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Application of Jacobi's Representation Theorem to locally multiplicatively convex topological $\mathbb{R}$-Algebras

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In this talk, we intend to present the main results of [5].

Notation: For C a cone in $\mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \ldots, X_n]$, τ a locally convex topology on $\mathbb{R}[\underline{X}]$ and $K \subseteq \mathbb{R}^n$ a closed set, denote by $\overline{C}^\tau$ the closure with respect to the topology τ, and by $\mathrm{Pos}(K)$ the set of polynomials nonnegative on K.

I. Motivation

It was known to Hilbert [8] that a nonnegative real multivariable polynomial $f = \sum_\alpha f_\alpha \underline{X}^\alpha \in \mathbb{R}[\underline{X}]$ is not necessarily a sum of squares of polynomials. However, every such polynomial can be approximated by elements of $\sum \mathbb{R}[\underline{X}]^2$ (the cone of sums of squares of polynomials) with respect to the topology induced by the norm $\|\sum_\alpha f_\alpha \underline{X}^\alpha\|_1 := \sum_\alpha |f_\alpha|$. This follows immediately from a result of [1, Theorem 9.1] stating that $\overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_1} = \mathrm{Pos}([-1,1]^n)$. Moreover, there is an effective way of approximating in $\|\cdot\|_1$ any $f \in \mathrm{Pos}([-1,1]^n)$ by sums of squares [10, Theorem 3.9]: for every $\epsilon > 0$, there exists $N > 0$ depending on n, ϵ, $\deg f$ and the f_α's such that for every integer $r \geq N$, the polynomial $f_{\epsilon,r} := f + \epsilon(1 + \sum_{i=1}^n X_i^{2r}) \in \sum \mathbb{R}[\underline{X}]^2$. Applying Stone-Weierstrass, one shows that $\overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_\infty} = \mathrm{Pos}([-1,1]^n)$ holds for the coarser norm $\|f\|_\infty := \sup_{x \in [-1,1]^n} |f(x)|$ as well. However in practice, finding $\|f\|_\infty$ is a computationally difficult optimization problem, whereas $\|f\|_1$ is easy to compute. Therefore to gain more computational flexibility it is interesting to study such closures with respect to various norms on $\mathbb{R}[\underline{X}]$. The closure of $\sum \mathbb{R}[\underline{X}]^2$ with respect to the family of weighted $\|\cdot\|_p$-norms has been studied in [3]. In this paper, we consider the following more general set-up.

II. Setup

Let C be a cone in $\mathbb{R}[\underline{X}]$, τ a locally convex topology on $A := \mathbb{R}[\underline{X}]$ and $K \subseteq \mathbb{R}^n$ be a closed set. Consider the condition:

$$\overline{C}^\tau \supseteq \mathrm{Pos}(K), \tag{1}$$

An application of Hahn-Banach Separation Theorem together with Haviland's Theorem shows that (1) holds if and only if for every τ-continuous linear functional L with $L(C) \subseteq [0, \infty)$, there exists a Borel measure μ on K such that

$$\forall f \in A \quad L(f) = \int_K f \, d\mu. \tag{2}$$

This is a joint work with Mehdi Ghasemi and Murray Marshall (Department of Mathematics and Statistics, University of Saskatchewan)

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Let A be any commutative unital real algebra. For an integer $d \geq 1$, $\sum A^{2d}$ denotes the set of all finite sums of $2d$ powers of elements of A. A $\sum A^{2d}$*-module* of A is a subset S of A such that $1 \in S$, $S + S \subseteq S$ and $a^{2d} \cdot S \subseteq S$ for each $a \in A$. In this paper, we study closure results of type (1) and their corresponding representation results of type (2). The cones we consider are the $\sum A^{2d}$-modules of A, $d \geq 1$.

III. Results

In [4, Theorem 5.3] we applied T. Jacobi's representation theorem [9] to compute the closure of the cone $\sum A^{2d}$ with respect to the topology induced by a submultiplicative norm on the $\mathbb{R}$-algebra A. In the present paper we determine the closure of a $\sum A^{2d}$-module S of A with respect to any locally multiplicatively convex topology. We show that for any $d \geq 1$, this closure is exactly the set of all elements $a \in A$ such that $\alpha(a) \geq 0$ for every continuous $\mathbb{R}$-algebra homomorphism $\alpha : A \longrightarrow \mathbb{R}$ with $\alpha(S) \subseteq [0, \infty)$. We obtain a representation of any linear functional $L : A \longrightarrow \mathbb{R}$ which is continuous with respect to any such topology and non-negative on S as integration with respect to a unique Radon measure on the space of all real valued $\mathbb{R}$-algebra homomorphisms on A, and we characterize the support of the measure obtained in this way.

Our main result applies to the case of a (unital, commutative) $*$-algebra equipped with a submultiplicative $*$-seminorm, thereby generalizing results on $*$-semigroup algebras in [2, Theorem 4.2.5].

Our main result can viewed as a strengthening (in the commutative case) of the result in [11, Lemma 6.1 and Proposition 6.2] for enveloping algebras of Lie algebras.

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