

Computing the Codimension of the Singularity at the Origin for Delay Systems: The Missing Link with Birkhoff Incidence Matrices

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Abstract—A standard framework in analyzing Time-delay systems consists first, in identifying the associated crossing roots and secondly, then, in characterizing the local bifurcations of such roots with respect to small variations of the system parameters. Moreover, the dynamics of such spectral values are strongly related to their multiplicities (algebraic/geometric). This paper focuses on an interesting type of such singularities; that is when the zero spectral value is multiple. The simplest case, which is quite common in applications, is characterized by an algebraic multiplicity two and a geometric multiplicity one known as Bogdanov-Takens singularity. Unlike finite dimensional systems, the algebraic multiplicity of the zero spectral value may exceed the dimension of the delay-free system of differential equations. To the best of the authors' knowledge, the bound of such a multiplicity for Time-delay systems was not deeply investigated in the literature. Our contribution is two fold. First, we emphasize the link between the multiplicity characterization and Birkhoff matrices. Secondly, we elaborate a constructive bound for the zero spectral value in the regular case; i.e. when the delay polynomials of a given quasipolynomial are complete, as well as in the singular case; i.e. when such polynomials are sparse. In the last case, the established bound is sharper than Polya-Szegö generic bound.

I. INTRODUCTION

Consider the following infinite-dimensional system with N constant delays:

$$\dot{x} = \sum_{k=0}^N A_k x(t - \tau_k) \quad (1)$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ denotes the state-vector, under appropriate initial conditions belonging to the Banach space of continuous functions $\mathcal{C}([-\tau_N, 0], \mathbb{R}^n)$. Here τ_j , $j = 1 \dots N$ are strictly increasing positive constant delays with $\tau_0 = 0$ and $0 < \tau_1 < \tau_2 < \dots < \tau_N$, the matrices $A_j \in \mathcal{M}_n(\mathbb{R})$ for $j = 0 \dots N$. It is well known that the asymptotic behavior of the solutions is determined from the roots of the characteristic equation [1], [2], that is a transcendental equation in the Laplace variable λ in which appears exponential terms induced by delays. More precisely, system (1) has a characteristic function $\Delta : \mathbb{C} \times \mathbb{R}_+^N \rightarrow \mathbb{C}$ of the form:

$$\Delta(\lambda, \tau) = \det \left(\lambda I - A_0 - \sum_{k=1}^N A_k e^{-\tau_k \lambda} \right) \quad (2)$$

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or shorter, denoted $\Delta(\lambda)$, which gives

$$\left. \begin{aligned} \Delta(\lambda) &= P_0(\lambda) + \sum_{M^k \in S_{N,n}} P_{M^k}(\lambda) e^{\sigma_{M^k} \lambda} \\ &= P_0(\lambda) + \sum_{k=1}^{\tilde{N}_{N,n}} P_{M^k}(\lambda) e^{\sigma_k \lambda} \end{aligned} \right\} \quad (3)$$

where $\sigma_{M^k} = -M^k \tau^T$, $\tau = (\tau_1, \dots, \tau_N)$ is the delays vector and $S_{N,n}$ is the set of all the possible row vectors $M^k = (M_1^k, \dots, M_N^k)$ belonging to \mathbb{N}^N such that $1 \leq M_1^k + \dots + M_N^k \leq n$ and $\tilde{N}_{N,n} = \#(S_{N,n})$. For instance,

$$S_{3,2} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (2, 0, 0), (1, 1, 0), (1, 0, 1), (0, 2, 0), (0, 1, 1), (0, 0, 2)\},$$

is ordered first by increasing sums ($\sum_{i=1}^N M_i^k$) then by lexicographical order, in this case one has:

$$M^2 = (0, 1, 0) \quad \text{and} \quad \tilde{N}_{3,2} = 9.$$

Without any loss of generality, assume that P_0 is a monic polynomial of degree n in λ with unitary leading monomial and the polynomials P_{M^k} are such that $\deg(P_{M^k}) = n - \sum_{s=1}^N M_s^k \leq (n-1) \forall M^k \in S_{N,n}$ and D_q will designate the degree of the quasipolynomial $\sum_{k=1}^{\tilde{N}_{N,n}} P_{M^k}$. One can prove that the quasipolynomial function (3) admits an infinite number of zeros, see [3], [2]. The study of zeros of entire function [4] in the form (3) plays a crucial role in the analysis of asymptotic stability of the zero solution of given system (1). Indeed, the zero solution is asymptotically stable if all the zeros of (3) are in the open left-half complex plane [5]. Accordingly to this observation, the parameter space which is spanned by the coefficients of the polynomials P_i , can be split into stability and instability domains (Nothing else that the so-called D-decomposition, see for instance [5] and references therein). These two domains are separated by a boundary, called the *critical boundary*, corresponding to the spectra consisting in roots with zero real parts. When the intersection of the spectrum with such a boundary is nonempty then the equilibrium point is said to be *nonhyperbolic*. The local behavior at a nonhyperbolic singularity is described by the versal deformation of the singularity; that is, replacing the original vector field $f(\cdot)$ by a perturbation-dependent vector field $g(\cdot, \epsilon)$ such that when the vector parameter vanish $\epsilon = (\epsilon_1, \dots, \epsilon_k) = 0$ one has $f(\cdot) = g(\cdot, \epsilon)|_{\epsilon=0}$. This deformation g is said to be *versal* if any other deformation occurs as a deformation induced from it g and the number of its parameters k is minimal. The *codimension* of such a singularity is nothing else than the integer k . For instance,

singularities of codimension 1 are characterized by a simple zero spectral value or a simple pair of imaginary roots.

From another viewpoint, matrices arising from a wide range of applications typically display characteristic structures, for instance sparsity patterns. Such structures induce appropriate algebraic restrictions that often allow to multiple roots. In this paper we are concerned by multiple-zero spectral values. The typical example for non-simple zero spectral value is the Bogdanov-Takens singularity which is characterized by an algebraic multiplicity two and a geometric multiplicity one (codimension two singularity). Cases with higher order multiplicities of the zero spectral value are known to us as generalized Bogdanov-Takens singularities. Those types of configurations are not necessarily synthetic and are involved in concrete applications. Indeed, the Bogdanov-Takens singularity is identified in [6] where the case of two coupled scalar delay equations modeling a physiological control problem is studied. In [7], this type of singularity is also encountered in the study of coupled axial-torsional vibrations of an oilwell rotary drilling system. Moreover, the paper [8] is dedicated to this type of singularities, codimensions two and three are studied and the associated center manifolds are explicitly computed.

Commonly, the time-delay induces desynchronizing and/or destabilizing effect on the dynamics. However, new theoretical developments in control of finite-dimensional dynamical systems suggest the use of delays in the control laws for stabilization purposes. For instance, the papers [9], [10] are concerned by the stabilization of the inverted pendulum by delayed control laws and furnish concrete situations where the codimension of the zero spectral value exceeds the number of the coupled scalar equations modeling the inverted pendulum on cart. In [9], the authors prove that delayed proportional-derivative (PD) controller stabilize the inverted pendulum by identifying a codimension three singularity for a system of two coupled delayed equations. In [10], the same singularity is characterized by using a particular delay block configuration. It is shown that two delay blocks offset a PD delayed controller. By the present work we investigate the link between the system parameters (delays and coefficients) and the upper bound for the codimension of the zero spectral value.

The remaining paper is organized as follows: Section 2 is dedicated to the background on some old problems, namely, the zeros of entire functions as well as the problem of multivariate interpolation. The challenges of the cited problems motivate the present investigation. Section 3, includes some important results from [11] (an extended version of the present work) allowing to recover the generic Pólya-Szegő bound \sharp_{PS} . A resulting constructive framework is presented. Next, under some sparsity patterns, the main result is proposed and proved in section 4. A control oriented illustrative example and some concluding remarks end the paper.

II. PREREQUISITES AND MOTIVATIONS

Although the algebraic multiplicity of each spectral value of a time-delay system is finite (a direct consequence of Rouché Theorem, see [12]), to the best of the authors' knowledge, the estimation of *the upper bound of the codimension of the zero spectral value* did not receive a complete characterization especially when the physical parameters of a given time-delay model are subject to algebraic constraints. It is worthy to note that, the root at the origin is invariant with respect to the delay parameters, however, its multiplicity is strongly dependent on the existing links between the delays and the other parameters of the system.

In this paper, we investigate this type of singularity and give an answer to the question above. This work is motivated by the fact that the knowledge of such information is crucial: first, in the linear analysis for time-delay systems, for instance, the analysis of sensitivity as well as the study local bifurcation. Secondly, when dealing with a nonlinear analysis and the center manifold computations are involved. Indeed, when the zero spectral value is the only eigenvalue with zero real part, then the center manifold dimension is none other than the codimension of the generalized Bogdanov-Takens singularity [13], [14], [15].

The following result in [12] gives some valuable information allowing to have a first estimation of such a bound for the multiplicity.

Proposition 1 (Pólya-Szegő, [12], pp. 144). *Let τ_1, \dots, τ_N denote real numbers such that*

$$\tau_1 < \tau_2 < \dots < \tau_N,$$

and d_1, \dots, d_N positive integers satisfying

$$d_1 \geq 1, d_2 \geq 1 \dots d_N \geq 1, \quad d_1 + d_2 + \dots + d_N = D + N.$$

Let $f_{i,j}(s)$ stands for the function $f_{i,j}(s) = s^{j-1} e^{\tau_i s}$, for $1 \leq j \leq d_i$ and $1 \leq i \leq N$.

Let \sharp be the number of zeros of the function

$$f(s) = \sum_{1 \leq i \leq N, 1 \leq j \leq d_i} c_{i,j} f_{i,j}(s),$$

that are contained in the horizontal strip $\alpha \leq \mathcal{I}(z) \leq \beta$.

Assuming that

$$\sum_{1 \leq k \leq d_1} |c_{1,k}| > 0, \dots, \sum_{1 \leq k \leq d_N} |c_{N,k}| > 0,$$

then

$$\frac{(\tau_N - \tau_1)(\beta - \alpha)}{2\pi} - D + 1 \leq \sharp \leq \frac{(\tau_N - \tau_1)(\beta - \alpha)}{2\pi} + D + N - 1.$$

See also [16] for a modern formulation of the mentioned result. The proof of Pólya-Szegő result is mainly based on Rouché Theorem. It can be generically exploited to establish a bound for the multiplicity of the zero spectral value that we denote by \sharp_{PS} . Indeed, setting $\alpha = \beta = 0$ allows to $\sharp_{PS} \leq D + N - 1$ where D stands for the degree of the quasipolynomial function f and N designate the associated number of polynomials. This gives a sharp bound in the case of *complete polynomials* i. e. polynomials having all their

terms ordered from the greatest degree up to the independent term. Nevertheless, it is obvious that the Pólya-Szegő bound remains unchanged when certain coefficients $c_{i,j}$ vanish without affecting the degree of the quasipolynomial function. Such a remark allows us to claim that Pólya-Szegő bound does not take into account the algebraic constraints on the parameters. However, such constraints are commonly encountered in control problems due to models structures: explicit situations will be given in the next section concerned by motivating examples. Moreover, when one needs the conditions insuring a given multiplicity bounded by \sharp_{PS} , then computations of the successive differentiations of the quasipolynomial have to be made.

By the present paper, we emphasize a systematic approach allowing to a sharper bound for the zero spectral value multiplicity. Indeed, the proposed approach does not only take into account the algebraic constraints on the coefficients $c_{i,j}$ but it also furnishes appropriate conditions guaranteeing such a multiplicity. Furthermore, the symbolic approach we adopt in this study underlines the connexion between the codimension of the zero singularity problem and *incidence matrices* of the so-called *Confluent Vandermonde Matrix* as well as the *Birkhoff Matrix*, see for instance [17], [18], [19], [20], [11]. To the best of the author knowledge, the first time the Vandermonde matrix appears in a control problem is reported in [21], where the controllability of a finite dimensional dynamical system is guaranteed by the invertibility of such a matrix, see [21, p. 121]. Next, in the context of time-delay systems, the use of Vandermonde matrix properties was proposed by [22], [5] when controlling one chain of integrators by delay blocks. Here we further exploit the algebraic properties of such matrices into a different context.

Initially, Birkhoff and Vandermonde matrices are derived from the problem of polynomial interpolation of an unknown function g , that can be presented in a general way by describing the interpolation conditions in terms of *Birkhoff incidence matrices*, see for instance [23]. For a given integers $n \geq 1$ and $r \geq 0$, the matrix

$$\mathcal{E} = \begin{pmatrix} e_{1,0} & \cdots & e_{1,r} \\ \vdots & & \vdots \\ e_{n,0} & \cdots & e_{n,r} \end{pmatrix},$$

is called an incidence matrix if $e_{i,j} \in \{0,1\}$ for every i and j . Such a matrix contains the data providing the known information about the function g . Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ such that $x_1 < \dots < x_n$, the problem of determining a polynomial $\hat{P} \in \mathbb{R}[x]$ with degree less or equal to r that interpolates g at (x, \mathcal{E}) , i.e. which satisfies the conditions:

$$\hat{P}^{(j)}(x_i) = g^{(j)}(x_i)$$

is known as the *Birkhoff interpolation problem*. An incidence matrix \mathcal{E} is said to be *poised* if such a polynomial \hat{P} is unique. This amounts to saying that the coefficients of the interpolating polynomial \hat{P} are solutions of a linear square system with associated square matrix $\Upsilon_{\mathcal{E}}$ that we call in the

sequel by *Birkhoff matrix*. This matrix is parametrized in x and is shaped by \mathcal{E} . It turns out that the incidence matrix \mathcal{E} is poised if and only if the Birkhoff matrix $\Upsilon_{\mathcal{E}}$ is non singular for all x such that $x_1 < \dots < x_n$. The characterization of poised incidence matrices is solved for interpolation problem for low degrees, for instance, the problem still unsolved for any degree $n \geq 6$, see for instance [20], [24]. As an illustration of the above notions, let consider the reduced example from [24] with the incidence matrix

$$\mathcal{E} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (4)$$

for which we associate the Birkhoff matrix

$$\Upsilon_{\mathcal{E}}^T = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 0 & 2 & 6x_2 \\ 0 & 1 & 2x_3 & 3x_3^2 \end{pmatrix}.$$

The interpolation problem is solvable if and only if

$$12x_3x_2 + 6x_1^2 - 12x_2x_1 - 6x_3^2$$

does not vanish for all values of x such that $x_1 < x_2 < x_3$. For the sake of the space limit, one can afford to replace in the sequel the incidence matrix \mathcal{E} by an appropriate vector $\mathcal{V}_{\mathcal{E}}$ reproducing exactly the same information, for instance, in the case of (4), one has $\mathcal{V}_{\mathcal{E}} = (x_1, x_1, *, *, x_2, *, x_3)$. We point out that when no stars appear in $\mathcal{V}_{\mathcal{E}}$, and if in addition no any variable is repeated in the sequence defining $\mathcal{V}_{\mathcal{E}}$ then we are dealing with the classical Vandermonde matrix, otherwise, (there are at least a repeated variable in $\mathcal{V}_{\mathcal{E}}$) the matrix $\Upsilon_{\mathcal{E}}$ is the so called Confluent Vandermonde matrix.

In the sequel, by *generalized Birkhoff matrix* we associate to a given positive integer $s \geq 0$ and an incidence matrix \mathcal{E} (or equivalently $\mathcal{V}_{\mathcal{E}}$) the square matrix $\Upsilon_{\mathcal{E}}^s$ defined by:

$$\Upsilon_{\mathcal{E}}^s = [\Upsilon_1 \Upsilon_2 \dots \Upsilon_M] \in \mathcal{M}_{\delta}(\mathbb{R}) \quad (5)$$

where

$$\Upsilon_i = [\kappa^{(k_{i1})}(x_i) \kappa^{(k_{i2})}(x_i) \dots \kappa^{(k_{id_i})}(x_i)] \quad (6)$$

such that $k_{i,l} \geq 0$ for all $(i,l) \in \{1, \dots, M\} \times \{1, \dots, d_i\}$ and $\sum_{i=1}^M d_i = \delta$ where

$$\kappa(x_i) = [x_i^s \dots x_i^{\delta+s-1}]^T, \quad \text{for } 1 \leq i \leq M. \quad (7)$$

When $s = 0$, the matrix $\Upsilon_{\mathcal{E}}$ is nothing else than the standard Birkhoff matrix and thus $\kappa(x_i) = [1 x_i \dots x_i^{\delta-1}]^T$. If in addition $\mathcal{V}_{\mathcal{E}}$ does not contain stars then we recover the confluent Vandermonde matrix. The particular case $d_i = 1$ for $i = 1 \dots N$ is associated to the standard Vandermonde matrix and in this case $M = \delta$ since $\Upsilon_{\mathcal{E}}$ is assumed to be a square matrix.

The explicit development of numeric/symbolic algorithms for LU-factorization and inversion of the Vandermonde and confluent Vandermonde matrices [25], [26] is still an attracting topic due to their specific structure and their implications in various applications, see for instance [21], [27] and references therein. The authors propose in [11] an

explicit recursive formula for the LU-factorization for three configurations of the generalized Birkhoff matrix defined by (5)-(7). To the best of the authors' knowledge, such an explicit formulas seems to be unavailable in the mathematical literature and linear algebra textbooks, see for instance [20]. The Birkhoff matrix configurations we consider are: the first one, the regular case, no stars in $\mathcal{V}_\mathcal{E}$ that is the generalized confluent Vandermonde matrix. The second configuration is when the polynomials associated with the delays in the quasipolynomial are sparse, that is, $\mathcal{V}_\mathcal{E}$ containing stars. For instance, that is the case for the variable blocks x_2 and x_3 in the example (4).

Furthermore, as a byproduct of the approach, we will present first a different proof for the Polya-Szegö bound \sharp_{PS} of the origin multiplicity deduced from proposition 1, then, we will establish sharper bound for such a multiplicity under the nondegeneracy of an appropriate Birkhoff matrix.

To summarize, the contribution of the present paper is threefold:

- 1) In the general case, the Birkhoff interpolation problem may or may not have a unique solution. No general form for its determinant is known, and thus no general formula for the interpolating polynomial (when it exists) is known. The problem still unsolved [20], [24] since such a formulae depends directly in the chosen incidence matrix among a multitude of configurations. With this connexion, we present an explicit recursive formula for the LU-factorization of the generalized confluent Vandermonde matrix [11].
- 2) We identify the link between the multiplicity of the zero singularity associated with time-delay systems (even in the presence of coupling delays) and the generalized Birkhoff matrix.
- 3) In the generic case (all the polynomials $P_{M_k k \geq 0}$ are complete), the Polya-Szegö bound \sharp_{PS} is completely recovered using an alternative method Vandermonde-based. Moreover, when at least one of the polynomials contains a star, then under the nondegeneracy of an appropriate Birkhoff matrix we establish a bound for the multiplicity which is sharper than the Polya-Szegö bound \sharp_{PS} .

It is worthy to note that, in the simplest case of uncoupled delays, such a bound was recovered in [28] and a simplified framework is presented.

The following notations are adopted. Let ξ stands for the vector composed from x_i counting their repetition d_i through columns of Υ , that is

$$\xi = \underbrace{(x_1, \dots, x_1)}_{d_1}, \dots, \underbrace{(x_M, \dots, x_M)}_{d_M}.$$

For instance one has $\xi_1 = x_1$ and $\xi_{d_1+d_2+1} = \xi_{d_1+d_2+d_3} = x_3$. In the light of the above notations and under the setting $d_0 = 0$, without any loss of generality: $\xi_k = \xi_{d_0+\dots+d_r+\alpha} = \xi_{\sum_{i=0}^{\varrho(k)-1} d_i + \varkappa(k)}$, where $0 \leq r \leq M-1$ and $\alpha \leq d_{r+1}$, here $\varrho(k)$ denotes the index of component of x associated with ξ_k , that is $x_{\varrho(k)} = \xi_k$ and by $\varkappa(k)$ the order of ξ_k

in the sequence of ξ composed only by $x_{\varrho(k)}$. Obviously, $\varrho(k) = r+1$ and $\varkappa(k) = \alpha$.

III. RECOVERING POLYA-SEGÖ GENERIC BOUND

In this section we focus on *the regular case*, that is when all the polynomials of the delayed part of the studied quasipolynomial are complete. However, the complementary configuration, when the polynomials of the delayed part are *sparse*, that is, when the incidence vector $\mathcal{V}_\mathcal{E}$ contains a star or a sequence of successive stars will be considered in the next section.

We start this section by defining some results on generalized confluent Vandermonde matrices that will be useful for the remaining paper. For the sake of simplicity, since we are concerned by the regular case, $\Upsilon_\mathcal{E}$ will be denoted Υ .

It is well known that Vandermonde and confluent Vandermonde matrices V can be factorized into a lower triangular matrix L and an upper triangular matrix U where $V = LU$, see for instance [29], [30]. In what follows, we show that the same applies for the generalized confluent Vandermonde matrix (5)-(7) by establishing explicit formulas for L and U where $\Upsilon = LU$. The factorization is *unique* if no row or column interchanges are made and if it is specified that the diagonal elements of L are unity. The following theorem concerning (5)-(7) with $s = n+1$ will be used in the sequel, but by the same way it can be easily adapted for any positive integer s . The following result is proved in [11] using a total 2D recurrence.

Theorem 2 ([11]). *Given the generalized confluent Vandermonde matrix (5)-(7) with incidence vector $\mathcal{V}_\mathcal{E}$ wanting stars, the unique LU-factorization with unitary diagonal elements $L_{i,i} = 1$ is given by the formulae:*

$$\begin{cases} L_{i,1} = x_1^{i-1} & \text{for } 1 \leq i \leq \delta, \\ U_{1,j} = \Upsilon_{1,j} & \text{for } 1 \leq j \leq \delta, \\ L_{i,j} = L_{i-1,j-1} + L_{i-1,j} \xi_j & \text{for } 2 \leq j \leq i, \\ U_{i,j} = (\varkappa(j) - 1) U_{i-1,j-1} + U_{i-1,j} (x_{\varrho(j)} - \xi_{i-1}) & \\ & \text{for } 2 \leq i \leq j. \end{cases} \quad (8)$$

The explicit computation determinant of the generalized confluent Vandermonde matrix Υ follows directly from (8):

Corollary 3 ([11]). *The determinant of the generalized confluent Vandermonde matrix Υ is given by:*

$$\det(\Upsilon) = \prod_{j=1}^{\delta} (U_{j,j}),$$

where $U_{j,j}$ for $1 \leq j \leq \delta$ are defined by:

$$\begin{cases} U_{1,1} = x_1^{n+1}, \\ U_{j,j} = U_{j-1,j} (x_{\varrho(j)} - \xi_{j-1}) & \text{when } j > 1 \text{ and } \varkappa(j) = 1, \\ U_{j,j} = (\varkappa(j) - 1) U_{j-1,j-1} & \text{otherwise.} \end{cases}$$

Moreover, the diagonal elements of the matrix U associated with the generalized confluent Vandermonde matrix Υ

are obtained as follows:

$$\begin{cases} U_{1,1} = x_1^{n+1}, \\ U_{j,j} = x_{k+1}^{n+1} \prod_{l=1}^k (x_{k+1} - x_l)^{d_l} \\ \quad \text{when } j = 1 + d_k \text{ for } 1 \leq k \leq M-1, \\ U_{j,j} = (j-1-d_k)U_{j-1,j-1} \\ \quad \text{when } d_k+1 < j \leq d_{k+1} \text{ for } 1 \leq k \leq M-1, \end{cases}$$

Moreover, the generalized confluent Vandermonde matrix Υ is invertible if and only if $\forall 1 \leq i \neq j \leq \delta$ we have $x_i \neq 0$ and $x_i \neq x_j$.

In view of the obtained results on generalized confluent Vandermonde matrix we are now able to prove the following proposition. Let us define $a_{i,j}$ the coefficient of the monomial λ^j for the polynomial P_{M^i} for $1 \leq i \leq \tilde{N}_{N,n}$ and denote $P_{M^0} = P_0$. Thus, $a_{0,n} = 1$ and $a_{i,k} = 0 \forall k \geq d_i = n - \sum_{s=1}^N M_s^i$, here $d_i - 1$ is nothing else than the degree of P_{M^i} .

Proposition 4 ([11]). *The multiplicity of the zero root for the generic quasipolynomial function (3) cannot be larger than $\sharp_{PS} = D + \tilde{N}_{N,n}$, where D is the degree of the quasipolynomial and $\tilde{N}_{N,n} + 1$ the number of the associated polynomials. Moreover, such a bound is reached if and only if the parameters of (3) satisfy simultaneously for $0 \leq k \leq \sharp_{PS} - 1$:*

$$a_{0,k} = - \sum_{i \in S_{N,n}} \left(a_{i,k} + \sum_{l=0}^{k-1} \frac{a_{i,l} \sigma_i^{k-l}}{(k-l)!} \right). \quad (9)$$

Remark 1. In the generic case, the Polya-Szegö bound \sharp_{PS} is completely recovered. The proof of Proposition 4 gives an alternative method for identifying such a bound.

Remark 2. When the coefficients of a given time-delay system (1) are fixed, it turns the same to consider the generic case accompanied with an appropriate algebraic constraint additionally to an inequality constraint due to dealing with positive delays. When written in terms of the coefficients of the associated quasipolynomial (3), the algebraic constraint becomes $\mathfrak{C}(a) = 0$ additionally to the inequality constraint $\tau_k > 0$.

Remark 3. The above claim can be interpreted as follows. Under the hypothesis:

$$\Delta(i\omega) = 0 \Rightarrow \omega = 0 \quad (\text{H})$$

that is all the imaginary roots are located at the origin, then the dimension of the projected state on the center manifold associated with zero singularity for equation (3) is less or equal to its number of nonzero coefficients minus one. Indeed, under (H), the codimension of the zero spectral value \equiv the dimension of the state on the center manifold since in general the state's dimension on the center manifold is none other than the sum of the dimensions of the generalized eigenspaces associated with the spectral values having a zero real part.

We need first to introduce some notations. Let denote by $\Delta^{(k)}(\lambda)$ the k -th derivative of $\Delta(\lambda)$ with respect to the variable λ . We say that zero is an eigenvalue of algebraic multiplicity $m \geq 1$ for (1) if $\Delta(0) = \Delta^{(k)}(0) = 0$ for all $k = 1, \dots, m-1$ and $\Delta^{(m)}(0) \neq 0$. We assume in what follows that $\sigma_k \neq \sigma_{k'}$ for any $k \neq k'$ where $k, k' \in S_{N,n}$. Indeed, when for some value of the delay vector τ there exists some $k \neq k'$ such that $\sigma_k = \sigma_{k'}$ then the number of auxiliary delays and the number of polynomials is reduced by considering a new family of polynomials \tilde{P} such that $\tilde{P}_{M^k} = P_{M^k} + P_{M^{k'}}$.

Since we are dealing only by the values of $\Delta_k(0)$, we suggest to translate the problem into the parameter space (the space of the coefficients of the P_i), this will be more appropriate and will consider parametrization by σ . In the appendix we introduce a lemma that allows to establish an m -set of multivariate algebraic functions (polynomials) vanishing at zero when the multiplicity of the zero root of the transcendental equation $\Delta(\lambda) = 0$ is equal to m .

Proof: [Proof of Proposition 4:] The condition (9) follows directly from Lemma 1, (see Appendix). In what follows, we recover the bound \sharp_{PS} by using explicitly the Vandermonde matrices. Then, when assuming that some coefficients of the quasipolynomial vanish without affecting its degree, we show that a sharper bound can be related to the number of nonzero parameters rather than the degree.

We shall consider the variety associated with the vanishing of the polynomials ∇_k (defined in Lemma 1 in the appendix), that is $\nabla_0(0) = \dots = \nabla_{m-1}(0) = 0$ and $\nabla_m(0) \neq 0$ and we aim to find the maximal m (codimension of the zero singularity).

Let us exhibit the first elements from the family ∇_k

$$\begin{cases} \nabla_0(0) = \sum_{s=0}^{\tilde{N}_{N,n}} a_{s,0} = 0, \\ \nabla_1(0) = \sum_{s=0}^{\tilde{N}_{N,n}} a_{s,1} + \sum_{s=1}^{\tilde{N}_{N,n}} a_{s,0} \sigma_s = 0, \\ \nabla_2(0) = 2 \sum_{s=0}^{\tilde{N}_{N,n}} a_{s,2} + 2 \sum_{s=1}^{\tilde{N}_{N,n}} a_{s,1} \sigma_s + \sum_{s=1}^N a_{s,0} \sigma_s^2 = 0, \end{cases}$$

if we consider $a_{i,j}$ and σ_k as variables, the obtained algebraic system is nonlinear and solving it in all generality (without attributing values for n and N) becomes a very difficult task. Indeed even by using Gröbner basis methods [31], this task is still complicated since the set of variables depends on N and n . However, considering $a_{i,j}$ as variables and σ_k as parameters gives the problem a linear aspect as it can be seen from (9). Let adopt the following notation: $a_0 = (a_{0,0}, a_{0,1}, \dots, a_{0,n-1})^T$ and $a_i = (a_{i,0}, a_{i,1}, \dots, a_{i,d_i-1})^T$ for $1 \leq i \leq \tilde{N}_{N,n}$. Next, denote by $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{\tilde{N}_{N,n}})$ and $a = (a_1, a_2, \dots, a_{\tilde{N}_{N,n}})^T$.

Consider now the ideal I_1 generated by the n polynomials $\langle \nabla_0(0), \nabla_1(0), \dots, \nabla_{n-1}(0) \rangle$. As it can be seen from (9) and Lemma 1 (see appendix), the variety V_1 associated

with the ideal I_1 has the following linear representation $a_0 = \Upsilon_1 a$ such that $\Upsilon_1 \in \mathcal{M}_{n, D_q + \tilde{N}_{N,n}}(\mathbb{R}[\sigma])$ where D_q is the degree of $\sum_{k=1}^{\tilde{N}_{N,n}} P_{M_k}$ and $D_q = D - n$ (D the degree of the quasipolynomial (3)). In some sense, in this variety there are no any restriction on the components of a if a_0 is left free. Since $a_{0,k} = 0$ for all $k > n$, the remaining equations consist of an algebraic system only in a and parametrized by σ . Consider now the ideal denoted I_2 and generated by the $D_q + \tilde{N}_{N,n}$ polynomials defined by $I_2 = \langle \nabla_{n+1}(0), \nabla_{n+2}(0), \dots, \nabla_{D+\tilde{N}_{N,n}}(0) \rangle$. It can be observed that the variety V_2 associated with I_2 can be written as $\Upsilon_2 a = 0$ which is nothing else than an homogeneous linear system with $\Upsilon_2 \in \mathcal{M}_{D_q + \tilde{N}_{N,n}}(\mathbb{R}[\sigma])$. More precisely, Υ_2 is nothing else than the generalized confluent Vandermonde matrix (5)-(7) with $x = \sigma$, $s = n$, $M = \tilde{N}_{N,n}$ and $\delta = D_q + \tilde{N}_{N,n}$ which is associated with some incidence vector:

$$\mathcal{V}_{\mathcal{E}} = \left(\underbrace{\sigma_{M^1}, \dots, \sigma_{M^1}, \dots, \dots, \sigma_{M^{\tilde{N}_{N,n}}}, \dots, \sigma_{M^{\tilde{N}_{N,n}}}}_{n - \sum_{s=1}^{\tilde{N}_{N,n}} M_s^1} \right). \quad (10)$$

Now, using Corollary 3 and the assumption that σ_i are distinct non zero auxiliary delays we can conclude that the determinant of Υ_2 can not vanish. Thus the only solution for this subsystem is the zero solution, that is $a = 0$.

Finally, consider the polynomial defined by $\nabla_n(0)$, by lemma 1 (see appendix)

$$\nabla_n(0) = 0 \Leftrightarrow 1 = - \sum_{i=1}^{\tilde{N}_{N,n}} \sum_{s=0}^{n-1} \frac{a_{i,s} \sigma_i^{n-s}}{(n-s)!}$$

substituting the unique solution of V_2 into the last equality leads to an incompatibility result. In conclusion, the maximal codimension of the zero singularity is less or equal to $D_q + \tilde{N}_{N,n} + n$ which is exactly Polya-Szegö bound $\sharp_{PS} = \underbrace{D_q + (n+1)}_{D + \tilde{N}_{N,n}}$ proving *i*).

Remark 4. It is noteworthy that the codimension of the zero singularity may decrease if the vector parameter a_0 is not left free. Indeed, if some parameter component $a_{0,k}$ is fixed for $0 \leq k \leq n-1$, then the variety associated to the first ideal I_1 may impose additional restrictions on the vector parameter a .

IV. MAIN RESULT: ON BEYOND OF POLYA-SEGÖ BOUND

Polynomials in nature (e.g. from applications) are not necessarily generic they often have some additional structure which we would like to take into account showing what it reflects in the multiplicity bound.

Proposition 5. Consider a quasipolynomial function (3) containing one or several incomplete polynomials, for which we associate an incidence vector $\mathcal{V}_{\tilde{\mathcal{E}}}$ which is nothing than (10) such that the vanishing coefficients are replaced by stars.

When the associated generalized Birkhoff matrix $\Upsilon_{\tilde{\mathcal{E}}}$ is nonsingular then the multiplicity of the zero root for the

quasipolynomial function (3) cannot be larger than n plus the number of nonzero coefficients of the polynomial family $(P_{M^k})_{M^k \in S_{N,n}}$.

Proof: [Proof of Proposition 5] By the same way as for the proof of proposition 4: when z coefficients from the polynomial family $(P_{M^k})_{M^k \in S_{N,n}}$ vanish without affecting the degree of the quasipolynomial, then $a^T \in \mathbb{R}^{D_q + \tilde{N}_{N,n} - z}$ and thus the matrix Υ_2 of the proof of proposition 4 becomes $\Upsilon_{\tilde{\mathcal{E}}} \in \mathcal{M}_{D_q + \tilde{N}_{N,n} - z}(\mathbb{R}[\sigma])$. For such proving that the maximal codimension of the zero singularity is less or equal to $D_q + \tilde{N}_{N,n} - z + n < \sharp_{PS}$.

Remark 5. Obviously, the number of non-zero coefficients of a given quasipolynomial function is bounded by its degree plus its number of polynomials. Thus, the bound elaborated in Proposition 5 is sharper than \sharp_{PS} , even in the generic case, that is when all the parameters of the quasipolynomial are left free, these two bounds are equal. Indeed, in the generic case, that is when the number of the left free parameters is optimal, the Polya-Szegö bound $\sharp_{PS} = D + \tilde{N}_{N,n} = n + D_q + \tilde{N}_{N,n}$ which is nothing else than n plus the number of parameters of the polynomial family $(P_{M^k})_{M^k \in S_{N,n}}$.

V. ILLUSTRATION ON INVERTED PENDULUM: AN EFFECTIVE APPROACH VS POLYA-SZEGÖ BOUND

A natural consequence of propositions 4-5 is to explore the situation when the codimension of zero singularity reaches its upper bound. Starting the section by a generic example, we show the convenience of the proposed approach even in the case of coupling delays. Then the obtained symbolic results are applied to identify an effective sharp bound in the case of concrete physical system (with constraints on the coefficients). Namely, the stabilization of an inverted pendulum on cart via a multi-delayed feedback.

We associate to the general planar time-delay system with two positive delays $\tau_1 \neq \tau_2$ the quasipolynomial function:

$$\begin{aligned} \Delta(\lambda) = & \lambda^2 + a_{0,0,1} \lambda + a_{0,0,0} + (a_{1,0,0} + a_{1,0,1} \lambda) e^{\lambda \sigma_{1,0}} \\ & + (a_{0,1,0} + a_{0,1,1} \lambda) e^{\lambda \sigma_{0,1}} \\ & + a_{2,0,0} e^{\lambda \sigma_{2,0}} + a_{1,1,0} e^{\lambda \sigma_{1,1}} + a_{0,2,0} e^{\lambda \sigma_{0,2}}. \end{aligned} \quad (11)$$

Generically, the multiplicity of the zero singularity is bounded by $\sharp_{PS} = 9$. However, in what follows, we present two configurations where such a bound cannot be reached. The first, corresponds to the case when $\sigma_i = \sigma_j$ for $i \neq j$ and the second, when some components of the coefficient vector $a = (a_{1,0,0}, a_{1,0,1}, a_{0,1,0}, a_{0,1,1}, a_{2,0,0}, a_{1,1,0}, a_{0,2,0})^T$ vanish.

Formula (9) allows us to explicitly computing the confluent Vandermonde matrices Υ_1 and Υ_2 and the expression of $\nabla_2(0)$ from the proof of Proposition 4 such that $\Upsilon_1 a = a_0$, $\nabla_2(0) = 0$ and $\Upsilon_2 a = 0$ where $a_0 = (a_{0,0,0}, a_{0,0,1})^T$:

$$\Upsilon_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ \sigma_{1,0} & 1 & \sigma_{0,1} & 1 & \sigma_{2,0} & \sigma_{1,1} & \sigma_{0,2} \end{bmatrix},$$

$$\nabla_2(0) - 2 = \begin{bmatrix} \sigma_{1,0}^2 & 2\sigma_{1,0} & \sigma_{0,1}^2 & 2\sigma_{0,1} & \sigma_{2,0}^2 & \sigma_{1,1}^2 & \sigma_{0,2}^2 \end{bmatrix} a$$

$$\Upsilon_2 = \begin{bmatrix} \sigma_{1,0}^3 & 3\sigma_{1,0}^2 & \sigma_{0,1}^3 & 3\sigma_{0,1}^2 & \sigma_{2,0}^3 & \sigma_{1,1}^3 & \sigma_{0,2}^3 \\ \sigma_{1,0}^4 & 4\sigma_{1,0}^3 & \sigma_{0,1}^4 & 4\sigma_{0,1}^3 & \sigma_{2,0}^4 & \sigma_{1,1}^4 & \sigma_{0,2}^4 \\ \sigma_{1,0}^5 & 5\sigma_{1,0}^4 & \sigma_{0,1}^5 & 5\sigma_{0,1}^4 & \sigma_{2,0}^5 & \sigma_{1,1}^5 & \sigma_{0,2}^5 \\ \sigma_{1,0}^6 & 6\sigma_{1,0}^5 & \sigma_{0,1}^6 & 6\sigma_{0,1}^5 & \sigma_{2,0}^6 & \sigma_{1,1}^6 & \sigma_{0,2}^6 \\ \sigma_{1,0}^7 & 7\sigma_{1,0}^6 & \sigma_{0,1}^7 & 7\sigma_{0,1}^6 & \sigma_{2,0}^7 & \sigma_{1,1}^7 & \sigma_{0,2}^7 \\ \sigma_{1,0}^8 & 8\sigma_{1,0}^7 & \sigma_{0,1}^8 & 8\sigma_{0,1}^7 & \sigma_{2,0}^8 & \sigma_{1,1}^8 & \sigma_{0,2}^8 \\ \sigma_{1,0}^9 & 9\sigma_{1,0}^8 & \sigma_{0,1}^9 & 9\sigma_{0,1}^8 & \sigma_{2,0}^9 & \sigma_{1,1}^9 & \sigma_{0,2}^9 \end{bmatrix}.$$

As shown in the proof of Proposition 4, Υ_2 is a singular matrix when $\sigma_i = \sigma_j$ for $i \neq j$. For instance, when $\sigma_{2,0} = \sigma_{0,1}$ that is $2\tau_1 = \tau_2$, then the bound of multiplicity of the zero singularity decrease since the polynomials $P_{2,0}$ and $P_{0,1}$ will be collected $\tilde{P}_{0,1} = P_{0,1} + P_{2,0}$.

Consider now a system of two coupled equations with two delays modeling a friction free inverted pendulum on cart. The adopted model is studied in [32], [9], [33], [10] and in the sequel we keep the same notations. In the

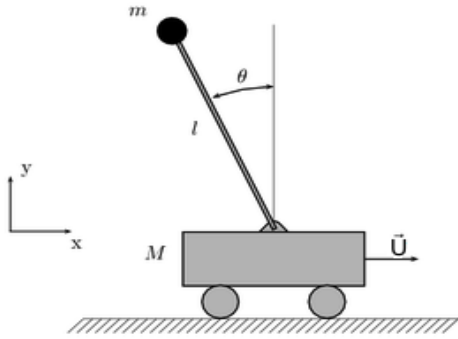


Fig. 1. Inverted Pendulum on a cart

dimensionless form, the dynamics of the inverted pendulum on a cart in figure 1 is governed by the following second-order differential equation:

$$\left(1 - \frac{3\epsilon}{4} \cos^2(\theta)\right) \ddot{\theta} + \frac{3\epsilon}{8} \dot{\theta}^2 \sin(2\theta) - \sin(\theta) + U \cos(\theta) = 0, \quad (12)$$

where $\epsilon = m/(m + M)$, M the mass of the cart and m the mass of the pendulum and D represents the control law that is the horizontal driving force. A generalized Bogdanov-Takens singularity with codimension three is identified in [9] by using $U = a\theta(t - \tau) + b\dot{\theta}(t - \tau)$. Motivated by the technological constraints, it is suggested in [10] to avoid the use of the derivative gain that requires the estimation of

the angular velocity that can induce harmful errors for real-time simulations and propose a multi-delayed-proportional controller $U = a_{1,0}\theta(t - \tau_1) + a_{2,0}\dot{\theta}(t - \tau_2)$, this choice is argued by the accessibility of the delayed state by some simpler sensor. By this last controller choice and by setting $\epsilon = \frac{3}{4}$, the associated quasipolynomial function Δ becomes:

$$\Delta(\lambda) = \lambda^2 - \frac{16}{7} + \frac{16a_1}{7} e^{-\lambda\tau_1} + \frac{16a_2}{7} e^{-\lambda\tau_2}.$$

A zero singularity with codimension three is identified in [10]. Moreover, it is shown that the upper bound of the codimension for the zero singularity for (12) is three (can be easily checked by (9)) and this configuration is obtained when the gains and delays satisfy simultaneously:

$$a_{1,0} = -\frac{7}{-7 + 8\tau_1}, \quad a_{2,0} = \frac{8\tau_1^2}{-7 + 8\tau_1^2}, \quad \tau_2 = \frac{7}{8\tau_1}.$$

However, using Polya-Szegö result, one has $\sharp_{PS} = D - 1 = (3 + 2 + 2) - 1 = 6$ exceeding the effective bound which is three. This is a further justification for the algebraic constraints on the parameters imposed by the physical model, for instance the vanishing of $a_{0,1}$.

Let consider now the sparse case associated with the control law $U = a_{1,0}\theta(t - \tau_1) + a_{2,1}\dot{\theta}(t - \tau_2)$. The quasipolynomial function Δ becomes:

$$\Delta(\lambda) = \lambda^2 - \frac{16}{7} + a_{1,0}e^{-\lambda\tau_1} + \lambda a_{2,1}e^{-\lambda\tau_2}.$$

using Polya-Szegö result, one has $\sharp_{PS} = D - 1 = (3 + 2 + 3) - 1 = 7$ however, using the proposition 5, one knows that the zero multiplicity can not be larger than 4. Indeed, the multiplicity 4 is reached only when $a = \frac{16}{7}$, $b = \frac{4\sqrt{42+28\sqrt{3}}}{7}$, $\tau_1 = \frac{\sqrt{42+28\sqrt{3}}}{4}$, $\tau_2 = \frac{1}{336} (42 + 28\sqrt{3})^{3/2} - \frac{\sqrt{42+28\sqrt{3}}}{8}$.

Remark 6. The obtained framework can be useful in the analysis of a wide range of applications modeled by time-delay systems. For instance, the analysis of a double-inverted pendulum is given in [28] and a biological model describing a vector disease is given in [11].

VI. CONCLUDING REMARKS

This paper addressed the problem of identifying the maximal dimension of the generalized eigenspace associated with a zero singularity for time-delay systems as well as the explicit conditions guaranteeing such a dimension. Under the assumption that all the imaginary roots are located at the origin, our result gives the relation between d the maximal dimension of the projected state on the center manifold associated with the generalized Bogdanov-Takens singularities from one side and N the number of the delays and n the degree of the polynomial P_0 from the other side. The presented upper bound is sharper than the one deduced from Polya-Szegö result [12], since it relies on the number of nonzero coefficients rather than the degree of the quasipolynomial. Moreover, our approach takes into account the possible algebraic constraints on the system coefficients, for instance, the vanishing of certain coefficients. Finally,

the effective method elaborated in this paper emphasizes the connexions between the codimension problem and incidence matrices of a class of generalized Birkhoff matrices for which we presented in different significant configurations an explicit LU-factorization.

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APPENDIX

Lemma 1. Zero is a root of $\Delta^{(k)}(\lambda)$ for $k \geq 0$ if and only if the coefficients of P_{M^j} for $0 \leq j \leq \tilde{N}_{N,n}$ satisfy the following assertion

$$a_{0,k} = - \sum_{i \in S_{N,n}} \left[a_{i,k} + \sum_{l=0}^{k-1} \frac{a_{i,l} \sigma_i^{k-l}}{(k-l)!} \right]. \quad (A.1)$$

Proof: We define the family ∇_k for all $k \geq 0$ by

$$\begin{aligned} \nabla_k(\lambda) = & \sum_{i=0}^{\tilde{N}_{N,n}} \frac{d^k}{d\lambda^k} P_{M^i}(\lambda) \\ & + \sum_{j=0}^{k-1} \left(\binom{k}{j} \sum_{i=1}^{\tilde{N}_{N,n}} \sigma_i^{k-j} \frac{d^j}{d\lambda^j} P_{M^i}(\lambda) \right), \end{aligned} \quad (A.2)$$

here, $M^0 \triangleq 0$ and $\frac{d^0}{d\lambda^0} f(\lambda) \triangleq f(\lambda)$. Obviously, the defined family ∇_k is polynomial since P_i and their derivatives are polynomials. Moreover, zero is a root of $\Delta^{(k)}(\lambda)$ for $k \geq 0$ if and only if zero is a root of $\nabla_k(\lambda)$. This can be proved by induction. More precisely, differentiating k times $\Delta(\lambda)$ the following recursive formula is obtained:

$$\begin{aligned} \Delta^{(k)}(\lambda) = & \sum_{i=0}^{\tilde{N}_{N,n}} \frac{d^k}{d\lambda^k} P_{M^i}(\lambda) e^{\sigma_i \lambda} \\ & + \sum_{j=0}^{k-1} \left(\binom{k}{j} \sum_{i=1}^{\tilde{N}_{N,n}} \sigma_i^{k-j} \frac{d^j}{d\lambda^j} P_{M^i}(\lambda) e^{\sigma_i \lambda} \right). \end{aligned}$$

Since only the zero root is of interest, we can set $e^{\sigma_i \lambda} = 1$ which define the polynomial functions ∇_k . Moreover, careful inspection of the obtained quantities presented in (A.2) and substituting $\frac{d^k}{d\lambda^k} P_i(0) = k! a_{i,k}$ leads to the formula (A.1).

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