21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

Power Series Expansion of Noncommutative Rational Functions around a Matrix Point

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I. Noncommutative rational functions and system theory

The skew field of noncommutative (nc) rational functions over a field $\mathbb{K}$, $\mathbb{K}\langle\!\langle x_1,\ldots,x_d\rangle\!\rangle$ is the universal skew field of fractions of the ring of nc polynomials over $\mathbb{K}$, $\mathbb{K}\langle x_1,\ldots,x_d\rangle$. This involves some non-trivial details since unlike the commutative case, a nc rational function does not admit a canonical coprime fraction representation; see [1], [7], [9], [11] for some of the original constructions, and [26, Chapter 8] and [10], [12] for good expositions and background. The following quick description follows [19], [20], to which we refer for both details and further references. We first define (scalar) nc rational expressions by starting with nc polynomials and then applying successive arithmetic operations — addition, multiplication, and inversion. A nc rational expression r can be evaluated on a d-tuple X of $n\times n$ matrices in its *domain of regularity*, $\operatorname{dom} r$, which is defined as the set of all d-tuples of square matrices of all sizes such that all the inverses involved in the calculation of $r(X)$ exist. (We assume that $\operatorname{dom} r\neq\emptyset$, in other words, when forming nc rational expressions we never invert an expression that is nowhere invertible.) Two nc rational expressions r_1 and r_2 are called *equivalent* if $\operatorname{dom} r_1\cap\operatorname{dom} r_2\neq\emptyset$ and $r_1(Z)=r_2(Z)$ for all d-tuples $Z\in\operatorname{dom} r_1\cap\operatorname{dom} r_2$. We define a *nc rational function* $\mathfrak{r}$ to be an equivalence class of nc rational expressions; notice that it has a well-defined evaluation on $\bigcup_{r\in\mathfrak{r}}\operatorname{dom} r$ (in fact, on a somewhat larger set called the extended domain of regularity of $\mathfrak{r}$).

Noncommutative rational functions first appeared in system theory in the context of recognizable formal power series in noncommuting indeterminates in the theory of formal languages and finite automata; see Kleene [22], Schützenberger [27], [28], and Fliess [13], [14], [15] (where the motivation comes also from applications to certain classes of nonlinear systems), and Berstel–Reutenauer [8] for a survey. In particular, noncommutative rational functions admit a good state space realization theory. More recently, state space realizations of rational expressions in Hilbert space operators (modelling structured possibly time varying uncertainty) have figured prominently in work on robust control of linear systems, see Beck [5], Beck–Doyle–Glover [6], Lu–Zhou–Doyle [23].

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Another important application comes from the area of Linear Matrix Inequalities (LMIs); see, e.g., Nesterov–Nemirovski [25], Nemirovski [24], Skelton–Iwasaki–Grigoriadis [29]. As it turns out, most optimization problems appearing in systems and control are dimension-independent, i.e., the natural variables are matrices, and the problem involves rational expressions in these matrix variables which have therefore the same form independent of matrix sizes; see Helton [16], Helton–McCullough–Putinar–Vinnikov [17]. Realizations of rational functions in noncommuting indeterminates are exactly what is needed here to convert (numerically unmanageable) rational matrix inequalities into (highly manageable) linear matrix inequalities, see Helton–McCullough–Vinnikov [18].

For a recent systematic study of noncommutative realization theory, we refer to the series of papers of Ball–Groenewald–Malakorn [2], [4], [3].

NC rational functions are also an instance of nc functions, which are defined as functions on tuples of matrices of all sizes satisfying certain compatibility conditions as we vary the size of matrices (they respect direct sums and simultaneous similarities); see Kaliuzhnyi-Verbovetskyi–Vinnikov [21].

II. Power series expansion around a scalar point

A matrix-valued noncommutative rational expression which is regular at zero determines a noncommutative formal power series with matrix coefficients. This correspondence is defined recursively by inverting formal power series with invertible constant term (the coefficient for $z^{\emptyset}$); see, e.g., [8]. Furthermore, R_1 and R_2 are equivalent if and only if the corresponding formal power series coincide, so that the noncommutative formal power series expansion of a matrix-valued noncommutative rational function which is regular at zero is well defined; see [19, Remark 2.14].

By translation, we obtain a power series expansion of a $p\times q$ matrix-valued nc rational function $\mathfrak{R}$ around any scalar point λ in its domain of regularity,

$$\mathfrak{R}(X)=\sum_{w\in\mathbf{F}_d}(X-\lambda)^w\otimes\mathfrak{R}_w. \qquad (1)$$

Here $X=(X_1,\ldots,X_d)$, $\lambda=(\lambda_1,\ldots,\lambda_d)\in\mathbb{K}^d$, and $\mathfrak{R}_w\in\mathbb{K}^{p\times q}$. Further, $\mathbf{F}_d$ is the free semigroup on d generators $g_1,\ldots,g_d$, we use the usual nocommutative multipower notation:

$$Z^w=Z_{i_1}\cdots Z_{i_l}$$

ISBN: 978-90-367-6321-9

for $Z = (Z_1, \ldots, Z_d)$ and $w = g_{i_1} \cdots g_{i_l}$, and if X is a d-tuple of $n \times n$ matrices then $X - \lambda$ stands for $(X_1 - \lambda_1 I_n, \ldots, X_d - \lambda_d I_n)$.

From the point of view of nc function theory, (1) is the Taylor–Taylor (TT) power series expansion of $\mathfrak{R}$ around λ. In particular, the coefficients $\mathfrak{R}_w$ can be calculated by means of the nc difference-differential calculus: $\mathfrak{R}_w = \Delta^w \mathfrak{R}(\lambda, \ldots, \lambda)$, and in case $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, the series converges normally on any open nc ball contained in the extended domain of regularity of $\mathfrak{R}$.

III. Power series expansion around a matrix point

NC function theory provides an immediate generalization of (1) to the case of a matrix point $Y = (Y_1, \ldots, Y_d) \in (\mathbb{K}^{s\times s})^d$ in the (extended) domain of regularity of $\mathfrak{R}$, that holds in all matrix dimensions n that are multiples of s, $n = sm$. Namely,

$$\mathfrak{R}(X) = \sum_{w \in \mathbf{F}_d} \left(X - \bigoplus_{\alpha=1}^{m} Y \right)^{\odot_s w} \mathfrak{R}_w. \qquad (2)$$

Here for $Z = (Z_1, \ldots, Z_d)$ and $w = g_{i_1} \cdots g_{i_l}$,

$$Z^{\odot_s w} = Z_{i_1} \odot_s \cdots \odot_s Z_{i_l},$$

and $\odot_s$ denotes the product of $sm \times sm$ matrices viewed as $m \times m$ matrices over the tensor algebra of $\mathbb{K}^{s\times s}$. In other words, for $A = [A_{ij}]_{i,j=1,\ldots,m}$ and $B = [B_{ij}]_{i,j=1,\ldots,m}$, with $A_{ij}, B_{ij} \in \mathbb{K}^{s\times s}$,

$$A \odot_s B = \left[\sum_{j=1}^{m} A_{ij} \otimes B_{jk} \right]_{i,k=1,\ldots,m}$$

where $\otimes$ denotes tensor product of vectors in the vector space $\mathbb{K}^{s\times s} \otimes \mathbb{K}^{s\times s}$ rather than Kronecker product of matrices.

With this notation, the terms

$$\left(X - \bigoplus_{\alpha=1}^{m} Y \right)^{\odot_s w} \qquad (3)$$

appearing in the expansion (2) are $m \times m$ matrices over $(\mathbb{K}^{s\times s})^{\otimes l}$, where l is the length of the word $w \in \mathbf{F}_d$. Now, *the coefficient* $\mathfrak{R}_w$ *is a* l*-linear mapping from* $(\mathbb{K}^{s\times s})^l$ *to* $\mathbb{K}^{ps\times qs}$. Such a multilinear mapping can be viewed alternatively as a linear mapping from $(\mathbb{K}^{s\times s})^{\otimes l}$ to $\mathbb{K}^{ps\times qs}$, that we apply to every entry of the $m\times m$ matrix (3), yielding an $m \times m$ over $\mathbb{K}^{ps\times qs}$, i.e., a $psm \times qsm$ matrix over $\mathbb{K}$ — which is where the value $\mathfrak{R}(X)$ lies.

Using the fact that a multilinear mapping M from $(\mathbb{K}^{s\times s})^l$ to $\mathbb{K}^{s\times s}$ is given by a finite sum

$$M(W_1, \ldots, W_l) = \sum_s A_{(s),0} W_1 A_{(s),1} \cdots W_l A_{(s),l},$$

for some $A_{(s),0}, A_{(s),1}, A_{(s),l} \in \mathbb{K}^{s\times s}$, one can rewrite the series (2) as a generalized nc power series, see [21][Theorem 4.6]. A major difference with the case $s = 1$ is that the coefficients $\mathfrak{R}_w$ have to satisfy very restrictive recursive equations, see [21][Remark 4.3].

IV. Conclusions

Rational functions in noncommuting indeterminates occur in many areas of system theory: most control problems involve rational expressions in matrix parameters. It is well known that rational nc power series provide a powerful tool for studying nc rational functions that are regular at 0. It is therefore natural to study power series expansion of a nc rational function around a matrix point, since

- the matrix point (rather than 0 or another scalar point) may be the point of interest for the problem at hand;
- the function may not be regular at any scalar point (e.g., $(x_1x_2 - x_2x_1)^{-1}$).

In particular such expansions can provide a starting point for a suitable realization theory.

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