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Damping optimization in vibrational systems based on amplitude

Jonas Denißen[1] and Antti Koskela[2] and Hermann Mena[3] and Zoran Tomljanović[4]

***Abstract*— We consider the optimal damping problem for a linear vibrational system $M\ddot{x} + D\dot{x} + Kx = 0$, where M and K are positive definite matrices. For the damping optimization we use a criterion based on minimization of the integral of the solution's amplitude over a given time interval. Finding the optimal damping D is a very demanding problem, and using this approach the computational cost comes mainly from a large number of matrix exponential computations. We propose an efficient numerical scheme to accelerate these computations. The performance of our approach is illustrated by numerical results for an n-mass oscillator.**

I. Introduction

We consider a mathematical model of a linear vibrational system described by the differential equation

$$M\ddot{x} + D\dot{x} + Kx = 0, \tag{1}$$

with initial conditions $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$. The matrices M, D and K (called mass, damping and stiffness matrices, respectively) are real, symmetric $n \times n$ matrices. Moreover, M and K are positive definite. In vibrational systems the damping matrix is of the form

$$D = C_u + C_{\text{ext}},$$

where C_u represents the internal damping, while C_{ext} describes the external damping. The external damping $C_{\text{ext}} = v_1C_1 + v_2C_2 + \cdots + v_kC_k$ is semidefinite matrix where v_i is the viscosity and C_i describes a geometry of the corresponding dampers position for $i = 1, \ldots, k$. The internal damping C_u is usually taken to be a small multiple of the critical damping, i.e.,

$$\begin{aligned} &C_u = \alpha C_{\text{crit}},\ \alpha \in [0.02, 0.1], \\ &C_{\text{crit}} = 2M^{1/2}\sqrt{M^{-1/2}KM^{-1/2}}M^{1/2}. \end{aligned} \tag{2}$$

For more details see, e.g., [1], [2], [3]. Equation (1) can be transformed into a system of first order differential equations. For that purpose let Φ be a matrix that simultaneously diagonalizes M and K, i.e.,

$$\Phi^T K \Phi = \Omega^2 = \operatorname{diag}(\omega_1^2, \ldots, \omega_n^2)$$

and

$$\Phi^T M \Phi = I.$$

Since the internal damping is a small multiple of critical damping, we see that Φ diagonalizes the internal damping too, that is, $\Phi^T C_u \Phi = \alpha_c \Omega$, where $\alpha_c = 2\alpha$. I.e., the internal damping is in modal coordinates a small multiple of Ω.

We write the differential equation (1) in the phase space as a linear ODE $\dot{y} = Ay$, $y(0) = y_0$, where

$$A = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\alpha\Omega - \Phi^T C_{\text{ext}} \Phi \end{bmatrix}$$

and

$$y_0 = \begin{bmatrix} \Omega\Phi^{-1}x_0 \\ \Phi^{-1}\dot{x}_0 \end{bmatrix}.$$

For more details, see [1], [4], and the references therein. The solution for this ODE is given by

$$y(t) = e^{At}y_0.$$

For given matrices M, D and K it can be shown that the eigenvalues of A are in the left half of the complex plane [5], i.e., A is stable. Thus, each component of the solution y tends to its equilibrium

[1]Jonas Denißen is with the Computational Methods in Systems and Control Theory group at Max Planck Institute for Dynamics of Complex Technical Systems, 39106 Magdeburg, Germany denissen@mpi-magdeburg.mpg.de

[2]Antti Koskela is with the Department of Mathematics, KTH Royal Institute of Technology, SE-10044 Stockholm, Sweden akoskela@kth.se

[3]Hermann Mena is with the Department of Mathematics, University of Innsbruck, A-6020 Innsbruck, Austria hermann.mena@uibk.ac.at

[4]Zoran Tomljanović is with the Department of Mathematics, Josip Juraj Strossmayer University of Osijek, HR-31000 Osijek, Croatia ztomljan@mathos.hr

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as t goes to infinity. In the following we assume that the coordinate system is set such that y is zero at the equilibrium.

The relation between the convergence speed of $\|y(t)\|_2$ and the damping problem is discussed in Section V.

The damping optimization problem as well as its technological motivation has been widely studied (see e.g. [1], [2], [3], [4], [6], [7]). In [8] the author presents a mathematical introduction to damped oscillations of linear systems. The main purpose of this paper is to study the damping optimization based on a criterion that uses the solution of a corresponding differential equation. In the next section we discuss this approach.

II. Damping optimization

The aim of damping optimization is to find the 'best' damping matrix D which ensures optimal evanescence of each component of y. In this paper we are interested in damping based on given initial conditions. Note that by optimizing D we optimize the external damping depending on a given geometry of the dampers positions. For this optimization problem we need an optimization criterion.

One criterion is the so-called spectral abscissa criterion which minimizes the spectral abscissa defined by

$$\eta(A) := \max_k \operatorname{Re} \lambda_k,$$

where λ_k's are the eigenvalues of the corresponding quadratic eigenvalue problem

$$(\lambda^2 M + \lambda D + K)x = 0.$$

This criterion considers the eigenvalues' behavior, meaning that 'dangerous' eigenvalues are the ones closest to imaginary axis. The aim is to determine such a damping that the eigenvalues are as far as possible from the imaginary axis. In the optimization process a large number of quadratic eigenvalue problems needs to be solved, and therefore numerical approximations for accelerating the optimization process are required. In [8] and [9] the authors consider the eigenvalue behavior especially for small and large viscosities, which can give insight into the quality of a given damping.

Another criterion minimizes the total energy of the system, i.e., the aim is to determine

$$\min_D \int_0^\infty E(t, D)\mathrm{d}t,$$

where the energy $E(t, D)$ is the sum of the kinetic and potential energy,

$$E(t, D) = \frac{1}{2}\dot{x}(t)^* M\dot{x}(t) + \frac{1}{2}x(t)^* Kx(t).$$

In this criterion, dependence on initial data can be avoided by averaging over all initial states of unit energy and a given frequency range. This leads to a Lyapunov equation which gives the total average energy of the system (see [1], [6], [2], [7]).

On the other hand, if one is interested in damping the system which is adapted for a particular initial data y_0, a criterion that includes the initial data might be considered.

Since the main problem is to find an optimal damping matrix D which ensures optimal evanescence of each component of $y(t)$ in time, we propose as a minimization criterion

$$\min_D \int_0^T \|y(t)\|_2 \,\mathrm{d}t = \min_D \int_0^T \|e^{At}y_0\|_2 \,\mathrm{d}t, \quad (3)$$

where T is defined for a feasible time interval $[0, T]$ in which we want to minimize $y(t)$. In particular, the amount of time T depends on inputs x_0 and $\dot{x}_0$. Taking T large enough we include all important information regarding the vibrations.

In general this is a Mixed-Integer Nonlinear Program (MINLP) and hence, it is $\mathbf{NP}$-hard. It combines the complexity of integer programming as well as global optimization of nonlinear constraints and a nonlinear objective function, see e.g. [10]. Therefore, even for moderate dimension this is a very demanding problem and one has to find ways to overcome the complexity issue as long as $\mathbf{P} \neq \mathbf{NP}$ [11].

An example of such a vibrational system is an n-mass oscillator or an oscillator ladder shown in Figure 1. This system describes the mechanical system of n masses $m_1, \ldots, m_n$ and $n+1$ springs with stiffnesses being equal to $k_1, \ldots, k_{n+1}$. The mass and stiffness matrices are given by

$$M = \operatorname{diag}(m_1, m_2, \ldots, m_n),$$

and

$$K = \begin{pmatrix} k_1+k_2 & -k_2 & & & & \\ -k_2 & k_2+k_3 & -k_3 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -k_{n-1} & k_{n-1}+k_n & -k_n & \\ & & & -k_n & k_n+k_{n+1} \end{pmatrix},$$

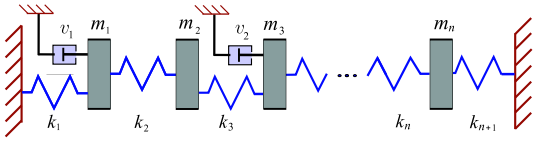

Fig. 1. An n-mass oscillator with two dampers, and different viscosities.

and the damping matrix is

$$D = C_u + C_{\text{ext}},$$

where the internal damping C_u is defined as in (2). Since we will consider two dampers of different viscosities, the external damping is defined by

$$C_{\text{ext}} = v_1 e_i e_i^T + v_2 e_j e_j^T, \quad 1 \le i < j \le n.$$

In order to determine the optimal damping one needs to optimize the dampers position i, j and the viscosities v_1, v_2 which in our case results in evaluations of the integral (3) for different parameters i, j, v_1 and v_2. This results furthermore in a large number of matrix exponential times vector products, which constitutes the major computational cost of the optimization process. In the next section we propose an algorithm for computing this criterion based on an efficient approximation of the exponential of the matrix A.

III. The Algorithms

In this section we describe the numerical algorithms that were used to implement the optimization procedure discussed in Section II.

For evaluating the integral

$$I(T) := \int_0^T \|y(t)\|_2 \,\mathrm{dt} = \int_0^T \|e^{At} y_0\|_2 \,\mathrm{dt} \quad (4)$$

we need to efficiently compute the integrand $\|e^{At}y_0\|_2$ for several values of t. Due to the complexity of damping optimization we consider moderate dimensions where the diagonalization $A = V\Lambda V^{-1}$ is affordable. Diagonalization can be performed up to a system dimension of 10 000 (although nowadays one can diagonalize larger matrices), but the limiting factor for the optimization is the determination of the damper's positions.

The integrand can be evaluated by using the fact that

$$e^{tA}y_0 = e^{tV\Lambda V^{-1}}y_0 = V\big((e^{t\Lambda})V^{-1}y_0\big).$$

This gives the following simple algorithm to evaluate the integral (4).

1) Diagonalize: $A = V\Lambda V^{-1}$.
2) Compute $\widetilde{y}_0 = V^{-1}y_0$.
3) Evaluate the integral (4), e.g. using the command `quad` of Matlab, and a fast evaluation by
$$e^{tA}y_0 = Ve^{t\Lambda}\widetilde{y}_0. \quad (5)$$

The computational complexity for diagonalization of the matrix A and solving a linear system of equations $V\widetilde{y}_0 = y_0$ is of $\mathcal{O}(n^3)$. By (5) the evaluation of the integrand requires only the evaluation of the diagonal matrix $e^{t\Lambda}$, the product $e^{t\Lambda}\widetilde{y}_0$, an additional matrix vector product and a computation of the norm. The computational complexity of the matrix-vector product and the computation of the norm is $\mathcal{O}(n^2)$ and $\mathcal{O}(n)$, respectively.

Numerical results in Section IV show drastic reductions in the running time compared to an implementation based on the Matlab command `expm` for computing $\|e^{At}y_0\|_2$. This command is based on the Padé approximation with scaling and squaring [12].

We also consider the problem of evaluating the integral (4) as a solution of the ODE

$$I'(t) = \|y(t)\|_2, \quad I(0) = 0, \quad (6)$$

where $I(t)$ is defined as in (4). In order to evaluate this integral, we compare the `quad` command of Matlab which uses an adaptive Simpson rule [13], and the integrator `ode45` [14], which is based on an explicit Runge-Kutta method with an adaptive step-size control. Both algorithms take desired error tolerances as parameters. For both algorithms, the integrand $\|y(t)\|_2$ is evaluated using the diagonalization of A as in (5).

For simplicity we assume that both dampers have the same viscosity, that is $C_{\text{ext}} = v(e_i e_i^T + e_j e_j^T)$, $1 \le i < j \le n$. This viscosity is optimized using Brent's method (see e.g. [15]) implemented in Matlab's command `fminbnd`. This

algorithm also takes an error tolerance as a parameter. The error tolerances we used are given in the numerical experiments.

IV. Numerical results

The numerical comparisons were performed on a dual core laptop with 2.26 GHz and 3GB RAM.

In the numerical examples for a given n we consider the following configurations:

$$\begin{aligned} m_i &= n - 2i, \quad i = 1, \ldots, n/4; \\ m_i &= i + n/4, \quad i = n/4 + 1, \ldots, n; \\ k_i &= 1, \quad i = 1, \ldots, n; \\ T &= 1000; \\ \alpha_c &= 0.02. \end{aligned}$$

A surface plot of the optimal integral value (4) as a function of the dampers' positions is given in Figure 2 ($n = 200$).

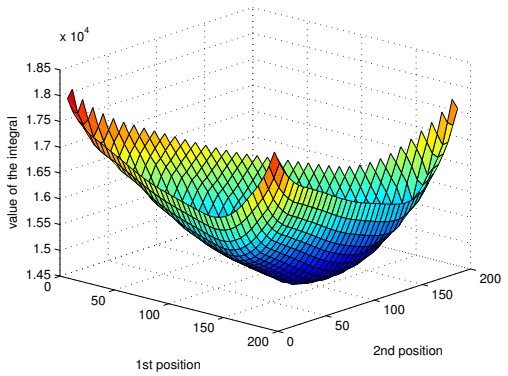

Fig. 2. x and y-axis: positions of the two dampers, z-axis: value for the minimal value of the integral (3).

We consider the example above and set $n = 20$. Implementing the evaluation of the integral using the `expm` and `quad`, we get the relative errors for the integral values and the timings given in Table I and II (with 2 different tolerances).

damp. pos.	relative error	time(s)
1, 1	5.62e-07	4.08
1, 6	3.46e-06	5.16
1, 11	4.63e-05	5.46
1, 16	5.30e-06	4.96
6, 6	4.45e-07	5.79
6, 11	7.41e-06	5.32
6, 16	9.72e-06	5.45
11, 11	7.76e-06	6.87
11, 16	3.06e-05	5.23
16, 16	1.10e-06	6.65

TABLE I

Relative errors and timings using the `expm` and `quad` commands ($tol = 10^{-4}$ for `quad` and $tol = 10^{-3}$ for `fminbnd`).

damp. pos.	relative error	time(s)
1, 1	1.22e-12	37.60
1, 6	3.94e-10	40.26
1, 11	4.92e-10	40.82
1, 16	1.92e-08	41.91
6, 6	8.48e-10	46.85
6, 11	2.07e-11	45.37
6, 16	4.33e-11	43.78
11, 11	1.34e-09	50.92
11, 16	1.61e-10	45.06
16, 16	8.90e-11	48.34

TABLE II

Relative errors and timings using the `expm` and `quad` command ($tol = 10^{-8}$ for `quad` and $tol = 10^{-7}$ for `fminbnd`).

Using the diagonalization of A combined with the `quad` command we get the errors and timings given in Table III and IV.

damp. pos.	relative error	time(s)
1, 1	5.62e-07	0.35
1, 6	3.46e-06	0.40
1, 11	4.63e-05	0.43
1, 16	5.30e-06	0.40
6, 6	4.45e-07	0.45
6, 11	7.41e-06	0.43
6, 16	9.72e-06	0.43
11, 11	7.76e-06	0.55
11, 16	3.06e-05	0.42
16, 16	1.10e-06	0.54

TABLE III

Relative errors and timings using the diagonalization of A and the `quad` command ($tol = 10^{-4}$ for `quad` and $tol = 10^{-3}$ for `fminbnd`).

damp. pos.	relative error	time(s)
1, 1	1.19e-12	3.87
1, 6	3.94e-10	3.78
1, 11	4.92e-10	3.24
1, 16	1.92e-08	3.36
6, 6	8.48e-10	3.79
6, 11	2.07e-11	3.53
6, 16	4.33e-11	3.41
11, 11	1.34e-09	4.34
11, 16	1.61e-10	3.43
16, 16	8.90e-11	4.09

TABLE IV

RELATIVE ERRORS AND TIMINGS USING THE DIAGONALIZATION OF A AND THE `quad` COMMAND ($tol = 10^{-8}$ FOR `quad` AND $tol = 10^{-7}$ FOR `fminbnd`).

When the integrator `ode45` is used we get the errors and timings given in Table V and VI. The same tolerances were used as for the `quad` command.

damp. pos.	relative error	time(s)
1, 1	2.86e-04	0.43
1, 6	1.85e-03	0.39
1, 11	1.26e-03	0.38
1, 16	4.74e-03	0.43
6, 6	9.95e-04	0.42
6, 11	1.10e-03	0.44
6, 16	1.64e-03	0.45
11, 11	5.63e-03	0.50
11, 16	1.99e-03	0.49
16, 16	9.81e-04	0.43

TABLE V

RELATIVE ERRORS AND TIMINGS USING THE DIAGONALIZATION OF A AND THE `quad` COMMAND ($tol = 10^{-4}$ FOR `ode45` AND $tol = 10^{-3}$ FOR `fminbnd`).

damp. pos.	relative error	time(s)
1, 1	6.79e-08	3.01
1, 6	7.28e-07	3.86
1, 11	8.68e-07	3.96
1, 16	1.34e-06	4.14
6, 6	1.99e-07	3.69
6, 11	1.39e-07	3.72
6, 16	1.61e-07	4.75
11, 11	2.20e-07	4.58
11, 16	6.11e-07	4.27
16, 16	1.01e-07	4.71

TABLE VI

RELATIVE ERRORS AND TIMINGS USING THE DIAGONALIZATION OF A AND THE `quad` COMMAND ($tol = 10^{-8}$ FOR `ode45` AND $tol = 10^{-7}$ FOR `fminbnd`).

As expected due to the adaptiveness of the algorithms, `quad` and `ode45` give similar performance. However, we see that the actual errors given by `ode45` match well with the set tolerances.

Using the diagonalization approach and `ode45` for the ODE (6), we are able compute the objective values also for larger systems. Table VII gives the timings for the n-mass oscillator of size $n = 200$.

damp. pos.	opt. visc.	obj. value	time (s)
1 1	42.14	18372.21	9.68
1 51	24.31	17097.94	12.03
1 101	26.43	16601.98	11.20
1 151	30.69	16460.77	12.14
51 51	11.38	17229.99	12.56
51 101	23.70	15483.49	11.19
51 151	26.71	15109.36	10.54
101 101	12.77	16738.54	12.87
101 151	26.81	14776.80	11.01
151 151	14.92	16603.04	12.45

TABLE VII

NUMERICAL RESULTS AND TIMINGS USING THE DIAGONALIZATION OF A AND A SIMPLE QUADRATURE RULE FOR THE INTEGRAL, WHEN $n = 200$ AND $tol = 10^{-4}$.

V. FUTURE PLANS

We will next describe some future plans to enhance the damping optimization algorithm.

The running time of the numerical computation of the matrix e^{tA} or of the vector $e^{tA}b$ strongly depends on the norm of tA. Because of this, it is important to evaluate $e^{tA}b$ as

$$e^{tA}b = e^{(t-t_0)A}(e^{t_0A}b)$$

if $e^{t_0A}b$ is at hand. Especially for problems where the diagonalization of A is not feasible (system dimension larger than 10000), this will be taken into account when implementing a step-size adaptive algorithm for the ODE (6).

For a normal and stable matrix A, the right hand side $\|y(t)\|_2$ of (6) is monotonously decreasing, as it can be seen by differentiating and bounding as follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\|y(t)\|_2 &= \frac{\langle y'(t), y(t)\rangle + \langle y(t), y'(t)\rangle}{2\|y(t)\|_2} \\ &= \frac{\langle y(t), \big((A+A^T)/2\big)y(t)\rangle}{\|y(t)\|_2} \\ &\le \mu(A)\|y(t)\|_2,\end{aligned} \tag{7}$$

where $\mu(A)$ is the logarithmic norm of A, i.e.,

$$\mu(A) = \lambda_{\max}(\tfrac{1}{2}(A + A^T)).$$

If A is normal, then $\mu(A) = \eta(A)$ (see e.g. [16]) and from (7) it follows that $\|y(t)\|_2$ is decreasing for a stable matrix A. However, in general the matrix A is nonnormal and this leads to a transient growth of $\|y(t)\|_2$ (see [17]). Therefore, we will seek a norm $\|\cdot\|$ in which the solution decreases monotonically, i.e., the vibrations are suppressed (see e.g. [18]). We have the additional requirement that the computation of the integral $\int_0^T \|e^{At}y_0\|\mathrm{dt}$ must be affordable.

By applying an explicit Runge-Kutta method to the ODE (6) combined with a step-size control, we may control the error of the optimization, and need fewer evaluations of the integrand (larger time steps can be taken when $\|y(t)\|_2$ becomes small). Moreover, this approach extends naturally to larger problems, where the diagonalization $A = V\Lambda V^{-1}$ is possibly not feasible and, e.g., Krylov subspace methods should be used to evaluate the product $e^{tA}b$ (see e.g. [19]).

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