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Team Theory of Stochastic Decision Systems with Decentralized Noiseless Feedback Information Structures via Girsanov's Measure Transformation

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Abstract— **In this paper we generalized static team theory to dynamic team theory, in the context of stochastic differential decision system with decentralized noiseless feedback information structures.**

We apply Girsanov's theorem to transformed the initial stochastic dynamic team problem to an equivalent team problem, under a reference probability space, with state process and information structures independent of any of the team decisions. Subsequently, we derive team and Person-by-Person (PbP) optimality conditions, via the stochastic Pontryagin's maximum principle, consisting of forward and backward stochastic differential equations, and a set of conditional variational Hamiltonians with respect to the information structures of the team members.

Finally, we relate the backward stochastic differential equation to the value process of the stochastic team problem.

I. Introduction

In classical stochastic control or decision theory the control actions or decisions applied by the multiple controllers or Decision Makers (DM) are based on the same information, see [1], [3], [4] for full information problems and [4] for partially observable problems.

In this paper, we deviate from the classical stochastic control formulation by consider a system operating over a finite time period $[0,T]$, with the following features.

1) There are N observation posts or stations collecting information;
2) There are N control stations, each having direct access to information collected by at most one observation post, without delay;
3) The observation stations may not communicate their information to the other control stations, or they may communicate their information to the other control stations by signaling part or all of their information to some of the control stations with delay;
4) The N control stations may not have perfect recall, that is, information which is available at any of the control stations at time $t \in [0,T)$ may not be available at any future time $\tau \geq t, \tau \in (0,T]$;
5) The control strategies applied at the N control stations have to be coordinated to optimize a common pay-off or reward.

In the above formulation we have assumed that one observation post is serving one control station without delay, and we allowed the possibility that a subset of the other observation posts signal their information to any of the control stations they are not serving subject to delay. Such signaling among the observation posts and control stations is called information sharing [5]–[8].

The elements of the proposed system of study are the following.

$\mathbb{Z}_N \triangleq \{1,\ldots,N\}$: Set of observation posts/control stations;
$x : [0,T] \times \Omega \longrightarrow \mathbb{R}^n$: Unobserved state process;
$W : [0,T] \times \Omega \longrightarrow \mathbb{R}^n$: State exogenous Brownian Motion (BM) process;
For $i = 1,\ldots,N$,
$u^i : [0,T] \longrightarrow \mathbb{A}^i$: Control process action space $\mathbb{A}^i \subseteq \mathbb{R}^{d_i}$ applied at the ith control station;
$z^i : [0,T] \times \Omega \longrightarrow \mathbb{R}^{k_i}$: Distributed observation process collected at the ith observation post;
$h^i : [0,T] \times C([0,T],\mathbb{R}^n) \longrightarrow \mathbb{R}^{k_i}$: Information functional generating z^i at the ith observation post;
$\mathbb{U}^{z^i}[0,T]$: Admissible strategies generating control actions at the ith control station based on $\{z^i(t) : t \in [0,T]\}$;
$J : \mathbb{A}^{(N)} \longrightarrow (-\infty,\infty]$: Team pay-off or reward.

We call as usual the information available as arguments of the control laws, which generate the control actions applied at the N control stations, "information Structure or Pattern".

Suppose, for now, there is no signaling of information from the observation posts to any of the control stations they are not serving, and let $\{z^i(t) : 0 \leq t \leq T\}$ denote the observation available to the ith control station to generate the control actions $\{u_t^i : 0 \leq t \leq T\}$, with corresponding control strategies by $\mathbb{U}^{z^i}[0,T]$, for $i = 1,\ldots,N$.

The performance of the collective decisions or control actions applied by the control stations, is formulated using dynamic team theory, as follows.

$$\inf \Big\{ J(u^1,\ldots,u^N) : (u^1,\ldots,u^N) \in \times_{i=1}^N \mathbb{U}^{z^i}[0,T] \Big\}, \tag{1}$$

$$J(u^1,\ldots,u^N) = \mathbf{E}\Big\{ \int_0^T \ell(t,x(t),u_t^1(z^1),\ldots,u_t^N(z^N))dt + \varphi(x(T)) \Big\}, \tag{2}$$

subject to stochastic Itô differential dynamics and distributed

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noiseless observations

$$dx(t) = f(t,x(t),u_t^1(z^1),\ldots,u_t^N(z^N))dt + \sigma(t,x(t))dW(t), \quad x(0)=x_0,\ t\in(0,T], \quad (3)$$

$$z^i(t) = h^i(t,x), \quad t\in[0,T],\ i=1,\ldots,N. \quad (4)$$

We call the stochastic differential decision system (1)-(4) with decentralized noiseless feedback information structures, $\{z^1(t),z^2(t),\ldots,z^N(t) : 0\le t\le T\}$, a stochastic dynamic team problem, and a $u^o \triangleq (u^{1,o},u^{2,o},\ldots,u^{N,o}) \in \times_{i=1}^N \mathbb{U}^{z^i}[0,T]$ which achieves the infimum in (1) a team optimal strategy.

Moreover, we call $u^o \triangleq (u^{1,o},u^{2,o},\ldots,u^{N,o}) \in \times_{i=1}^N \mathbb{U}^{z^i}[0,T]$ a PbP optimal strategy if

$$J(u^{1,o},\ldots,u^{N,o}) \le J(u^{1,o},\ldots,u^{i-1,o},u^i,u^{i+1,o},\ldots,u^{N,o}), \quad \forall u^i\in\mathbb{U}^i[0,T], \forall i=1,\ldots,N, \quad (5)$$

and the infimum subject to constraints (3), (4) is achieved. In team theory terminology $\{u^1,\ldots,u^N\}$ are called the DMs, agents or members of the team problem.

In this paper, we investigate the stochastic dynamic team problem (1)-(4), and its generalization when, there is information sharing from the observation posts to any of the control stations, and there is no perfect recall of information at the control stations. We apply techniques from classical strochastic control theory to generalize Marschak's and Radner's static team theory [9]–[11] to continuous-time stochastic differential decision systems with decentralized noiseless feedback information structures, defined by (1)-(4). Moreover, we discuss generalizations of (1)-(4), when there is information sharing from the observation posts to any of the control stations, and there is no perfect recall of information at the control stations.

Our methodology is based on deriving team and PbP optimality conditions, using stochastic Pontryagin's maximum principle, by utilizing the semi martingale representation method due to Bismut [2], under a weak formulation of the probability space by invoking Girsanov's theorem [12]. First, we apply Girsanov's theorem to transform the original stochastic dynamic team problem to an equivalent team problem, under a reference probability space in which the state process and the information structures are not affected by any the team decisions. Subsequently, we show the precise connection between Girsanov's measure transformation and Witsenhausen's notion of "Common Denominator Condition" and "Change of Variables" introduced in [13] to establish equivalence between static and dynamic team problems. Second, we derive optimality conditions based on stochastic variational methods, by taking advantage of the fact that under the reference measure, the state process and the information structures do not react to any perturbations of the team decisions. The optimality conditions are given by a "Hamiltonian System" consisting of a backward and forward stochastic differential equations, while the optimal team actions of the ith team member are determined by a conditional variational Hamiltonian, conditioned on the information structure of the ith team member, while the rest are fixed to their optimal values, for $i=1,\ldots,N$. Third, we show the connection between the backward stochastic differential equation and the value process of the stochastic dynamic team problem.

We point out that the approach we pursued in this paper is different from the various approaches pursued over the years to address stochastic dynamic decentralized decision systems, formulated using team theory in [13]–[21], and our recent treatment in [22]. Compared to [22], in the current paper we apply Girsanov's measure transformation, which allows us to derive the stochastic Pontryagin's maximum principle, for decentralized noiseless feedback information structures, instead of nonanticipative information structures adapted to a sub-filtration of the fixed filtration generated by the Brownian motion $\{W(t) : t\in[0,T]\}$ (e.g., $u_t = \mu(t,W)$) [22].

II. Equivalent Stochastic Dynamic Team Problems

In this section, we consider the stochastic dynamic team problem (1)-(4), and we apply Girsanov's theorem, to transformed it to an equivalent team problem under a reference probability measure, in which the information structures are functionals of Brownian motion, and hence independent of any of the team decisions.

Let $C([0,T],\mathbb{R}^n)$ denote the space of continuous real-valued $n-$dimensional functions defined on the time interval $[0,T]$, and $\mathcal{B}(\mathbb{R}^n)$ its canonical Borel filtration.

Let $L^2_{\mathbb{F}_T}([0,T],\mathbb{R}^n) \subset L^2(\Omega\times[0,T],d\mathbb{P}\times dt,\mathbb{R}^n) \equiv L^2([0,T],L^2(\Omega,\mathbb{R}^n))$ denote the space of $\mathbb{F}_T-$adapted random processes $\{z(t) : t\in[0,T]\}$ such that

$$\mathbb{E}\int_{[0,T]} |z(t)|^2_{\mathbb{R}^n} dt < \infty,$$

which is a Hilbert subspace of $L^2([0,T],L^2(\Omega,\mathbb{R}^n))$.
$L^2_{\mathbb{F}_T}([0,T],\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)) \subset L^2([0,T],L^2(\Omega,\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)))$ denotes the space of $\mathbb{F}_T-$adapted $n\times m$ matrix valued random processes $\{\Sigma(t) : t\in[0,T]\}$ such that

$$\mathbb{E}\int_{[0,T]} |\Sigma(t)|^2_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)} dt \triangleq \mathbb{E}\int_{[0,T]} tr(\Sigma^*(t)\Sigma(t))dt < \infty.$$

Let $B^\infty_{\mathbb{F}_T}([0,T],L^2(\Omega,\mathbb{R}^n))$ denote the space of $\mathbb{F}_T$-adapted $\mathbb{R}^n-$ valued second order random processes endowed with the norm topology $\|\cdot\|$ defined by

$$\|\phi\|^2 \triangleq \sup_{t\in[0,T]} \mathbb{E}|\phi(t)|^2_{\mathbb{R}^n}.$$

Next, we introduce conditions on the coefficients $\{f,\sigma,h^i, i\in\mathbb{Z}_N \triangleq \{1,\ldots,N\}\}$, which are partly used to derive the results of this section.

Assumptions 1: (Main assumptions) The following maps are Borel measurable: $f : [0,T]\times\mathbb{R}^n\times\mathbb{A}^{(N)} \longrightarrow \mathbb{R}^n$, $\sigma : [0,T]\times\mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$, $h^i : [0,T]\times C([0,T],\mathbb{R}^n) \longrightarrow \mathbb{R}^{k_i}$, $\forall i\in\mathbb{Z}_N$. Moreover,

(A0) $\mathbb{A}^i \subseteq \mathbb{R}^{d_i}$ is nonempty, $\forall i\in\mathbb{Z}_N$.

There exists a $K>0$ such that

(A1) $|f(t,x,u)|_{\mathbb{R}^n} \leq K(1+|x|_{\mathbb{R}^n}+|u|_{\mathbb{R}^d}), \forall t \in [0,T]$;

(A2) $|\sigma(t,x)|_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)} \leq K(1+|x|_{\mathbb{R}^n}), \forall t \in [0,T]$;

(A3) $|\sigma(t,x)-\sigma(t,y)|_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)} \leq K|x-y|_{\mathbb{R}^n}, \forall t \in [0,T]$;

(A4) $\sigma(t,x)$ is invertible $\forall (t,x) \in [0,T]\times\mathbb{R}^n$;

(A5) $|\sigma^{-1}(t,x)f(t,x,u)|^2_{\mathbb{R}^n} < K$, uniformly in $(t,x,u) \in [0,T]\times\mathbb{R}^n\times\mathbb{A}^{(N)}$;

(A6) $|\sigma(t,x)|_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)} \geq K(1+|x|^q_{\mathbb{R}^n}), \forall t \in [0,T], \quad q \geq 1$;

(A7) $|\sigma^{-1}(t,x)f(t,x,u)|_{\mathbb{R}^n} \leq K(1+|x|_{\mathbb{R}^n}+|u|_{\mathbb{R}^d}), \forall t \in [0,T]$;

(A8) $|\sigma^{-1}(t,x)f(t,x,u) - \sigma^{-1}(t,z)f(t,z,v)|_{\mathbb{R}^n} \leq (1+|x-z|_{\mathbb{R}^n}+|u-v|_{\mathbb{R}^d})$.

A. Equivalent Stochastic Team Problems via Girsanov's

Next, we define the dynamic team problem (1)-(4) using the weak Girsanov's change of measure approach.

We start with a canonical space $\Big(\Omega,\mathbb{F},\mathbb{P}\Big)$ on which $(x_0, \{W(t): t\in[0,T]\})$ are defined by

(WP1) $x(0)=x_0$: an $\mathbb{R}^n$-valued Random Variable with distribution $\Pi_0(dx)$;

(WP2) $\{W(t): t \in [0,T]\}$: an $\mathbb{R}^m$-valued standard Brownian motion, independent of $x(0)$;

We introduce the Borel $\sigma-$algebra $\mathcal{B}(C[0,T],\mathbb{R}^n))$ on $C([0,T],\mathbb{R}^n)$ generated by $\{W(t): 0\leq t\leq T\}$, and let $\mathbb{P}^W$ its Wiener mesure on it. Further, we introduce the filtration $\mathcal{F}^W_T \triangleq \{\mathcal{F}^W_{0,t}: t\in[0,T]\}$ generated by truncations of $W \in C([0,T],\mathbb{R}^n)$. Next, we define

$$\Omega \triangleq \mathbb{R}^n \times C([0,T],\Re^n), \quad \mathbb{F} \triangleq \mathcal{B}(\mathbb{R}^n)\otimes\mathcal{B}(C([0,T],\mathbb{R}^n)),$$
$$\mathbb{F}_{0,t} \triangleq \mathcal{B}(\mathbb{R}^n)\otimes\mathcal{F}^W_{0,t}, \quad \mathbb{P} \triangleq \Pi_0\times\mathbb{P}^W.$$

On the probability space $(\Omega,\mathbb{F},\{\mathbb{F}_{0,t}: t\in[0,T]\},\mathbb{P})$ we define the stochastic differential equation

$$dx(t) = \sigma(t,x(t))dW(t), \qquad x(0)=x_0. \quad t\in(0,T]. \tag{6}$$

By Assumptions 1, **(A2), (A3)**, and for any initial condition satisfying $\mathbb{E}|x(0)|^q_{\mathbb{R}^n}, q\geq 1$, (6) has a unique strong solution [12], $x(\cdot) \in C([0,T],\mathbb{R}^n) - \mathbb{P} - a.s.$ and $x(\cdot) \in B^\infty_{\mathbb{F}_T}([0,T],L^2(\Omega,\mathbb{R}^n))$.
We also introduce the $\sigma-$algebra $\mathcal{F}^x_{0,t}$ defined by

$$\mathcal{F}^x_{0,t} \triangleq \sigma\Big\{\{x\in C([0,T],\mathbb{R}^n): x(s)\in A\}: 0\leq s\leq t, \quad A\in\mathcal{B}(\mathbb{R}^n)\Big\} \equiv \mathcal{B}_t(C([0,T],\mathbb{R}^n)). \tag{7}$$

Hence, $\mathcal{F}^x_T \triangleq \{\mathcal{F}^x_{0,t}: t\in[0,T]\}$ is the canonical Borel filtration generated by $x(\cdot)\in C([0,T],\mathbb{R}^n) - \mathbb{P} - a.s.$ satisfying (6). From (6), and the additional Assumptions 1, **(A4)** on σ, it can be shown that $\mathcal{F}^x_{0,t} = \mathbb{F}_{0,t} \equiv \mathcal{F}^{x(0)} \bigvee \mathcal{F}^W_{0,t}, \forall t \in [0,T]$, and this $\sigma-$algebra is independent of any of the team decisions u. Note that for the feedback information structures to be independent of any of the team decisions u, it is necessary that under the reference probability measure, $\mathbb{P}$ the state process $x(\cdot)$ is independent of u, which is indeed the case.

Next we prepare to define three sets of admissible team strategies. We define the Borel $\sigma-$algebras generated by projections of $x\in\mathbb{R}^n$ on any of its subspaces say, $x^i \triangleq \mathbf{\Pi}^i(x)$, and the distributed observations process $\{z^i(t) \triangleq h^i(t,x): t\in[0,T]\}$ as follows.

$$\mathcal{G}^{x^i(t)} \triangleq \sigma\Big\{\{x^i\in C([0,T],\mathbb{R}^{n_i}): x^i(t)\in A\}: A\in\mathcal{B}(\mathbb{R}^{n_i})\Big\}, \quad t\in[0,T], \quad \forall i\in\mathbb{Z}_N \tag{8}$$

$$\mathcal{G}^{z^i}_{0,t} \triangleq z^{i,-1}\Big(\mathcal{B}_t([0,T])\otimes\mathcal{B}_t(C([0,T],\mathbb{R}^n))\Big), \quad t\in[0,T], \ \forall i\in\mathbb{Z}_N. \tag{9}$$

Further, define $\mathcal{G}^{z^i}_T \triangleq \{\mathcal{G}^{z^i}_{0,t}: t\in[0,T]\}, \mathcal{G}^{z^i}_{0,t} \subseteq \mathcal{F}^x_{0,t}, \forall t\in[0,T]\}$, the canonical Borel filtration generated by $\{z^i(t): 0\leq t\leq T\}$, for $i=1,\ldots,N$. Define the delayed sharing information structure at the ith control station by $\mathcal{G}^{I^i}_T \triangleq \{\mathcal{G}^{I^i}_{0,t}: t\in[0,T]\}$, which is the minimum filtration generated by the Borel $\sigma-$algebra at the ith observation post $\{z^i(s): 0\leq s\leq t\}$, and the delayed sharing information signaling, $\Big\{z^j(s-\epsilon_j): \epsilon_j>0, j\in\mathcal{O}(i), 0\leq s\leq t\Big\}$, $t\in[0,T]$, from the observation posts $\mathcal{O}(i)\subset\{1,2,\ldots,i-1,i+1,\ldots,N\}$, to the control station i, for $i=1,\ldots,N$. Next, we define the three classes of information structures we consider in this paper.

Definition 1: (Noiseless Feedback Admissible Strategies)
Without Signaling: If there is no signaling from the observation posts to any of the other control stations, then

$$\mathbb{U}^{z^i}[0,T] \triangleq \Big\{u^i: [0,T]\times\Omega \longrightarrow \mathbb{A}^i \subseteq \mathbb{R}^{d_i}: u^i \text{ is } \{\mathcal{G}^{z^i}_{0,t}: t\in[0,T]\}-\text{Progressively Measurable (PM)} \text{ and } \mathbb{E}\int_0^T \Lambda^u(t)|u_t|^2_{\mathbb{R}^d}dt < \infty\Big\}, \quad \forall i\in\mathbb{Z}_N. \tag{10}$$

A team strategy is $(u^1,\ldots,u^N) \in \mathbb{U}^{(N),z}[0,T] \triangleq \times^N_{i=1}\mathbb{U}^{z^i}[0,T]$.

With Signaling: If there is delayed sharing information signaling from the other observation posts, the set of admissible strategies at the ith control station is

$$\mathbb{U}^{I^i}[0,T] \triangleq \Big\{u^i: [0,T]\times\Omega \longrightarrow \mathbb{A}^i \subseteq \mathbb{R}^{d_i}: u^i \text{ is } \{\mathcal{G}^{I^i}_{0,t}: t\in[0,T]\}-\text{PM} \text{ and } \mathbb{E}\int_0^T \Lambda^u(t)|u_t|^2_{\mathbb{R}^d}dt < \infty\Big\}, \ \forall i\in\mathbb{Z}_N. \tag{11}$$

A team strategy is $(u^1,\ldots,u^N) \in \mathbb{U}^{(N)}[0,T] \triangleq \times^N_{i=1}\mathbb{U}^{I^i}[0,T]$.

Without Perfect Recall $\sim$ Markov: If the distributed observation process collected at the ith observation post is $z^i = x^i$, and there is no perfect recall, the set of admissible strategies at the ith control station is

$$\mathbb{U}^{x^i}[0,T] \triangleq \Big\{u^i : [0,T]\times\mathbb{R}^{n_i} \longrightarrow \mathbb{A}^i \subseteq \mathbb{R}^{d_i} : \text{ for any } t\in[0,T],\ u_t^i \text{ is } \mathcal{G}^{x^i(t)}-\text{measurable and } \mathbb{E}\int_0^T \Lambda^u(t)|u_t|^2_{\mathbb{R}^d}dt<\infty\Big\},\ \forall i\in\mathbb{Z}_N. \quad (12)$$

A team strategy is $(u^1,\ldots,u^N) \in \mathbb{U}^{(N),x}[0,T] \triangleq \times_{i=1}^N \mathbb{U}^{x^i}[0,T]$.

The results derived in this paper hold for other variations of the information structures, such as, control stations without perfect recall based on delayed information $\mathcal{G}^{x^i(t-\delta_i)}, \delta_i\geq 0, i=1,\ldots,N$, etc.

The importance of condition $\mathbb{E}\int_0^T \Lambda^u(t)|u_t|^2_{\mathbb{R}^d}dt < \infty$ will be clarified shortly. Thus, an admissible strategy, say, $u \equiv (u^1,\ldots,u^N) \in \mathbb{U}^{(N)}[0,T]$ is a family of N functions, say, $\Big(\mu_t^1(\cdot),\mu_t^2(\cdot),\ldots,\mu_t^N(\cdot)\Big), t\in[0,T]$, which are progressively measurable (nonanticipative) with respect to the delayed sharing noiseless feedback information structure $\{\mathcal{G}_{0,t}^{I^i} : t\in[0,T]\}, i=1,2,\ldots,N$.

For any $u \in \mathbb{U}^{(N)}[0,T]$ (we can also consider $\mathbb{U}^{(N),z}[0,T], \mathbb{U}^{(N),x}[0,T]$) we define on $\Big(\Omega,\mathbb{F},\{\mathbb{F}_{0,t} : t\in[0,T]\},\mathbb{P}\Big)$ the exponential function

$$\Lambda^u(t) \triangleq \exp\Big\{\int_0^t f^*(s,x(s),u_s)a^{-1}(s,x(s))dx(s) - \frac{1}{2}\int_0^t f^*(s,x(s))a^{-1}(s,x(s))f(s,x(s))ds\Big\}, \quad (13)$$
$$a(t,x)=\sigma(t,x)\sigma^*(t,x), \quad t\in[0,T].$$

Under the additional Assumptions 1, **(A5)**, by Itô's differential rule $\{\Lambda^u(t) : t\in[0,T]\}$ it is the unique $\{\mathbb{F}_{0,t} : t\in[0,T]\}-$adapted, $\mathbb{P}-$a.s. continuous solution [12] of the stochastic differential equation

$$d\Lambda^u(t) = \Lambda^u(t)f^*(t,x(t),u_t)a^{-1}(t,x(t))dx(t), \quad (14)$$

with initial conditions $\Lambda^u(0) = 1$. Given any $u\in\mathbb{U}^{(N)}[0,T]$ we define the reward of the team problem under $\Big(\Omega,\mathbb{F},\{\mathbb{F}_{0,t} : t\in[0,T]\},\mathbb{P}\Big)$ by

$$J(u^*) \triangleq \inf_{u\in\mathbb{U}^{(N)}[0,T]} \mathbb{E}\Big\{\int_0^T \Lambda^u(t)\ell(t,x(t),u_t)dt + \Lambda^u(T)\varphi(x(T))\Big\}, \quad (15)$$

where $\ell : [0,T]\times\mathbb{R}^n\times\mathbb{A}^{(N)} \longrightarrow (-\infty,\infty], \varphi : [0,T]\times\mathbb{R}^n \longrightarrow (-\infty,\infty]$ will be such that (15) is finite.

For any admissible strategy $u\in\mathbb{U}^{(N)}[0,T]$, by Assumptions 1, **(A5)**, Novikov condition [12] is satisfied, hence $\{\Lambda^u(t) : 0\leq t\leq T\}$ is an $\Big(\{\mathbb{F}_{0,t} : t\in[0,T]\},\mathbb{P}\Big)$-martingale, $\forall t\in[0,T]$, and by the martingale property, $\int_\Omega \Lambda^u(t,\omega)d\mathbb{P}(\omega) = 1, \forall t\in[0,T]$, Therefore, we can utilize $\Lambda^u(\cdot)$ which represents a version of the Radon-Nikodym derivative, to define a probability measure $\mathbb{P}^u$ on $\Big(\Omega,\mathbb{F},\{\mathbb{F}_{0,t} : t\in[0,T]\}\Big)$ by setting

$$\frac{d\mathbb{P}^u}{d\mathbb{P}}\Big|_{\mathbb{F}_{0,t}} = \Lambda^u(t), \quad t\in[0,T]. \quad (16)$$

Moreover, by Girsanov's theorem under the probability space $\Big(\Omega,\mathbb{F},\{\mathbb{F}_{0,t} : t\in[0,T]\},\mathbb{P}^u\Big)$, the process $\{W^u(t) : t\in[0,T]\}$ is a standard Brownian motion and it is defined by

$$W^u(t) \triangleq W(t) - \int_0^t \sigma^{-1}(s,x(s))f(s,x(s),u_s)ds, \quad (17)$$

$t\in[(0,T]$, and the distribution of $x(0)$ is unchanged.
Therefore, under Assumptions 1, **(A1)-(A5)** we have constructed the probability space $\Big(\Omega,\mathbb{F},\{\mathbb{F}_{0,t} : t\in[0,T]\},\mathbb{P}^u\Big)$, the Brownian motion $\{W^u(t) : t\in[0,T]\}$ defined on it, and the state process $x(\cdot)$ which is a weak solution of

$$dx(t) = f(t,x(t),u_t)dt + \sigma(t,x(t))dW^u(t),\ x(0)=x_0, \quad (18)$$

$t\in(0,T]$, unique in probability law, having the properies $x(\cdot)\in C([0,T],\mathbb{R}^n) - \mathbb{P}^u - a.s.$ and $x(\cdot)\in B^\infty_{\mathbb{F}_T}([0,T],L^2(\Omega,\mathbb{R}^n))$.
By substituting (16) into (15), under the probability measure $\mathbb{P}^u$, the team game reward is given by

$$J(u^*) = \inf_{u\in\mathbb{U}^{(N)}[0,T]} \mathbb{E}^u\Big\{\int_0^T \ell(t,x(t),u_t)dt + \varphi(x(T))\Big\}. \quad (19)$$

From the definition of the Radon-Nikodym derivative (16), for any admissible strategy, say, $u\in\mathbb{U}^{(N)}[0,T]$ we also have $\mathbb{E}\int_0^T \Lambda^u(t)|u_t|^2_{\mathbb{R}^d}dt = \mathbb{E}^u\int_0^T |u_t|^2_{\mathbb{R}^d}dt < \infty$.

Remark 1: Assumptions 1, **(A5)** is satisfied if the following alternative conditions hold.

(A5)(a) (A4), (A6) holds and either (i) **(A1)** is replaced by $|f(t,x,u)|_{\mathbb{R}^n} \leq K(1+|x|_{\mathbb{R}^n}), K>0, \forall t\in[0,T]$, or (ii) $\mathbb{A}^{(N)}$ is bounded;

Remark 2: The Girsanov's measure transformation is precisely the continuous-time counterpart of so called "Common Denominator Condition and Change of Variables" (i.e. [Sections 4, 5, [13]]), of Witsenhausen's discrete-time stochastic control problems with finite decisions. Witsenhausen in [13] called any discrete-time stochastic dynamical decentralized decision problem which can be transformed via the "Common Denominator Condition and Change of Variables" to observations which are not affected by any of the team decisions "Static". The main point we wish to make regarding [13] is the following.

The main problem with developing the team and PbP optimality conditions based on variational methods, under the original probability space $\Big(\Omega,\mathbb{F},\{\mathbb{F}_{0,t} : t\in[0,T]\},\mathbb{P}^u\Big)$, is the definition of admissible strategies, which states that $\{u_t^i : t\in[0,T]\}$ is adapted to feedback information $\{\mathcal{G}_{0,t}^{I^i} : t\in[0,T]\} \subset \{\mathcal{F}_{0,t}^x : t\in[0,T]\}, i=1,\ldots,N$, and hence affected by the team decisions. Therefore, if one invokes weak or needle variations of $u\in\mathbb{U}^{(N)}[0,T]$, to compute the Gateaux derivative of the pay-off, then one needs the variational equation of the unobserved state $x(\cdot)$

satisfying (19), which implies that one should differentiate $\{u_t^i \equiv \mu(t, I^i) : t \in [0,T]\}, i = 1, \ldots, N$ with respect to x, because $\{I^1(t), \ldots, I^N(t) : t \in [0,T]\}$ are affected by the decisions. Therefore, the classical methods which assume nonanticipative strategies adapted to $\{\mathcal{F}_{0,t}^W : t \in [0,T]\}$ [23] or any sub-$\sigma-$algebra of this [24], in general do not apply. One approach to circumvent this technicality is to show that feedback strategies are dense in noanticipatve or open loop strategies, and the pay-off is continuously dependent on $u \in \mathbb{U}^{(N)}[0,T]$ as in [22]. Another approach is to use Girsanov's theorem.

Before we proceed we show, in the next theorem, that Girsanov's change of probability measure, holds under more general conditions than the uniform bounded condition given by Assumptions 1, **(A5)**.

Theorem 1: (Equivalence of Dynamic Team Problems) Suppose $\mathbb{E}|x(0)|_{\mathbb{R}^n} < \infty$, Assumptions 1, **(A1), (A2), (A7)**, hold, and consider any of the admissible strategies of Definition 1. Then $\mathbb{E}\Big(\Lambda^u(t)\Big) = 1, \forall t \in [0,T]$, and the dynamic team problem with pay-off (19) subject to $x(\cdot)$ satisfying (18) is equivalent to the dynamic team problem with pay-off (15) with $(x(\cdot), \Lambda(\cdot))$ satisfying (6), (14).

Proof: See [25]. ∎

III. Dynamic Team Optimality Conditions

In this section we derive the team and PbP optimality conditions, under the reference probability measure $\mathbb{P}$, and then we translate the results under the original probability measure $\mathbb{P}^u$. For the derivation of stochastic optimality conditions we shall require the following stronger regularity conditions.

Assumptions 2: $\mathbb{A}^i$ is a closed, bounded and convex subset of $\mathbb{R}^{d_i}, \forall i \in \mathbb{Z}_N$, $\mathbb{E}|x(0)|^2_{\mathbb{R}^n} < \infty$, the maps $\{f, \sigma, \ell, \varphi\}$ are Borel measurable, $\{h^i : i = 1, \ldots, N\}$ are progressively measurable, defined by

$$f : [0,T] \times \mathbb{R}^n \times \mathbb{A}^{(N)} \longrightarrow \mathbb{R}^n, \ \sigma : [0,T] \times \mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n),$$
$$\varphi : \mathbb{R}^n \longrightarrow R, \quad \ell : [0,T] \times \mathbb{R}^n \times \mathbb{A}^{(N)} \longrightarrow \mathbb{R},$$
$$h^i : [0,T] \times C([0,T], \mathbb{R}^n) \longrightarrow \mathbb{R}^{k_i},$$

and they satisfy the following conditions.

(C1) The map σ satisfies **(A2), (A3), (A4)** and the map $\sigma^{-1}f$ satisfies **(A5)**;

(C2) The map f is once continuously differentiable with respect to $u \in \mathbb{A}^{(N)}$, and the first derivative of $\sigma^{-1}f$ with respect to is u is bounded uniformly in $(t,x,u) \in [0,T] \times \mathbb{R}^n \times \mathbb{A}^{(N)}$;

(C3) The maps ℓ is once continuously differentiable with respect to $u \in \mathbb{A}^{(N))}$, and there exists a $K > 0$ such that

$$\Big(1 + |x|^2_{\mathbb{R}^n} + |u|^2_{\mathbb{R}^d}\Big)^{-1} |\ell(t,x,u)|_{\mathbb{R}}$$
$$+ \Big(1 + |x|_{\mathbb{R}^n} + |u|_{\mathbb{R}^d}\Big)^{-1} |\ell_u(t,x,u)|_{\mathbb{R}^d} \leq K,$$
$$\Big(1 + |x|^2_{\mathbb{R}^n}|\Big)^{-1} \varphi(x)|_{\mathbb{R}} \leq K;$$

A. Necessary Conditions for Team Optimality

Next, we prepare to give the variational equation under the reference probability.

Minimum Principle Under Reference Probability Space $(\Omega, \mathbb{F}, \{\mathbb{F}_{0,t} : t \in [0,T]\}, \mathbb{P})$**.**

Define the Hamiltonian of the augmented system (6), (14), (15).

$$\mathcal{H} : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathcal{L}(\mathbb{R}^n, \mathbb{R}) \times \mathbb{A}^{(N)} \longrightarrow \mathbb{R}$$
$$\mathcal{H}(t,x,\Lambda,\Psi,Q,u) \stackrel{\triangle}{=} \Lambda Q \sigma^{-1}(t,x) f(t,x,u) + \Lambda \ell(t,x,u). \tag{20}$$

For any $u \in \mathbb{U}^{(N)}[0,T]$, the adjoint process $(\Psi, Q) \in L^2_{\mathbb{F}_T}([0,T], \mathbb{R}) \times L^2_{\mathbb{F}_T}([0,T], \mathcal{L}(\mathbb{R}^n, \mathbb{R}))$ satisfies the following backward stochastic differential equations

$$\begin{aligned} d\Psi(t) =& -\ell(t,x(t),u_t)dt - Q(t)\sigma^{-1}(t,x(t)) f(t,x(t),u_t) \\ &+ Q(t)dW(t), \\ =& -\mathcal{H}_\Lambda(t,x(t),\Lambda(t),\Psi(t),Q(t),u_t)dt \\ &+ Q(t)dW(t), \quad t \in [0,T), \ \Psi(T) = \varphi(x(T)), \end{aligned} \tag{21}$$

The state process satisfies the stochastic differential equation (14) expressed in terms of the Hamiltonian as follows.

$$\begin{aligned} d\Lambda(t) =& \Lambda(t) f^*(t,x(t),u_t)\sigma^{*,-1}(t,x(t))dW(t), \\ =& \Lambda(t) f^*(t,x(t),u_t)\sigma^{*,-1}(t,x(t))dW(t), \ \Lambda(0) = 1. \end{aligned} \tag{22}$$

Moreover, under measure $\mathbb{P}$, the process $\{x(t) : t \in [0,T]\}$ is not affected by $u \in \mathbb{U}^{(N)}[0,T]$ and satisfies

$$dx(t) = \sigma(t,x(t))dW(t), \qquad t \in (0,T], \quad x(0) = x_0 \tag{23}$$

Next, we state the the necessary conditions for an element $u^o \in \mathbb{U}^{(N)}[0,T]$ to be team optimal.

Theorem 2: (Necessary Conditions for Team Optimality under Reference Measure)

Suppose Assumptions 2 hold. Then we have the following..

Necessary Conditions. For an element $u^o \in \mathbb{U}^{(N)}[0,T]$ with the corresponding solution $\Lambda^o \in B^\infty_{\mathbb{F}_T}([0,T], L^2(\Omega, \mathbb{R}))$ to be team optimal, it is necessary that the following hold.

(1) There exists a semi martingale $m^o \in \mathcal{SM}^2_0[0,T]$ (1-dimensional) with the intensity process $(\Psi^o, Q^o) \in L^2_{\mathbb{F}_T}([0,T], \mathbb{R}) \times L^2_{\mathbb{F}_T}([0,T], \mathcal{L}(\mathbb{R}^n, \mathbb{R}))$.

(2) The variational inequalities are satisfied:

$$\begin{aligned} &\sum_{i=1}^N \mathbb{E}\Big\{ \int_0^T \mathcal{H}(t,x(t),\Lambda^o(t),\Psi^o(t),Q^o(t),u_t^{-i,o},u_t^i)dt \Big\} \geq \\ &\sum_{i=1}^N \mathbb{E}\Big\{ \int_0^T \mathcal{H}(t,x(t),\Lambda^o(t),\Psi^o(t),Q^o(t),u_t^{-i,o},u_t^{i,o}))dt \Big\}, \\ &\forall u \in \mathbb{U}^{(N)}[0,T], \end{aligned} \tag{24}$$

$$\begin{aligned} &\mathbb{E}\Big\{ \int_0^T \mathcal{H}(t,x(t),\Lambda^o(t),\Psi^o(t),Q^o(t),u_t^{-i,o},u_t^i)dt \Big\} \\ &\geq \mathbb{E}\Big\{ \int_0^T \mathcal{H}(t,x(t),\Lambda^o(t),\Psi^o(t),Q^o(t),u_t^{-i,o},u_t^{i,o}))dt \Big\}, \\ &\qquad \forall u^i \in \mathbb{U}^{I^i}[0,T], \quad \forall i \in \mathbb{Z}_N. \end{aligned} \tag{25}$$

(3) The process $(\Psi^o, Q^o) \in L^2_{\mathbb{F}_T}([0,T],\mathbb{R}) \times L^2_{\mathbb{F}_T}([0,T],\mathcal{L}(\mathbb{R}^n,\mathbb{R}))$ is a unique solution of the backward stochastic differential equation (21) such that $u^o \in \mathbb{U}^{(N)}[0,T]$ satisfies the point wise almost sure inequalities with respect to the σ-algebras $\mathcal{G}^{I^i}_{0,t} \subset \mathbb{F}_{0,t}$, $t \in [0,T], i = 1,2,\ldots,N$:

$$\begin{aligned}&\mathbb{E}\Big\{\mathcal{H}(t,x(t),\Lambda^o(t),\Psi^o(t),Q^o(t),u_t^{-i,o},u_t^i)|\mathcal{G}^{I^i}_{0,t}\Big\}\\&\quad\geq \mathbb{E}\Big\{\mathcal{H}(t,x(t),\Lambda^o(t),\Psi^o(t),Q^o(t),u_t^o)|\mathcal{G}^{I^i}_{0,t}\Big\},\\&\quad\forall u^i \in \mathbb{A}^i, a.e.t \in [0,T], \mathbb{P}|_{\mathcal{G}^{I^i}_{0,t}} - a.s., \forall i \in \mathbb{Z}_N.\end{aligned} \tag{26}$$

(4) For admissible strategies $\mathbb{U}^{(N),z}[0,T], \mathbb{U}^{(N),x}[0,T]$ the conditional expectation in (26) is taken with respect to the information structures $\mathcal{G}^{z^i}_{0,t}, \mathcal{G}^{x^i(t)}$, respectively.

Proof: See [25]. ■

The important point to be made regarding Theorem 2 is that its derivation is based on applying, under the new (reference) probability space $\Big(\Omega,\mathbb{F},\mathbb{F}_T,\mathbb{P}\Big)$, any method based on strong formulation (in our case [22], [24]), but with u adapted to feedback information.

We also point out that the necessary conditions for a $u^o \in \mathbb{U}^{(N)}[0,T]$ to be a PbP optimal can be derived following the procedure described in Theorem 2, and that these necessary conditions are equivalent to the necessary conditions for team optimality, as expected.

Minimum Principle Under Original Probability Space $(\Omega,\mathbb{F},\{\mathbb{F}_{0,t} : t \in [0,T]\},\mathbb{P}^u)$

Next, we express the optimality conditions with respect to the original probability space $\Big(\Omega,\mathbb{F},\{\mathbb{F}_{0,t} : t \in [0,T]\},\mathbb{P}^u\Big)$. Since the Hamiltonian under the reference probability measure (20), appearing in Theorem 2 is multiplied by $\Lambda(\cdot)$, then we can write

$$\begin{aligned}\mathcal{H}(t,x,\Lambda,\Psi,Q,u) =&\Lambda\Big\{Q\sigma^{-1}(t,x)f(t,x,u)\\&+\ell(t,x,u)\Big\}\end{aligned} \tag{27}$$

Define the Hamiltonian under the original probability measure $\mathbb{P}^u$ by

$$\begin{aligned}&\mathbb{H} : [0,T]\times\mathbb{R}^n\times\mathcal{L}(\mathbb{R}^n,\mathbb{R})\times\mathbb{A}^{(N)} \longrightarrow \mathbb{R}\\&\mathbb{H}(t,x,Q,u) \triangleq \ell(t,x,u) + Q\sigma^{-1}(t,x)f(t,x,u).\end{aligned} \tag{28}$$

Since $\Lambda(T) = \frac{d\mathbb{P}^u}{d\mathbb{P}}|_{\mathbb{F}_T}$, under the original probability measure $\Big(\Omega,\mathbb{F},\{\mathbb{F}_{0,t} : t \in [0,T]\},\mathbb{P}^u\Big)$, the adjoint process $\{\Psi(t),Q(t) : t \in [0,T]\}$ is a solution of the backward and forward stochastic differential equation

$$\begin{aligned}d\Psi(t) =& -\ell(t,x(t),u_t)dt\\&+Q(t)dW^u(t), \quad \Psi(T) = \varphi(x(T)), \quad t \in [0,T),\end{aligned} \tag{29}$$

and the process $\{x(t) : t \in [0,T]\}$ is a solution of the following forward equation.

$$dx(t) = f(t,x(t),u_t)dt + \sigma(t,x(t))dW^u(t),\ x(0) = x_0. \tag{30}$$

Moreover, the conditional variational Hamiltonian is given by

$$\begin{aligned}&\mathbb{E}^{u^o}\Big\{\mathbb{H}(t,x^o(t),Q^o(t),u_t^{-i,o},u_t^i)|\mathcal{G}^{I^i}_{0,t}\Big\}\\&\quad\geq \mathbb{E}^{u^o}\Big\{\mathbb{H}(t,x^o(t),Q^o(t),u_t^{-i,o},u_t^{i,o})|\mathcal{G}^{I^i}_{0,t}\Big\},\\&\quad\forall u^i \in \mathbb{A}^i,\ a.e.t,\ \mathbb{P}^{u^o}|_{\mathcal{G}^{I^i}_{0,t}} - a.s., \forall i \in \mathbb{Z}_N.\end{aligned} \tag{31}$$

For admissible strategies $\mathbb{U}^{(N),z}[0,T], \mathbb{U}^{(N),x}[0,T]$ the conditional expectation in (31) is taken with respect to the information structures $\mathcal{G}^{z^i}_{0,t}, \mathcal{G}^{x^i(t)}$, respectively.

B. Value Processes of Team Problems

In this section, we first show that the solution of the Backward stochastic differential equation is the value process of the stochastic dynamic team problem, lifted to a conditioning with respect to the centralized information structure. Then we use the lifted value process to show that the necessary conditions (i.e. (31)) for PbP optimality are also sufficient.

Define the sample pay-off over the interval $[t,T]$ by

$$\mathcal{J}_{t,T}(u^1,\ldots,u^N) \triangleq \int_t^T \ell(s,x(s),u_s)ds + \varphi(x(T)). \tag{32}$$

and its conditional expectation with respect to $\mathcal{G}^{I^i}_{0,t}, i = 1,\ldots,N$, by

$$J^i_{t,T}(u) \triangleq \mathbb{E}^u\Big\{\mathcal{J}_{t,T}(u^1,\ldots,u^N)|\mathcal{G}^{I^i}_{0,t}\Big\},\ u \in \mathbb{U}^{(N)}[t,T], \tag{33}$$

where $\mathbb{U}^{(N)}[t,T]$ is the restriction of the strategies $\mathbb{U}^{(N)}[0,T]$, to the interval $[t,T]$.
PbP optimality seeks admissible strategies $u^i \in \mathbb{U}^{I^i}[t,T], i = 1,\ldots,N$ to minimize the pay-off, in the sense,

$$\begin{aligned}&\mathbb{E}^{u^{-i,o},u^{i,o}}\Big\{\mathcal{J}_{t,T}(u^{-i,o},u^{i,o})|\mathcal{G}^{I^i}_{0,t}\Big\}\\&\quad\leq \mathbb{E}^{u^{-i,o},u^i}\Big\{\mathcal{J}_{t,T}(u^{-i,o},u^i)|\mathcal{G}^{I^i}_{0,t}\Big\},\ \forall u^i \in \mathbb{U}^{I^i}[t,T].\end{aligned}$$

This means that when team members employ strategies, $u^{-i,o} \in \times_{j=1,j\neq i}^N \mathbb{U}^{I^j}[t,T]$, team member u^i minimizes the reward $\mathbb{E}^{u^{-i,o},u^i}\Big\{\mathcal{J}_{t,T}(u^{-i,o},u^i)|\mathcal{G}^{I^i}_{0,t}\Big\}$ over all strategies $\mathbb{U}^{I^i}[0,T]$. The set of all such strategies $(u^{1,o},\ldots,u^{N,o}) \in \times_{i=1}^N \mathbb{U}^{I^i}[t,T]$ is called PbP optimal.
We denote the value processes of the team problem for each team member by

$$V^i(t) \triangleq \mathbb{E}^{u^{-i,o},u^{i,o}}\Big\{\mathcal{J}_{t,T}(u^{-i,o},u^{i,o})|\mathcal{G}^{I^i}_{0,t}\Big\},\ i = 1,\ldots,N. \tag{34}$$

Consider the solution of the backward stochastic differential equation (21)

$$\Psi^u(t) = \Psi^u(T) + \int_t^T \mathbb{H}(s, x(s), Q^u(s), u_s)ds - \int_t^T Q^u(s)dW(s), \qquad t \in [0,T). \tag{35}$$

For $u = u^o$ this is the lifted value process of the team pay-off with respect to the information $\mathbb{F}_{0,t}, t \in [0,T]$. From (35) we have

$$\Psi^u(t) = \Psi^u(T) + \int_t^T \ell((s, x(s), u_s)ds - \int_t^T Q^u(s)dW^u(s), \qquad t \in [0,T), \tag{36}$$

and by taking conditional expectation $\mathbb{E}^u\Big\{ \cdot |\mathbb{F}_{0,t}\Big\}$ of both sides of (36), and using $\Psi^u(T) = \varphi(x(T))$, we obtain

$$\Psi^u(t) = \mathbb{E}^u\Big\{ \int_t^T \ell(s, x(s), u_s)ds + \varphi(x(T))|\mathbb{F}_{0,t}\Big\}. \tag{37}$$

Hence,

$$J^i_{t,T}(u) = \mathbb{E}^u\Big\{\Psi^u(t)|\mathcal{G}^{I^i}_{0,t}\Big\}, \ u \in \mathbb{U}^{(N)}[0,T], \forall i \in \mathbb{Z}_N. \tag{38}$$

Now, we state the main theorem.

Theorem 3: (Sufficient Conditions for PbP Optimality) Let

$$(\Psi^u, Q^u) \in L^2([0,T], L^2(\Omega,\mathbb{R})) \times L^2([0,T], L^2(\Omega, \mathcal{L}(\mathbb{R}^n,\mathbb{R})))$$

be a solution of the backward stochastic differential equation (35).

If $u^{i,o} \in \mathbb{U}^{I^i}[t,T]$ satisfy the conditional variational inequalities (31), then $(u^{1,o}, \ldots, u^{N.o}) \in \times_{i=1}^N \mathbb{U}^{I^i}[t,T]$ is a PbP optimal.

Moreover, $a.e. t \in [0,T], \mathbb{P}^{u^{-i,o}, u^{i,o}}|_{\mathcal{G}^{I^i}_{0,t}}$ −a.s. we have

$$\begin{aligned} V^i(t) =& \mathbb{E}^{u^{-i,o},u^{i,o}}\Big\{\mathbb{E}^{u^{-i,o},u^{i,o}}\Big\{\Psi^{u^{-i,o},u^{i,o}}(t)|\mathbb{F}_{0,t}\Big\}|\mathcal{G}^{I^i}_{0,t}\Big\} \\ =& \mathbb{E}^{u^{-i,o},u^{i,o}}\Big\{\Psi^{u^{-i,o},u^{i,o}}(t)|\mathcal{G}^{I^i}_{0,t}\Big\}, \ i = 1,\ldots,N. \end{aligned} \tag{39}$$

Proof: [25]. ■

IV. Conclusions and Future Work

This paper generalizes static team theory to stochastic differential decision system with decentralized noiseless feedback information structures. We have applied Girsanov's theorem to transformed the initial dynamic team problem to an equivalent team problem, under a reference probability space, with state process independent of any of the team decisions. Then, we described the connection to static team theory discussed by Witsenhausen in [13], and we proceeded further to derive team and PbP optimality conditions, using the stochastic Pontryagin's maximum principle. We also discussed the connection between the backward stochastic differential equation and the value process of the team problem.

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