)\phi = \frac{1}{Pe}\frac{d^2}{d\xi^2}\phi(\xi) - \frac{d}{d\xi}\phi(\xi), \quad \phi \in H^2(\Omega),$$

for $Pe \in \Theta$. The space H is equipped with the standard inner product of a weighted integral

$$\langle \phi_1, \phi_2 \rangle = \int_\Omega \rho(\xi)\phi_1^*(\xi)\phi_2(\xi)\,\mathrm{d}\xi, \;\; \phi_1, \phi_2 \in H,$$

and with norm $|\phi| = \sqrt{\langle \phi, \phi \rangle}$. The spatial function $\rho(\xi)$ denotes the weight function of the weighted inner product.

Information of the process (2), or its abstract form (3) is made possible via a finite number of measurements obtained by N in-domain sensing devices

$$y(t) = \int_\Omega c(\xi;\boldsymbol{\xi}_s)u(t,\xi)\,\mathrm{d}\xi = \begin{bmatrix} \int_\Omega c_1(\xi;\xi_{s1})u(t,\xi)\,\mathrm{d}\xi \\ \vdots \\ \int_\Omega c_N(\xi;\xi_{sN})u(t,\xi)\,\mathrm{d}\xi \end{bmatrix} \quad (4)$$

The output vector $y(t)$ denotes the vector of measured variables and the N dimensional vector of spatial functions

$c(\xi;\boldsymbol{\xi}_s)$ denotes the sensor shape model. The N-dimensional vector $\boldsymbol{\xi}_s \in (\Omega)^N$ denotes the vector of sensor locations or sensor centroids.

Typical choice for the spatial distribution of the sensing devices $c_i(\xi;\xi_{si})$ is the Dirac delta function $c_i(\xi;\xi_{si}) = \delta(\xi-\xi_{si})$ which provides the state information at the spatial location ξ_{si},

$$y_i(t) = \int_\Omega \delta(\xi-\xi_{si})u(t,\xi)\,\mathrm{d}\xi = u(t,\xi_{si}),\ i = 1\ldots,N.$$

We let $\boldsymbol{\xi}_s = (\xi_{s1},\xi_{s2},\ldots,\xi_{sN})$ denote the vector of the sensor locations and explicitly parameterize the output vector by $\boldsymbol{\xi}_s$ in order to emphasize the dependence on sensor location

$$y(t;\boldsymbol{\xi}_s) = \begin{bmatrix} y_1(t;\xi_{s1}) \\ \vdots \\ y_N(t;\xi_{sN}) \end{bmatrix} = \begin{bmatrix} \int_\Omega \delta(\xi-\xi_{s1})u(t,\xi)\,\mathrm{d}\xi \\ \vdots \\ \int_\Omega \delta(\xi-\xi_{sN})u(t,\xi)\,\mathrm{d}\xi \end{bmatrix} \quad (5)$$

The above can be written in terms of the output operator, parameterized by both the sensor location $\boldsymbol{\xi}_s$ and the Péclet number

$$y(t;\boldsymbol{\xi}_s) = \mathcal{C}(\boldsymbol{\xi}_s;Pe)u(t). \quad (6)$$

We can now state the design objective:

Problem statement: Given the parameter-dependent hybrid system (3) with location-parameterized measurements (5),

1) Find the optimal sensor location that corresponds to each switch mode of the parameter-dependent process (3); in other words, for each value of the parameter $Pe \in \Theta = \{Pe_a, Pe_d\}$, find the corresponding set of sensor locations that render the pair $(\mathcal{A}(Pe), \mathcal{C}(\boldsymbol{\xi}_s;Pe))$ approximately observable and optimize an appropriate measure of observability and filter performance
2) For each value of $Pe \in \Theta$, design the optimal filter associated with the pair $(\mathcal{A}(Pe), \mathcal{C}(\boldsymbol{\xi}_s;Pe))$ and schedule the set of sensors that correspond to that value of Pe.

II. Spatial discretization and finite dimensional representation

An appropriate approximation that plays a central role in defining the optimization measures is based on spectral expansion, where the state admits the expansion

$$u(t,\xi) = \sum_{i=1}^{\infty} \phi_i(\xi)x_i(t).$$

The spatial functions $\phi_i(\xi)$, are the eigenfunctions of the spatial operator, and are used as a basis for the space H in order to express the system state, u, as a function of system eigenmodes $x_i(t)$. Please note that these eigenfunctions depend on the parameter Pe.

To analyze the evolution equation (3), the linear component of $\mathcal{F}(u) \approx ku$ in (3) is retained, resulting in the following infinite dimensional ODE in H

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}(Pe)x(t) + kx(t) + \mathcal{D}(Pe)w(t) \\ y(t;\boldsymbol{\xi}_s) &= \mathcal{C}(\boldsymbol{\xi}_s;Pe)\,x(t). \end{aligned} \quad (7)$$

Following our earlier work [5], one can obtain a transfer function model for the ith eigenmode, via the use of Laplace transforms with $X_i(s) = \mathrm{L}[x_i(t)]$

$$X_i(s) = \frac{1}{s-(\lambda_i+k)}x_i(0) + \frac{\langle\phi_i,d\rangle}{s-(\lambda_i+k)}W(s),\ \ i = 1,\ldots\infty,$$

where λ_i denotes the ith eigenvalue of the operator $\mathcal{A}(Pe)$. In a similar fashion, one can obtain the expression for the Laplace transform of the measured output (6)

$$\begin{aligned} Y(s;\boldsymbol{\xi}_s) &= \sum_{i=1}^{\infty}\left(\int_\Omega c(\xi;\boldsymbol{\xi}_s)\phi_i(\xi)\,\mathrm{d}\xi\right)X_i(s) \\ &= \begin{bmatrix} \sum_{i=1}^{\infty}\left(\int_\Omega \delta(\xi-\xi_{s1})\phi_i(\xi)\,\mathrm{d}\xi\right)X_i(s) \\ \vdots \\ \sum_{i=1}^{\infty}\left(\int_\Omega \delta(\xi-\xi_{sN})\phi_i(\xi)\,\mathrm{d}\xi\right)X_i(s) \end{bmatrix} \end{aligned}$$

To present the output in a compact form and to facilitate the computation of an associated spatial norm, we define the N-dimensional output matrix

$$C_i(\boldsymbol{\xi}_s) = \int_\Omega c(\xi;\boldsymbol{\xi}_s)\phi_i(\xi)\,\mathrm{d}\xi,\ \ i = 1,\ldots,\infty$$

which in view of the specific sensor $c_i(\xi;\xi_{si}) = \delta(\xi-\xi_{si})$ takes the form

$$C_i(\boldsymbol{\xi}_s) = \begin{bmatrix} \phi_i(\xi_{s1}) \\ \vdots \\ \phi_i(\xi_{sN}) \end{bmatrix} \quad i = 1,\ldots\infty.$$

Similarly, we define the disturbance distribution matrix, where D_i represents the projection of the spatial distribution of the disturbances $d(\xi)$ with respect to the basis function set $\{\varphi_i(\xi)\}_{i=1}$, i.e.

$$D_i = \int_\Omega \rho(\xi)\phi_i^*(\xi)d(\xi)\,\mathrm{d}\xi = \langle\phi_i,d\rangle,\ \ i = 1,\ldots\infty.$$

The above enable the Laplace transform of the N-dimensional output vector expressed as

$$\begin{aligned} Y(s;\boldsymbol{\xi}_s) &= \sum_{i=1}^{\infty} C_i(\boldsymbol{\xi}_s)\frac{x_i(0)}{s-(\lambda_i+k)} + \sum_{i=1}^{\infty} C_i(\boldsymbol{\xi}_s)\frac{D_i}{s-(\lambda_i+k)}W(s) \\ &= \sum_{i=1}^{\infty}\begin{bmatrix} \frac{\phi_i(\xi_{s1})x_i(0)}{s-(\lambda_i+k)} \\ \vdots \\ \frac{\phi_i(\xi_{sN})x_i(0)}{s-(\lambda_i+k)} \end{bmatrix} + \sum_{i=1}^{\infty}\begin{bmatrix} \frac{\phi_i(\xi_{s1})\langle\phi_i,d\rangle}{s-(\lambda_i+k)} \\ \vdots \\ \frac{\phi_i(\xi_{sN})\langle\phi_i,d\rangle}{s-(\lambda_i+k)} \end{bmatrix} W(s). \end{aligned}$$

In more compact form, it is written as

$$\begin{aligned} Y(s;\boldsymbol{\xi}_s) &= \sum_{i=1}^{\infty} G_i(s;\boldsymbol{\xi}_s)x_i(0) + \sum_{i=1}^{\infty} H_i(s;\boldsymbol{\xi}_s)W(s) \\ &= G(s;\boldsymbol{\xi}_s,Pe)x(0) + H(s;\boldsymbol{\xi}_s,Pe)W(s), \end{aligned} \quad (8)$$

where $G_i(s;\boldsymbol{\xi}_s)$ or $H_i(s;\boldsymbol{\xi}_s)$ denote the transfer function between the measurement vector or the disturbance and the i^{th} eigenmode, respectively. Equation (8) can be used to provide the optimal sensor locations using open-loop techniques; essentially the optimal sensor locations are found so that the system becomes "more observable". Appropriate measures of enhanced observability may include the observability Gramian, the spatial H_2 and H_∞ norms.

The above open loop measures may be used to obtain the set of *admissible* sensor locations that can subsequently be used to design optimal state estimators. A state estimator for (7), parameterized by the sensor locations, is given by

$$\begin{aligned}\dot{\widehat{x}}(t) = \Big(\mathcal{A}(Pe) + kI - \mathcal{L}(\boldsymbol{\xi}_s, Pe)\mathcal{C}(\boldsymbol{\xi}_s, Pe)\Big)\widehat{x}(t) \\ + \mathcal{L}(\xi_s, Pe)y(t;\xi_s),\end{aligned} \tag{9}$$

where $\widehat{x}(t,\xi)$ is the estimate of the state $x(t)$ and $\mathcal{L}(\boldsymbol{\xi}_s, Pe)$ denotes the location-parametrized filter gain. The design of the filter gain is not explicitly defined, it may be a Kalman filter design, or simply based on Luenberger observer design.

The state error $e(t) = x(t) - \widehat{x}(t)$ associated with the above estimator is given by

$$\dot{e}(t) = \Big(\mathcal{A}(Pe) + kI - \mathcal{L}(\boldsymbol{\xi}_s, Pe)\mathcal{C}(\boldsymbol{\xi}_s, Pe)\Big)e(t) + \mathcal{D}w(t). \tag{10}$$

The state error (10) can be used to find the optimal sensor locations; it can be used to find the locations $\boldsymbol{\xi}_s$ that minimize the effects of the disturbance $w(t)$ on the state error and also to minimize an energy cost associated with the state error.

The sensor location optimization for the state error must also take into account the hybrid nature of the process (7). An optimal sensor location for a given value of the parameter Pe may not be an optimal value for a different value. This aspect must be considered in the sensor optimization problem.

In order to quantify the optimization objectives, we proceed with an assumption similar to the one made about the expansion of the process state. We assume that the spatially distributed state estimation error $e(t,\xi)$ can be expanded as

$$e(t,\xi) = \sum_{i=1}^{\infty} \varphi_i(\xi) e_i(t),$$

where $\varphi_i(\xi)$ is the ith eigenfunction of the parameter-dependent closed loop operator $\mathcal{A}(Pe) + kI - \mathcal{L}(\boldsymbol{\xi}_s, Pe)\mathcal{C}(\boldsymbol{\xi}_s, Pe)$. However, to obtain expressions similar to the sensor sensitivities to specific sensor locations, we consider the output estimation error

$$e(t;\boldsymbol{\xi}_s) = \sum_{i=1}^{\infty} \left(\int_\Omega c(\xi;\boldsymbol{\xi}_s)\varphi_i(\xi)\,\mathrm{d}\xi\right) e_i(t)$$

We also define the vector of the (closed-loop) modes at each of the N sensor locations

$$\overline{C}_i(\boldsymbol{\xi}_s) = \begin{bmatrix} \varphi_i(\xi_{s1}) \\ \vdots \\ \varphi_i(\xi_{sN}) \end{bmatrix} \quad i = 1, \ldots \infty,$$

Using the expansion and taking Laplace transforms of (10), we arrive at an expression similar to that for the process state

$$\begin{aligned}\mathcal{E}(s;\boldsymbol{\xi}_s) &= \sum_{i=1}^{\infty} \frac{\overline{C}_i(\boldsymbol{\xi}_s)e_i(0)}{s - \mu_i(\boldsymbol{\xi}_s)} + \sum_{i=1}^{\infty} \frac{\overline{C}_i(\boldsymbol{\xi}_s)D_i}{s - \mu_i(\boldsymbol{\xi}_s)} W(s) \\ &= \sum_{i=1}^{\infty} J_i(s;\boldsymbol{\xi}_s)e(0) + \sum_{i=1}^{\infty} K_i(s;\boldsymbol{\xi}_s)W(s) \\ &= J(s;\boldsymbol{\xi}_s, Pe)e(0) + K(s;\boldsymbol{\xi}_s, Pe)W(s).\end{aligned} \tag{11}$$

Similar to the definition of λ_i, the location-parameterized $\mu_i(\boldsymbol{\xi}_s)$ denotes the eigenvalues of the (parameter-dependent) closed loop operator $\mathcal{A}(Pe) + kI - \mathcal{L}(\boldsymbol{\xi}_s, Pe)\mathcal{C}(\boldsymbol{\xi}_s, Pe)$.

In view of (11), one can now quantify the first design objective presented in the previous section. The sensor location optimization can now be stated as that of finding the optimal locations $\boldsymbol{\xi}_s$ for each value $Pe \in \Theta$ so that

- it maximizes the spatial H_2 norm of $G(s;\boldsymbol{\xi}_s, Pe)$ in (8), i.e. it maximizes the value of information,
- it minimizes the spatial H_2 norm of $K(s;\boldsymbol{\xi}_s, Pe)$ in (11), i.e. it enhances the estimator performance.

Please note that the optimization of the spatial H_2 norm of $G(s;\boldsymbol{\xi}_s, Pe)$ is considered an "open-loop" optimization since it only improves the observability of the sensor locations. The minimization of the spatial H_2 norm of $K(s;\boldsymbol{\xi}_s, Pe)$ chooses from the admissible set of sensor locations that have already enhanced observability and then improves the performance of the state estimator.

The next two sections will summarize the sensor placement scheme and the sensor scheduling scheme.

III. Optimal sensor placement scheme

Once the state error is represented in (11) explicitly in terms of the initial conditions and the disturbances, the sensor placement can be presented. The location optimization can be posed as a constrained nonlinear optimization problem. To do so, we must first define the objective function and then present the optimization formulation.

A. Observability measures

Following our earlier fundamental work [5], we provide a quantitative metric for observability, which is based on the spatial H_2 norms first presented in [6]. The spatial $\mathcal{H}_2$ norm at sensor location $\boldsymbol{\xi}_s$ of $G(s,\xi;\boldsymbol{\xi}_s)$ in (8) is defined as

$$\|G\|_{\mathcal{H}_2}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_\Omega \rho(\xi)tr\left\{G^*(j\omega,\xi;\boldsymbol{\xi}_s)G(j\omega,\xi;\boldsymbol{\xi}_s)\right\}\mathrm{d}\xi\,\mathrm{d}\omega$$

and represents a measure of sensor sensitivity placed at location $\boldsymbol{\xi}_s$ over the entire spatial domain in an average sense.

Using the modal expansion for the process state, the spatial norm can be expressed in terms of the system modes

$$\begin{aligned}\|G\|_{\mathcal{H}_2}^2 &= \sum_{i=1}^{\infty}\int_{-\infty}^{\infty} tr\left\{G_i^*(j\omega;\boldsymbol{\xi}_s)G_i(j\omega;\boldsymbol{\xi}_s)\right\}d\xi \\ &= \sum_{i=1}^{\infty}\|G_i(s;\xi_s)\|_2^2.\end{aligned} \tag{12}$$

This then prompts the simplification involving the modal norms $f_i(\boldsymbol{\xi}_s, Pe) = \|G_i(s;\boldsymbol{\xi}_s, Pe)\|_2 = \left\|\dfrac{C_i(\boldsymbol{\xi}_s)}{s - (\lambda_i + k)}\right\|_2$.

The modal norm $f_i(\boldsymbol{\xi}_s, Pe)$ represents a measure of sensor network sensitivity placed at location $\boldsymbol{\xi}_s$ to the ith system mode. Each f_i is an N-dimensional vector.

The modal norm in (12) is an infinite dimensional vector. Since the higher modes become progressively more stable, then one can truncate the infinite series expansion and only consider a finite number of modes. Using the above, we have

$$\begin{aligned}\|G(s,\xi;\boldsymbol{\xi}_s,Pe)\|^2_{\mathcal{H}_2} =& \sum_{i=1}^{\infty}\|G_i(s;\boldsymbol{\xi}_s,Pe)\|_2^2 = \sum_{i=1}^{\infty} f_i^2(\boldsymbol{\xi}_s,Pe)\\ =&\sum_{i=1}^{M} f_i^2(\boldsymbol{\xi}_s,Pe)+\sum_{i=M+1}^{\infty} f_i^2(\boldsymbol{\xi}_s,Pe)\\ =&\mathbf{H}^2(\boldsymbol{\xi}_s,Pe)+\mathbf{S}^2(\boldsymbol{\xi}_s,Pe),\end{aligned} \tag{13}$$

where $\mathbf{H}^2(\boldsymbol{\xi}_s,Pe) = \sum_{i=1}^{M} f_i^2(\boldsymbol{\xi}_s,Pe)$ denotes the truncation of the $\mathcal{H}_2$ spatial norm to the first M slow modes and $\mathbf{S}^2(\xi_s,Pe) = \sum_{i=M+1}^{\infty} f_i^2(\boldsymbol{\xi}_s,Pe)$ denotes the "spillover" of the higher modes' dynamics on the sensor accuracy. With only open-loop measures, the sensor location optimization problem should maximize the value of information over the dominant modes, i.e. maximize $\mathbf{H}^2(\boldsymbol{\xi}_s,Pe)$ over a set of admissible locations, for each value $Pe \in \Theta$. Additionally, it should minimize the effects of the disturbance on the measurements. The later can be included in the minimization of the disturbance effects on the state estimation error (11).

B. Filter performance measures

Similar to the expansion (14), we can expand (11) in terms of the contributions of the individual modes of the closed-loop system. Using the orthogonality assumption of the closed loop modes and the fact that the higher mode of the closed loop system are also becoming progressively more stable, then in a similar fashion to (14), we have the truncated expressions for the spatial $\mathcal{H}_2$ norm

$$\begin{aligned}\|K(s;\boldsymbol{\xi}_s,Pe)\|^2_{\mathcal{H}_2} =& \sum_{i=1}^{\infty}\|K_i(s;\boldsymbol{\xi}_s,Pe)\|_2^2\\ =&\sum_{i=1}^{M}\|K_i(s;\boldsymbol{\xi}_s,Pe)\|_2^2+\sum_{i=M+1}^{\infty}\|K_i(s;\boldsymbol{\xi}_s,Pe)\|_2^2\\ =&\mathbf{L}^2(\boldsymbol{\xi}_s,Pe)+\mathbf{M}^2(\boldsymbol{\xi}_s,Pe),\end{aligned} \tag{14}$$

where $\mathbf{L}^2(\boldsymbol{\xi}_s,Pe) = \sum_{i=1}^{M}\|K_i(s;\boldsymbol{\xi}_s,Pe)\|_2^2$ denotes the truncation of the $\mathcal{H}_2$ spatial norm to the first slow M modes and $\mathbf{M}^2(\xi_s,Pe) = \sum_{i=M+1}^{\infty}\|K_i(s;\boldsymbol{\xi}_s,Pe)\|_2^2$ denotes the "spillover" of the higher modes' dynamics on the sensor accuracy.

C. Design of sensor network

Using both open-loop (enhanced observability) and close-loop (performance) measures, an optimization problem can be formulated as

$$\xi_s^* = \arg\max_{\xi_s\in\Omega_s}\left\{\mathbf{H}^2(\boldsymbol{\xi}_s)-\omega_L\mathbf{L}^2(\boldsymbol{\xi}_s)\right\} \tag{15}$$

leading to sensor locations with attributes:

- maximize the network sensitivity to dominant system dynamics
- minimize the effects of disturbances on observer performance

IV. Optimal sensor scheduling scheme

Once the optimal set os sensors (equiv. sensor centroids ξ_s) for each value of the parameter $Pe \in \Theta$, one can proceed with the scheduling policy of both the sensors and the estimator gains. We denote the output matrix associated with the optimal values of the sensor centroids for a given value of the parameter $Pe_k \in \Theta$ by

$$C(\xi_s^k;Pe_k), \quad Pe_k \in \Theta$$

and denote the optimal filter gain associated with the pair $(\mathcal{A}(Pe_k), C(\xi_s^k;Pe_k)$ by

$$L(\xi_s^k;Pe_k), \quad Pe_k \in \Theta.$$

Then, for each value of $Pe_k \in \Theta$ we implement the following estimator that utilizes the current set of sensors defined by $C(\xi_s^k;Pe_k)$ via

$$\begin{aligned}&\dot{\hat{x}}(t;\xi_s^k) = \left(\mathcal{A}(Pe_k)-L(\xi_s^k;Pe_k)C(\xi_s^k;Pe_k)\right)\hat{x}(t;\xi_s^k)\\ &+L(\xi_s^k;Pe_k)C(\xi_s^k;Pe_k)y(t;xi_s^k).\end{aligned} \tag{16}$$

Algorithm 1 Sensor scheduling for process monitoring

1: **measure** physical parameter $Pe_k \in \Theta$
2: **switch** to optimal sensors ξ_s^k associated with current $Pe_k \in \Theta$
3: **output** process measurements $y(t;\xi_s^k)$
4: **repeat** for next value of $Pe_\ell \in \Theta$

Algorithm 2 Estimation gain scheduling for state estimation

1: **read** physical parameter $Pe_k \in \Theta$
2: **obtain** process measurements with current sensor set ξ_s^k
3: **compute** filter gain $L(\xi_s^k;Pe_k)$ corresponding to current optimal sensor locations
4: **implement** state estimator (17)
5: **repeat** for next value of $Pe_\ell \in \Theta$

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